Integrable Discrete Equations Derived by Similarity Reduction of the Extended Discrete KP Hierarchy

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Abstract. We consider the extended discrete KP hierarchy and show that similarity reduction of its subhierarchies lead to purely discrete equations with dependence on some number of parameters together with equations governing deformations with respect to these parameters. It is written down discrete equations which naturally generalize the first discrete Painlevé equation $\mathrm{dP_I}$ in a sense that autonomous version of these equations admit the limit to the first Painlevé equation. It is shown that each of these equations describes Bäcklund transformations of Veselov–Shabat periodic dressing lattices with odd period known also as Noumi–Yamada systems of type $A^{(1)}_{2(n-1)}$.

Key words: extended discrete KP hierarchy; similarity reductions; discrete Painlevé equations

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1 Introduction

The main goal of the present paper is to exhibit an approach, which aims to construct a broad community of one-dimensional discrete systems over finite number of fields sharing the property of having Lax pair. It is achieved by similarity reduction of equations of the so-called extended discrete KP (edKP) hierarchy [18, 19, 20], which itself is proved to be a basis for construction of many classes of integrable differential-difference systems (see, for example, [20] and references therein). By analogy with the situation when Painlevé equations and its hierarchies arise as a result of similarity reduction of integrable partial differential evolution equations [2, 6], discrete equations (like $\mathrm{dP_I}$) and its corresponding hierarchies also can be obtained by application of group-theoretic methods.

In this connection it is necessary to mention various approaches aiming to select integrable purely discrete one-dimensional equations, for instance, by applying the confinement singularity test [9], from Bäcklund transformations for the continuous Painlevé equations [8, 14], by imposing integrable boundary conditions for two-dimensional discrete equations and reductions to one-dimensional ones [12, 11, 10, 15], deriving discrete systems from proper chosen representations of some affine Weyl group [16].

In the article we propose a scheme for constructing purely discrete equations by similarity reductions of equations of edKP hierarchy and then investigate more specific cases. Namely we analyze one-component generalizations of $\mathrm{dP_I}$ — discrete equations which turn out to serve as discrete symmetry transformation for Veselov–Shabat periodic dressing lattices with odd period. The latter also known in the literature as Noumi–Yamada systems of type $A^{(1)}_l$ due to series of works by Noumi with collaborators (see, for example, [17]) where they selected the systems of ordinary differential equations admitting a number of discrete symmetries which realized as automorphisms on field of rational functions of corresponding variables and constitute some representation of extended affine Weyl group $\tilde{W}(A_l^{(1)})$.

The paper is organized as follows. In Section 2, we give necessary information on Darboux-KP chain, its invariant manifolds and the edKP hierarchy. In Section 3, we show that self-similar ansatzes yield a large class of purely discrete systems supplemented by equations governing deformations with respect to parameters entering these systems. In Section 4, we show one-component discrete equations naturally generalizing $\mathrm{dP_I}$. We prove that these discrete equations together with deformation equations are equivalent to Veselov–Shabat periodic dressing chains. Finally, in this section we show that each of these discrete systems or more exactly its autonomous version has continuous limit to $\mathrm{P_I}$.

2 Darboux-KP chain and edKP hierarchy

2.1 Darboux-KP chain and its invariant submanifolds

In this section we give a sketch of the edKP hierarchy on the basis of approach using the notion of Darboux-KP (DKP) chain introduced in [13]. Equations of DKP chain are defined in terms of two bi-infinite sets of formal Laurent series. The first set

$$\left\{h(i) = z + \sum_{s=2}^{\infty} h_s(i) z^{-s+1} : i \in \mathbb{Z}\right\}$$

consists of generating functions for Hamiltonian densities of KP hierarchy, and second one

$$\left\{a(i) = z + \sum_{s=1}^{\infty} a_s(i) z^{-s+1} : i \in \mathbb{Z}\right\}$$

is formed by Laurent series $a(i)$ each of which relates two nearest neighbors $h(i)$ and $h(i+1)$ by Darboux map $h(i) \to h(i+1) = h(i) + a_x(i)/a(i)$. The DKP chain can be interpreted as a result of successive iterations of Darboux map applying to any fixed solution of KP hierarchy, say $h(0)$, in forward and backward directions. It is given by two equations

$$\begin{aligned} &\partial_k h(i) = \partial H^{(k)}(i), \\ &\partial_k a(i) = a(i)(H^{(k)}(i+1) - H^{(k)}(i)), \qquad \partial_k \equiv \partial/\partial t_k, \qquad \partial \equiv \partial/\partial x \end{aligned} \tag{1}$$

first of which defines evolution equations of KP hierarchy in the form of local conservation laws, and the second one serves as compatibility condition of KP flows with Darboux map. The Laurent series $H^{(k)} = H^{(k)}(i)$ in (1) is the current of corresponding conservation law constructed as special linear combination over Faà di Bruno polynomials $h^{(k)} = (\partial + h)^k(1)$ by requiring to be projection of z^k on $\mathcal{H}_+ = \langle 1, h, h^{(2)}, \ldots\rangle$ [4].

In [19, 20] we have exhibited two-parameter class of invariant submanifolds of DKP chain $\mathcal{S}_l^n$ each of which is specified by condition

$$z^{l-n+1} a^{[n]}(i) \in \mathcal{H}_+(i) \qquad \forall\, i \in \mathbb{Z}. \tag{2}$$

Here $a^{[k]} = a^{[k]}(i)$ is the discrete Faà di Bruno iteration calculated with the help of the recurrence relation $a^{[k+1]}(i) = a(i)a^{[k]}(i+1)$ with $a^{[0]} \equiv 1$.

The invariant submanifolds $\mathcal{S}_l^1$ were presented in [13]. It was shown there that the restriction of DKP chain on $\mathcal{S}_0^1$ is equivalent to discrete KP hierarchy [21]. Restriction of DKP chain on $\mathcal{S}_0^1$, $\mathcal{S}_0^2$, $\mathcal{S}_0^3$ and so on leads to the notion of edKP hierarchy which is an infinite collection of discrete KP like hierarchies attached to multi-times $t^{(n)} \equiv (t_1^{(n)}, t_2^{(n)}, \ldots)$ with $n \geq 1$. All these hierarchies "live" on the same phase-space $\mathcal{M}$ which can be associated with hyperplane whose points are parametrized by infinite number of functions of discrete variable: $\{a_k = a_k(i) : k \geq 1\}$. One can treat these functions as analytic ones whose the domain of definition is restricted to $\mathbb{Z}$. We have shown in [19] that the flows on $\mathcal{S}_0^n$ are given in the form of local conservation laws as

$$z^{k(n-1)}\partial_k^{(n)} \ln a(i) = K_{(n)}^{[kn]}(i+1) - K_{(n)}^{[kn]}(i), \tag{3}$$

that is on $\mathcal{S}_0^n$ one has $H^{(k)} = z^{k(1-n)}K_{(n)}^{[kn]}$ with

$$K_{(n)}^{[pn]} \equiv \sum_{k=0}^{p} q_k^{(n,pn)} z^{k(n-1)} a^{[(p-k)n]}.$$

Here the coefficients $q_k^{(n,r)}$ are some polynomials in variables $\{a_l^{(s)} \equiv \Lambda^s(a_l)\}$ (for definition, see below (5)). In what follows, we refer to (3) as nth subhierarchy of edKP hierarchy. In the following subsection we show Lax pair for edKP hierarchy and write down evolution equations on the functions $q_k^{(n,r)}$ generated by (3) in its explicit form.

2.2 Lax representation for edKP hierarchy

First, let us recall the relationship between Faà di Bruno iterations and formal Baker–Akhiezer function of KP hierarchy [4, 13] $\psi = \psi_i = \left(1 + \sum_{s\geq 1} w_s(i) z^{-s}\right) e^{\xi(t,z)}$ with $\xi(t,z) = \sum_{k\geq 1} t_k z^k$. One has

$$h^{(k)} = \frac{\partial^k(\psi)}{\psi}, \qquad H^{(k)} = \frac{\partial_k(\psi)}{\psi}, \qquad a^{[k]}(i) = \frac{\Psi_{i+k}}{\Psi_i}, \qquad \Psi_i \equiv z^i \psi_i.$$

These relations allow representing the DKP chain constrained by relation (2) as compatibility condition of auxiliary linear systems. As was shown in [19, 20], the restriction of DKP chain on $\mathcal{S}_0^n$ leads to linear discrete systems (see also [18])

$$Q_{(n)}^r \Psi = z^r \Psi, \qquad z^{k(n-1)}\partial_k^{(n)}\Psi = (Q_{(n)}^{kn})_+ \Psi \tag{4}$$

with the pair of discrete operators

$$Q_{(n)}^r \equiv \sum_{s\geq 0} q_s^{(n,r)} z^{s(n-1)} \Lambda^{r-sn}, \qquad (Q_{(n)}^{kn})_+ \equiv \sum_{s=0}^{k} q_s^{(n,kn)} z^{s(n-1)} \Lambda^{(k-s)n}.$$

Here Λ is a usual shift operator acting on arbitrary function $f = f(i)$ of the discrete variable as $(\Lambda f)(i) = f(i+1)$. The coefficients $q_s^{(n,r)} = q_s^{(n,r)}(i)$ are uniquely defined as polynomial functions on $\mathcal{M}$ in coordinates a_l through the relation[1]

$$z^r = \sum_{s\geq 0} z^{s(n-1)} q_s^{(n,r)} a^{[r-sn]}, \qquad r \in \mathbb{Z}. \tag{5}$$

We assign to $a_l^{(s)}$ its scaling dimension: $[a_l^{(s)}] = l$. One says that any polynomial Q_k in a_l is a homogeneous one with degree k if $Q_k \to \epsilon^k Q_k$ when $a_l \to \epsilon^l a_l$.

[1] As was mentioned above, in fact, these are polynomials in $a_l^{(s)}$. For notational convenience we assume that $q_0^{(n,r)} \equiv 1$ for all possible n and r.

The consistency condition for the pair of equations (4) reads as Lax equation

$$z^{k(n-1)}\partial_k^{(n)} Q_{(n)}^r = [(Q_{(n)}^{kn})_+, Q_{(n)}^r]$$

and can be rewritten in its explicit form as

$$\partial_k^{(n)} q_s^{(n,r)}(i) = Q_{s,k}^{(n,r)}(i) \equiv \sum_{j=0}^{k} q_j^{(n,kn)}(i)\cdot q_{s-j+k}^{(n,r)}(i+(k-j)n) - \sum_{j=0}^{k} q_j^{(n,kn)}(i+r-(s-j+k)n)\cdot q_{s-j+k}^{(n,r)}(i). \tag{6}$$

It is important also to take into account algebraic relations

$$q_k^{(n,r_1+r_2)}(i) = \sum_{j=0}^{k} q_j^{(n,r_1)}(i)\cdot q_{k-j}^{(n,r_2)}(i+r_1-jn) = \sum_{j=0}^{s} q_j^{(n,r_2)}(i)\cdot q_{k-j}^{(n,r_1)}(i+r_2-jn) \tag{7}$$

coded in permutability operator relation $Q_{(n)}^{r_1+r_2} = Q_{(n)}^{r_1} Q_{(n)}^{r_2} = Q_{(n)}^{r_2} Q_{(n)}^{r_1}$.

It is quite obvious that equations (6) and (7) admit reductions with the help of simple conditions $q_s^{(n,r)} \equiv 0$ ($\forall\, s \geq l+1$) for some fixed $l \geq 1$. As was shown in [19, 20], these reductions can be properly described in geometric setting as double intersections of invariant manifolds of DKP chain: $\mathcal{S}_{n,r,l} = \mathcal{S}_0^n \cap \mathcal{S}_{l-1}^{ln-r}$. The formula (6) when restricting to $\mathcal{S}_{n,r,l}$ is proved to be a container for many integrable differential-difference systems (lattices) which can be found in the literature (for reference see e.g. [20]).

2.3 Conservation laws for edKP hierarchy

The conserved densities for edKP hierarchy can be constructed in standard way as residues

$$h_s^{(n)} = \mathrm{Res}\left(Q_{(n)}^{sn}\right) = q_s^{(n,sn)}. \tag{8}$$

Corresponding currents are easily derived from (6). One has

$$\partial_k^{(n)} q_s^{(n,sn)}(i) = Q_{s,k}^{(n,sn)}(i) = J_{s,k}^{(n)}(i+n) - J_{s,k}^{(n)}(i) = I_{s,k}^{(n)}(i+1) - I_{s,k}^{(n)}(i) \tag{9}$$

with

$$J_{s,k}^{(n)}(i) = \sum_{l=0}^{k-1}\sum_{j=1}^{k-l} q_l^{(n,kn)}(i+(j+l-k-1)n)\cdot q_{s+k-l}^{(n,sn)}(i+(j-1)n)$$

and

$$I_{s,k}^{(n)}(i) = \sum_{j=1}^{n} J_{s,k}^{(n)}(i+j-1) = \sum_{l=0}^{k-1}\sum_{j=1}^{(k-l)n} q_l^{(n,kn)}(i+j+(l-k)n-1)\cdot q_{s+k-l}^{(n,sn)}(i+j-1).$$

We observe that by virtue of relations (7) with $r_1 = ln$, $r_2 = sn$ and $k = l+s$, the relation $Q_{l,s}^{(n,ln)} = Q_{s,l}^{(n,sn)}$ with arbitrary $l, s \in \mathbb{N}$ is identity and therefore we can write down exactness property relation

$$\partial_s^{(n)} h_l^{(n)} = \partial_l^{(n)} h_s^{(n)} \qquad \forall\, s, l \geq 1. \tag{10}$$

Looking at (9), it is natural to suppose that there exist some homogeneous polynomials $\xi_s^{(n)}$ in q_k such that

$$h_s^{(n)}(i) = \sum_{j=1}^{n} \xi_s^{(n)}(i+j-1) \qquad \forall\, s \geq 1. \tag{11}$$

If so, then one must recognize that $\{\xi_s^{(n)} : s \geq 1\}$ is an infinite collection of conserved densities of nth subhierarchy of edKP hierarchy, i.e.

$$\partial_k^{(n)} \xi_s^{(n)}(i) = J_{s,k}^{(n)}(i+1) - J_{s,k}^{(n)}(i).$$

Then the formula (11) says that each $\xi_s^{(n)}$ is equivalent to $h_s^{(n)}/n$ modulo adding trivial densities. Substituting (11) into (7) with $r_1 = 1$ and $r_2 = kn$ leaves us with the following equation[2]:

$$\xi_s^{(n)}(i+n) - \xi_s^{(n)}(i) = \sum_{j=1}^{s} q_j(i+jn) \cdot q_{s-j}^{(n,sn)}(i) - \sum_{j=1}^{s} q_j(i) \cdot q_{s-j}^{(n,sn)}(i+1-jn).$$

One can check that solution of this equation is given by

$$\xi_s^{(n)}(i) = \sum_{l=1}^{s} \sum_{j=1}^{l} q_l(i+(j-1)n) \cdot q_{s-l}^{(n,sn-1)}(i+1+(j-l-1)n). \tag{12}$$

For example, we can write down the following:

$$\xi_1^{(n)}(i) = q_1(i), \qquad \xi_2^{(n)}(i) = q_2(i) + q_2(i+n) + q_1(i) \cdot \sum_{j=1-n}^{n-1} q_1(i+j),$$

$$\begin{aligned} \xi_3^{(n)}(i) = {} & q_3(i) + q_3(i+n) + q_3(i+2n) + q_2(i) \cdot \sum_{j=1-2n}^{n-1} q_1(i+j) \\ & + q_2(i+n) \cdot \sum_{j=1-n}^{2n-1} q_1(i+j) + q_1(i) \cdot \left(\sum_{j=1-n}^{2n-1} q_2(i+j) \right. \\ & \left. + \sum_{l=1-n}^{n-1} \sum_{j=1-n}^{n-1} q_1(i+l) \cdot q_1(i+j+l) + \sum_{l=1}^{n-1} \sum_{j=l}^{n-1} q_1(i+l-n) \cdot q_1(i+j) \right). \end{aligned}$$

Now let us show that conserved densities $\xi_s^{(n)}$ satisfy some characteristic equations. From (8) one derives

$$\frac{\delta h_s^{(n)}}{\delta q_l}(i) = n \frac{\delta \xi_s^{(n)}}{\delta q_l}(i) = sn q_{s-l}^{(n,sn-1)}(i+1-ln).$$

Substituting the latter in (12) one gets

$$s \xi_s^{(n)}(i) = \sum_{l=1}^{s} \sum_{j=1}^{l} q_l(i+(j-1)n) \frac{\delta \xi_s^{(n)}}{\delta q_l}(i+(j-1)n).$$

[2] Here and in what follows $q_k(i) \equiv q_k^{(n,1)}(i)$. It is important to note that the mapping $\{a_k\} \to \{q_k\}$ is invertible and one can use $\{q_k\}$ as coordinates of $\mathcal{M}$.

3 Self-similar ansatzes for edKP hierarchy and integrable discrete equations

In what follows, we consider nth edKP subhierarchy restricted to the first p time variables $\{t_1^{(n)},\dots,t_p^{(n)}\}$ with $p\geq 2$. This system is obviously invariant under the group of scaling transformations

$$G_p=\left\{g:q_k\to\epsilon^k q_k\ \ (k\geq 1),\ \ t_l^{(n)}\to\epsilon^{-l}t_l^{(n)}\ \ (l=1,\dots,p)\right\}.$$

Since $q_k^{(n,r)}$, $Q_{k,s}^{(n,r)}$, $\xi_k^{(n)}$ are homogeneous polynomials in q_k of corresponding degrees, the transformation g yields

$$q_k^{(n,r)}\to\epsilon^k q_k^{(n,r)},\qquad Q_{k,s}^{(n,r)}\to\epsilon^{k+s}Q_{k,s}^{(n,r)},\qquad \xi_k^{(n)}\to\epsilon^k\xi_k^{(n)}.$$

Let us consider similarity reductions of nth edKP subhierarchy requiring

$$\left(k+\sum_{j=1}^{p}jt_j^{(n)}\partial_j^{(n)}\right)q_k=kq_k+\sum_{j=1}^{p}jt_j^{(n)}Q_{k,j}^{(n,1)}=0\qquad\forall\,k\geq 1. \tag{13}$$

Then any homogeneous polynomial in q_l satisfies corresponding self-similarity condition. In particular, one has

$$\left(k+\sum_{j=1}^{p}jt_j^{(n)}\partial_j^{(n)}\right)q_k^{(n,r)}=kq_k^{(n,r)}+\sum_{j=1}^{p}jt_j^{(n)}Q_{k,j}^{(n,r)}=0. \tag{14}$$

More explicitly, one can rewrite (14) as

$$\begin{aligned}q_k^{(n,r)}(i)\left(k+\alpha_i^{(n,n)}-\alpha_{i+r-kn}^{(n,n)}\right)&+\sum_{s=1}^{p}q_{k+s}^{(n,r)}(i+sn)\sum_{j=s}^{p}jt_j^{(n)}q_{j-s}^{(n,jn)}(i)\\&-\sum_{s=1}^{p}q_{k+s}^{(n,r)}(i)\sum_{j=s}^{p}jt_j^{(n)}q_{j-s}^{(n,jn)}(i+r-(k+s)n)=0\end{aligned}$$

with

$$\alpha_i^{(n,n)}\equiv\sum_{j=1}^{p}jt_j^{(n)}h_j^{(n)}(i)=\sum_{s=1}^{n}\alpha_{i+s-1}^{(n)},$$

where $\alpha^{(n)}\equiv\sum_{j=1}^{p}jt_j^{(n)}\xi_j^{(n)}$. We observe that $\alpha^{(n,n)}$ do not depend on evolution parameters $\{t_1^{(n)},\dots,t_p^{(n)}\}$ provided that (13) is valid. Indeed, taking into account exactness property (10) one has

$$\partial_s^{(n)}\alpha^{(n,n)}=sh_s^{(n)}+\sum_{j=1}^{p}jt_j^{(n)}\partial_s^{(n)}h_j^{(n)}=\left(s+\sum_{j=1}^{p}jt_j^{(n)}\partial_j^{(n)}\right)h_s^{(n)}=0.$$

For Baker–Akhiezer function, one has the corresponding self-similarity condition in the form

$$\left(z\partial_z-\sum_{j=1}^{p}jt_j^{(n)}\partial_j^{(n)}\right)\psi=0$$

or

$$z\partial_z\psi_i = \alpha_i^{(n,n)}\psi_i + \sum_{s=1}^{p}\psi_{i+sn}\sum_{j=s}^{p} jt_j^{(n)}q_{j-s}^{(n,jn)}.$$

One can rewrite the above formulas in terms of self-similarity ansatzes

$$T_l = \frac{t_l^{(n)}}{(pt_p^{(n)})^{l/p}}, \qquad l = 1,\dots,p-1, \qquad \xi = (pt_p^{(n)})^{1/p}z,$$
$$q_s^{(n,r)} = \frac{1}{(pt_p^{(n)})^{s/p}}x_s^{(n,r)}, \qquad Q_{s,k}^{(n,r)} = \frac{1}{(pt_p^{(n)})^{(s+k)/p}}X_{s,k}^{(n,r)}, \qquad \xi_s^{(n)} = \frac{1}{(pt_p^{(n)})^{s/p}}\zeta_s^{(n)}.$$

At this point we can conclude that similarity reduction of nth edKP subhierarchy yields the system of purely discrete equations

$$\begin{aligned} x_k^{(n,r)}(i)\left(k + \alpha_i^{(n,n)} - \alpha_{i+r-kn}^{(n,n)}\right) &+ \sum_{s=1}^{p-1} x_{k+s}^{(n,r)}(i+sn)\left(\sum_{j=s}^{p-1} jT_j x_{j-s}^{(n,jn)}(i) + x_{p-s}^{(n,pn)}(i)\right) \\ &- \sum_{s=1}^{p-1} x_{k+s}^{(n,r)}(i)\left(\sum_{j=s}^{p-1} jT_j x_{j-s}^{(n,jn)}(i+r-(k+s)n) + x_{p-s}^{(n,pn)}(i+r-(k+s)n)\right) \\ &+ x_{k+p}^{(n,r)}(i+pn) - x_{k+p}^{(n,r)}(i) = 0, \end{aligned} \tag{15}$$

$$\alpha^{(n)} = \sum_{j=1}^{p-1} jT_j\zeta_j^{(n)} + \zeta_p^{(n)} \tag{16}$$

supplemented by deformation equations

$$\partial_{T_k}x_s^{(n,r)} = X_{s,k}^{(n,r)}, \qquad k = 1,\dots,p-1.$$

These equations appear as a consistency condition of the following linear isomonodromy problem

$$\xi\psi_{i+r} + \sum_{k\ge1}\xi^{1-k}x_k^{(n,r)}\psi_{i+r-kn} = \xi\psi_i,$$

$$\partial_{T_k}\psi_i = \xi^k\psi_{i+kn} + \sum_{s=1}^{k}\xi^{k-s}x_s^{(n,kn)}\psi_{i+(k-s)n}, \qquad k = 1,\dots,p-1,$$

$$\xi\partial_\xi\psi_i = \alpha_i^{(n,n)}\psi_i + \sum_{s=1}^{p}\psi_{i+sn}\left(\sum_{j=s}^{p-1} jT_j x_{j-s}^{(n,jn)}(i) + x_{p-s}^{(n,pn)}(i)\right).$$

The pair of equations (15) and (16) when restricting dynamics on $\mathcal{S}_{n,r,l}$ becomes a system over finite number of fields. If one requires $x_k^{(n,r)}(i) \equiv 0$ ($\forall\, k \ge l+1$) then l-th equation in (15) is specified as

$$x_l^{(n,r)}(i)\cdot\left\{l + \sum_{s=1}^{n}\alpha_{i+s-1}^{(n)} - \sum_{s=1}^{n}\alpha_{i+r-ln+s-1}^{(n)}\right\} = 0.$$

Since it is supposed that $x_l^{(n,r)} \not\equiv 0$ then the constants $\alpha_i^{(n)}$ are forced to be subjects of constraint

$$l + \sum_{s=1}^{n}\alpha_{i+s-1}^{(n)} - \sum_{s=1}^{n}\alpha_{i+r-ln+s-1}^{(n)} = 0. \tag{17}$$

Remark that this equation makes no sense in the case $r = ln$. In the following section we consider restriction on $\mathcal{S}_{n,1,1}$ corresponding to Bogoyavlenskii lattice.

4 Examples of one-field discrete equations

4.1 One-field discrete system generalizing dP_I and its hierarchies

Let us consider reductions corresponding to $\mathcal{S}_{n,1,1}$ with arbitrary $n \geq 2$. For $p = 2$ one obtains one-field system[3]

$$T + \sum_{s=1-n}^{n-1} x_{i+s} = \frac{\alpha_i^{(n)}}{x_i}, \qquad 1 + \sum_{s=1}^{n-1} \alpha_{i+s}^{(n)} - \sum_{s=1}^{n-1} \alpha_{i-s}^{(n)} = 0 \tag{18}$$

together with deformation equation

$$x_i' = x_i \left(\sum_{s=1}^{n-1} x_{i+s} - \sum_{s=1}^{n-1} x_{i-s} \right) \tag{19}$$

which is nothing else but Bogoyavlenskii lattice. The pair of equations (18) is a specification of (16) and (17). In the case $n = 2$, (18) becomes dP_I while (19) turns into Volterra lattice. Hierarchy of dP_I appears to be related with a matrix model of two-dimensional gravity (see, for example, [7] and references therein). In [5] Joshi and Cresswell found out representation of dP_I hierarchy with the help of recursion operator.

We are in position to exhibit hierarchy of more general equation (18) by considering the cases $p = 3, 4$ and so on. From (12) one has

$$\zeta_s^{(n)}(i) = x_i x_{s-1}^{(n,sn-1)}(i + 1 - n).$$

Substituting this into (16) leads to

$$T_1 + \sum_{j=2}^{p-1} j T_j x_{j-1}^{(n,jn-1)}(i + 1 - n) + x_{p-1}^{(n,pn-1)}(i + 1 - n) = \frac{\alpha_i^{(n)}}{x_i}, \tag{20}$$

where $\alpha_i^{(n)}$'s are constants which are supposed to solve the same algebraic equations as in (18). As an example, for $p = 3$, one has

$$T_1 + 2T_2 \sum_{s=1-n}^{n-1} x_{i+s} + \sum_{s=1-n}^{n-1} \sum_{j=1-n}^{n-1} x_{i+s} x_{i+j+s} + \sum_{s=1}^{n-1} \sum_{j=s}^{n-1} x_{i+s-n} x_{i+j} = \frac{\alpha_i^{(n)}}{x_i}.$$

Equation (20) should be complemented by deformation ones

$$\partial_{T_k} x_i = x_i \left(\sum_{s=1}^{n-1} \zeta_k^{(n)}(i + s) - \sum_{s=1}^{n-1} \zeta_k^{(n)}(i - s) \right), \qquad k = 1, \ldots, p - 1$$

which are higher members in Bogoyavlenskii lattice hierarchy.

4.2 Equivalence (18) and (19) to Veselov–Shabat periodic dressing lattices

Here we restrict ourselves by consideration only (18) and (19) without their hierarchies. Our aim is to show equivalence of this pair of equations to well-known Veselov–Shabat periodic dressing lattices. To this end, we introduce the variables $\{r_0, \ldots, r_{2(n-1)}\}$ by identifying

$$x_{i+k-1} = -r_{2k} - r_{2k+1}, \qquad x_{i+n+k-2} = -r_{2k-1} - r_{2k}, \qquad k = 1, \ldots, n - 1,$$

[3] Here $x_i \equiv x_1^{(n,1)}(i)$ and $T \equiv T_1$.

$$\sum_{s=1}^{2(n-1)} x_{i+s-1} + T = r_0 + r_1,$$

where subscripts are supposed to be elements of $\mathbb{Z}/(2n-1)\mathbb{Z}$. One can show that with the constants

$$c_{2k} = \alpha^{(n)}_{i+k-1}, \qquad c_{2k-1} = -\alpha^{(n)}_{i+k+n-2}, \qquad k = 1, \ldots, n-1,$$
$$c_0 = 1 - \sum_{s=1}^{n-1} \alpha^{(n)}_{i+s-1} + \sum_{s=1}^{n-1} \alpha^{(n)}_{i+s+n-2}$$

the equations (18) and (19) can be rewritten as Veselov–Shabat periodic dressing chain [22, 1]

$$r'_k + r'_{k+1} = r^2_{k+1} - r^2_k + c_k, \qquad r_{k+N} = r_k, \qquad c_{k+N} = c_k \tag{21}$$

with $N = 2n-1$. For $N = 3$ this system is known to be equivalent to $\mathrm{P_{IV}}$. In the variables $f_k = r_k + r_{k+1}$ the system (21) has the form [22]

$$f'_k = f_k \left(\sum_{r=1}^{n-1} f_{k+2r-1} - \sum_{r=1}^{n-1} f_{k+2r} \right) + c_k, \qquad k = 0, \ldots, 2(n-1). \tag{22}$$

The system (22) for $n \geq 3$ can be considered as higher-order generalization of $\mathrm{P_{IV}}$ written in symmetric form. It was investigated in [17] where Noumi and Yamada showed its invariance with respect to discrete symmetry transformations (or Bäcklund–Schlesinger transformations) which are realized as a number of automorphisms $\{\pi, s_0, \ldots, s_{2(n-1)}\}$ on the field of rational functions in variables $\{c_0, \ldots, c_{2(n-1)}\}$ and $\{f_0, \ldots, f_{2(n-1)}\}$ as follows:

$$s_k(c_k) = -c_k, \qquad s_k(c_l) = c_l + c_k, \qquad l = k \pm 1, \qquad s_k(c_l) = c_l, \qquad l \neq k, k \pm 1,$$
$$s_k(f_k) = f_k, \qquad s_k(f_l) = f_l \pm \frac{c_k}{f_k}, \qquad l = k \pm 1, \qquad s_k(f_l) = f_l, \qquad l \neq k, k \pm 1,$$
$$\pi(f_k) = f_{k+1}, \qquad \pi(c_k) = c_{k+1}.$$

One can easily rewrite the formulas of action of automorphisms on generators r_k as follows:

$$s_k(r_k) = r_k - \frac{c_k}{r_k + r_{k+1}}, \qquad s_k(r_{k+1}) = r_k + \frac{c_k}{r_k + r_{k+1}},$$
$$s_k(r_l) = r_l, \qquad l \neq k, k+1, \qquad \pi(r_k) = r_{k+1}.$$

It is known by [16] that this set of automorphisms define a representation of the extended affine Weyl group $\tilde{W}(A^{(1)}_l) = \langle \pi, s_0, \ldots, s_l \rangle$ with $l = 2(n-1)$ whose generators satisfy the relations

$$s_k^2 = 1, \qquad s_k s_m = s_m s_k, \qquad m \neq k \pm 1, \qquad (s_k s_m)^3 = 1, \qquad m = k \pm 1$$

for $k, m = 0, \ldots, l$ and

$$\pi^{l+1} = 1, \qquad \pi s_k = s_{k+1} \pi.$$

The shift governed by discrete equations (18) affects on the system (21) as Bäcklund transformation

$$\overline{r}_0 = r_2 + \frac{c_1}{r_1 + r_2}, \qquad \overline{r}_k = r_{k+2}, \qquad k = 1, \ldots, 2n-3,$$
$$\overline{r}_{2(n-1)} = r_1 - \frac{c_1}{r_1 + r_2},$$
$$\overline{c}_0 = c_1 + c_2, \qquad \overline{c}_k = c_{k+2}, \qquad k = 1, \ldots, 2n-4,$$
$$\overline{c}_{2n-3} = c_0 + c_1, \qquad \overline{c}_{2(n-1)} = -c_{2(n-1)}.$$

By direct calculations one can check that it coincides with $\tau = s_1 \pi^2 \in \tilde{W}\big(A^{(1)}_{2(n-1)}\big)$.

4.3 Continuous limit of stationary version of (18)

Let us show in this subsection that stationary version of the equation (18)

$$T + \sum_{s=1-n}^{n-1} x_{i+s} = \frac{\alpha}{x_i}, \tag{23}$$

for any n, is integrable discretization of $\mathrm{P_I}$: $w'' = 6w^2 + t$.

One divides the real axis into segments of equal length ε. One considers discrete set of values $\{t = i\varepsilon : i \in \mathbb{Z}\}$. Values of the function w, respectively, are taken for all such values of the variable t. Therefore one can denote $w(t) = w_i$. Let

$$x_i = 1 + \varepsilon^2 w_i, \qquad \alpha = 1 - 2n - \varepsilon^4 t, \qquad T_1 = -2n + 1. \tag{24}$$

Substituting (24) in the equation (23) and taking into account the relations of the form

$$x_{i+1} = 1 + \varepsilon^2 w_{i+1} = 1 + \varepsilon^2 \left\{ w + \varepsilon w' + \frac{\varepsilon^2}{2} w'' + \cdots \right\}$$

and turning then ε to zero we obtain, in continuous limit, the equation

$$\sum_{s=1}^{n-1} (n-s)^2 \cdot w'' = -t - (2n-1)w^2$$

which, by suitable rescaling, can be deduced to canonical form of $\mathrm{P_I}$. This situation goes in parallel with that when Bogoyavlenskii lattice (19), for all values of $n \geq 2$, has continuous limit to Korteweg–de Vries equation [3].

5 Conclusion

We have considered in the paper edKP hierarchy restricted to p evolution parameters $\{t_1^{(n)}, \ldots, t_p^{(n)}\}$. It was shown that self-similarity constraint imposed on this system is equivalent to purely discrete equations (15) and (16) supplemented by $(p-1)$ deformation equations which in fact are evolution equations governing $(p-1)$ flows of nth edKP subhierarchy. What is crucial in our approach is that we selected quantities $\alpha^{(n,n)}$ which enter these discrete equations and turn out to be independent on evolution parameters due to exactness property for conserved densities.

Discrete systems over finite number of fields arise when one restricts edKP hierarchy on invariant submanifold $\mathcal{S}_{n,r,l}$. In the present paper we considered only one-field discrete equations corresponding to $\mathcal{S}_{n,1,1}$ with $n \geq 2$. It is our observation that in this case discrete equation under consideration supplemented by deformation one is equivalent to Veselov–Shabat periodic dressing lattice with odd period. It is known due to Noumi and Yamada that this finite-dimensional system of ordinary differential equations admit finitely generated group of Bäcklund transformations which realize representation of extended affine Weyl group $\tilde{W}(A_{2(n-1)}^{(1)})$. It is natural to expect that restriction of edKP hierarchy to other its invariant submanifolds also can yields finite-dimensional systems invariant under some finitely generated groups of discrete transformations. We are going to present relevant results on this subject in subsequent publications.

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[1] Adler V.E., Nonlinear chains and Painlevé equations, *Phys. D*, 1994, V.73, 335–351.

[2] Airault H., Rational solutions of Painlevé equations, *Stud. Appl. Math.*, 1979, V.61, 31–53.

[3] Bogoyavlenskii O.I., Breaking solitons: nonlinear integrable equations, Moscow, Nauka, 1991 (in Russian).

[4] Casati P., Falqui G., Magri F., Pedroni, M., The KP theory revisited I, II, III, IV, SISSA Preprint, 1996, SISSA/2-5/96/FM.

[5] Cresswell C., Joshi N., The discrete first, second and thirty-fourth Painlevé hierarchies, *J. Phys. A: Math. Gen.*, 1999, V.32, 655–669.

[6] Flaschka H., Newell A.C., Monodromy- and spectrum-preserving deformations I, *Comm. Math. Phys.*, 1980, V.76, 65–116.

[7] Fokas A.S., Its A.R., Kitaev A.R., The isomonodromy approach to matrix models in 2D quantum gravity, *Comm. Math. Phys.*, 1992, V.147, 395–430.

[8] Fokas A.S., Grammaticos B., Ramani A., From continuous to discrete Painlevé equations, *J. Math. Anal. Appl.*, 1993, V.180, 342–360.

[9] Grammaticos B., Ramani A., Papageorgiou V., Do integrable mapping have the Painlevé property?, *Phys. Rev. Lett.*, 1991, V.67, 1825–1828.

[10] Grammaticos B., Papageorgiou V., Ramani A., Discrete dressing transformations and Painlevé equations, *Phys. Lett. A*, 1997, V.235, 475–479.

[11] Grammaticos B., Ramani A., Satsuma J., Willox R., Carstea A.S., Reductions of integrable lattices, *J. Nonlinear Math. Phys.*, 2005, V.12, suppl. 1, 363–371.

[12] Kazakova T.G., Finite-dimensional reductions of the discrete Toda chain, *J. Phys. A: Math. Gen.*, 2004, V.37, 8089–8102.

[13] Magri F., Pedroni M., Zubelli J.P., On the geometry of Darboux transformations for the KP hierarchy and its connection with the discrete KP hierarchy, *Comm. Math. Phys.*, 1997, V.188, 305–325.

[14] Nijhoff F.W., Satsuma J., Kajiwara K., Grammaticos B., Ramani A., A study of the alternative discrete Painlevé-II equation, *Inverse Problems*, 1996, V.12, 697–716.

[15] Nijhoff F.W., Papageorgiou V.G., Similarity reductions of integrable lattices and discrete analogues of the Painlevé II equation, *Phys. Lett. A*, 1991, V.153, 337-344.

[16] Noumi M., Yamada Y., Affine Weyl groups, discrete dynamical systems and Painlevé equations, *Comm. Math. Phys.*, 1998, V.199, 281–295.

[17] Noumi M., Yamada Y., Higher order Painlevé equations of type $A_l^{(1)}$, *Funkcial. Ekvac.*, 1998, V.41, 483–503.

[18] Svinin A.K., Extension of the discrete KP hierarchy, *J. Phys. A: Math. Gen.*, 2002, V.35, 2045–2056.

[19] Svinin A.K., Extended discrete KP hierarchy and its reductions from a geometric viewpoint, *Lett. Math. Phys.*, 2002, V.61, 231–239.

[20] Svinin A.K., Invariant submanifolds of the Darboux–Kadomtsev–Petviashvili chain and an extension of the discrete Kadomtsev–Petviashili hierarchy, *Theor. Math. Phys.*, 2004, V.141, 1542–1561.

[21] Ueno K., Takasaki K., Toda lattice hierarchy. I, II, *Proc. Japan Acad. Ser. A Math. Sci.*, 1983, V.59, 167–170; 215–218.

[22] Veselov A.P., Shabat A.B., Dressing chains and the spectral theory of the Schrödinger operator, *Funct. Anal. Appl.*, 1993, V.27, 81–96.