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Representations of $U(2\infty)$ and the Value of the Fine Structure Constant

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Abstract. A relativistic quantum mechanics is formulated in which all of the interactions are in the four-momentum operator and Lorentz transformations are kinematic. Interactions are introduced through vertices, which are bilinear in fermion and antifermion creation and annihilation operators, and linear in boson creation and annihilation operators. The fermion-antifermion operators generate a unitary Lie algebra, whose representations are fixed by a first order Casimir operator (corresponding to baryon number or charge). Eigenvectors and eigenvalues of the four-momentum operator are analyzed and exact solutions in the strong coupling limit are sketched. A simple model shows how the fine structure constant might be determined for the QED vertex.

Key words: point form relativistic quantum mechanics; antisymmetric representations of infinite unitary groups; semidirect sum of unitary with Heisenberg algebra

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1 Introduction

With the exception of QCD and gravitational self couplings, all of the fundamental particle interaction Hamiltonians have the form of bilinears in fermion and antifermion creation and annihilation operators times terms linear in boson creation and annihilation operators. For example QED is a theory bilinear in electron and positron creation and annihilation operators and linear in photon creation and annihilation operators. The well-known nucleon-antinucleon-meson interactions are of this form as are the weak interactions. These interactions differ of course in the way the fermions are coupled to the bosons, including the way in which internal symmetries are incorporated. An exception is QCD, where due to the $SU(3)_{\rm color}$ symmetry which generates gluon self coupling terms, the gluon sector is no longer linear in creation and annihilation operators. The other exception is the gravitational interaction; since gravitons carry energy and momentum, they also can couple to themselves.

If $a^\dagger$, $b^\dagger$ and $c^\dagger$ denote, respectively, fermion, antifermion and boson creation operators, the aforementioned interactions can all be written as $(a^\dagger+b)(a+b^\dagger)(c^\dagger+c)$, while the "kinetic energy" terms are of the form $a^\dagger a - bb^\dagger + c^\dagger c$. Written in this way, the fermion-antifermion bilinears $a^\dagger a$, $bb^\dagger$, $a^\dagger b^\dagger$, and ba close to form a Lie algebra which, with indices attached to the creation and annihilation operators, is related to the group $U(2N)$. Similarly the boson operators $c^\dagger c$, $c^\dagger$ and c close to form a Lie algebra related to the semidirect sum of unitary with Heisenberg algebras. Then the aforementioned interactions can all be viewed as products of these two Lie algebras.

The goal of this paper is to exploit this Lie algebra structure to analyze eigensolutions of the four-momentum operator. The four-momentum operator P^μ will be written as the sum of free and interacting four-momentum operators $P^\mu = P^\mu(\mathrm{fr}) + P^\mu(I)$, where the free four-momentum operator has the "relativistic kinetic energy" form of creation and annihilation operators, while the interacting four-momentum operator has the Lie algebra product structure.

To guarantee the relativistic covariance of the theory, it is required that

$$[P^\mu, P^\nu] = 0, \tag{1}$$
$$U_\Lambda P^\mu U_\Lambda^{-1} = \left(\Lambda^{-1}\right)^\mu_\nu P^\nu, \tag{2}$$

where U_Λ is the unitary operator representing the Lorentz transformation Λ. These "point form" equations [1][1], in which all of the interactions are in the four-momentum operator and the Lorentz transformations are kinematic, lead to the eigenvalue problem

$$P^\mu |\Psi_p\rangle = p^\mu |\Psi_p\rangle, \tag{3}$$

where p^μ is the four-momentum eigenvalue and $|\Psi_p\rangle$ the eigenvector of the four-momentum operator, which is an element in a generalized fermion-antifermion-boson Fock space. The physical vacuum and physical bound and scattering states should then all arise as the appropriate solutions of the eigenvalue equation (3). How this might be done is sketched in the following sections.

2 Vertex Interactions

The free four-momentum operator is generated from irreducible representations of the Poincaré group[2]. The positive mass, positive spin representations act on a representation space that can be written as the product of square integrable functions over the forward hyperboloid times a spin space, $L^2(SO(1,3)/SO(3)) \otimes V^j$, with the group action on states given by

$$U_\Lambda |v,\sigma\rangle = \sum |\Lambda v, \sigma'\rangle D^j_{\sigma',\sigma}(R_W), \qquad U_a |v,\sigma\rangle = e^{imv\cdot a}|v,\sigma\rangle, \tag{4}$$

here a is a four-vector space-time translation, v is the four-velocity on the forward hyperboloid satisfying $v \cdot v = 1$ and σ is the relativistic spin projection ranging between $-j$ and j and generating a basis in the $2j+1$ dimensional spin space V^j. m is a mass parameter that (along with the spin j) labels an irreducible representation; the usual four-momentum is defined by $p = mv$. $D^j(\cdot)$ in equation (4) is a matrix element of the rotation group, with its argument a Wigner rotation defined by $R_W = B^{-1}(\Lambda v)\Lambda B(v)$ [5].

From this irreducible representation it is possible to define many-particle creation and annihilation operators which satisfy the following (anti)commutation relations:

$$[a_i, a^\dagger_{i'}]_+ = \delta_{i,i'}, \qquad [b_i, b^\dagger_{i'}]_+ = \delta_{i,i'}, \qquad [c_k, c^\dagger_{k'}]_- = \delta_{k,k'}, \tag{5}$$

the a_i, b_i, c_k are respectively bare fermion, antifermion and boson annihilation operators and satisfy appropriate commutation or anticommutation relations with the associated creation operators. They all transform in the same way as one-particle states. For example, $U_\Lambda a^\dagger_{v,\sigma} U_\Lambda^{-1} = \sum a^\dagger_{\Lambda v,\sigma'} D^j_{\sigma',\sigma}(R_W)$. The subscripts on these operators denote whatever set of labels is inherited from the one-particle representations. Thus, if the fermions are labelled by v and σ, the Kronecker delta on the right hand sides of equations (5) stands for $v_0\delta^3(v-v')\delta_{\sigma,\sigma'}$. It should be noted that the three (anti)commutation relations are all dimensionless; that is, by using the four-velocity rather than the four-momentum, all quantities are dimensionless.

The free four-momentum operator can be written in terms of the bilinears as

$$P^\mu(\mathrm{fr}) := m \sum \int dv\, v^\mu \big(a^\dagger_i a_i + b^\dagger_i b_i + \kappa c^\dagger_k c_k\big), \tag{6}$$

[1] For a discussion of the various forms of dynamics, see for example [2].

[2] The irreducible representations of the Poincaré group are worked out in many texts; good examples of group theory texts are the texts by A.O. Barut and R. Rączka [3] and W. Tung [4].

where, as stated, i stands for the collection of variables which includes the four-velocity and other discrete variables such as spin projections or internal symmetry labels. $dv = \frac{d^3v}{v_0}$ is the Lorentz invariant measure over the four-velocity, which is common to all three types of particles. k similarly always contains the four-velocity, along with other discrete variables that may differ from the fermionic variables. κ is a dimensionless relative bare boson mass parameter which is determined from solutions of the eigenvalue equation for physical particles. Thus, the only quantity with a dimension that appears in equation (6) is the bare fermion mass m; its value is determined by relating a physical mass such as the nucleon mass to the dimensionless eigenvalue of the corresponding stable particle. Because of the transformation properties of the creation and annihilation operators inherited from the one-particle states, the free four-momentum operator as defined in equation (6) satisfies the point form equations (1) and (2).

The interacting four-momentum operator is generated from vertices, products of free field operators themselves made from creation and annihilation operators. If the vertex operator is denoted by $V(x)$, where x is a space-time point, then the interacting four-momentum operator is obtained by integrating the vertex operator over the forward hyperboloid:

$$P^\mu(I) := g \int d^4x\, \delta\big(x \cdot x - \tau^2\big)\theta(x_0) x^\mu V(x) = g \int dx\, x^\mu V(x), \tag{7}$$

where g is a coupling constant and the measure dx is defined from equation (7). The vertex operator is required to be a scalar density under Poincaré transformations and have locality properties. That is,

$$U_a V(x) U_a^{-1} = V(x+a), \tag{8}$$

$$U_\Lambda V(x) U_\Lambda^{-1} = V(\Lambda x), \tag{9}$$

$[V(x), V(y)] = 0$ if $(x-y)^2$ is spacelike. Here $U_a = e^{-iP(\mathrm{fr})\cdot a}$ is the free four-translation operator, with $P^\mu(\mathrm{fr})$ defined in equation (6).

Making use of the fact that if x and y are two time-like four-vectors with the same length τ so that their difference is space-like, it follows that $[P^\mu(I), P^\nu(I)] = 0$. Also, from the Lorentz transformation properties of the vertex given in equation (9) it follows that the interacting four-momentum operator transforms as a four-vector. Thus $P^\mu(I)$ also satisfies the point form equations (1) and (2).

However, what is important for a relativistic theory is that the total four-momentum operator satisfy the point form equations. Making the four-translations in equation (8) infinitesimal gives $[P^\nu(\mathrm{fr}), P^\mu(I)] = \int dx\, x^\mu \frac{\partial}{\partial x_\nu} V(x)$ so that

$$[P^\nu(\mathrm{fr}), P^\mu(I)] - [P^\mu(\mathrm{fr}), P^\nu(I)] = g \int dx \left(x^\mu \frac{\partial}{\partial x_\nu} - x^\nu \frac{\partial}{\partial x_\mu} \right) V(x) = 0,$$

$$[P^\mu(\mathrm{fr}) + P^\mu(I), P^\nu(\mathrm{fr}) + P^\nu(I)] = [P^\mu, P^\nu] = 0.$$

Thus the total four-momentum operator also satisfies the point form equations.

As stated in the Introduction, the vertex is assumed to be bilinear in fermion-antifermion creation and annihilation operators, and linear in boson creation and annihilation operators. That is, $V(x) \sim (a^\dagger + b)(a + b^\dagger)(c + c^\dagger)$. For example, for electromagnetic interactions the vertex is the product of a local current operator times the photon field, $V(x) = J^\mu(x) A_\mu(x)$, where the current operator is a bilinear in fermion-antifermion creation and annihilation operators (for example electron and positron creation and annihilation operators) and the photon field is linear in photon creation and annihilation operators. Thus, the vertex operator has the structure of a $U(2\infty)$ Lie algebra (which is analyzed in the next section) times a Heisenberg-like algebra arising from the boson part of the vertex.

3 Representation structure of $U(2N)$

The fermion-antifermion creation and annihilation operators appear in the four-momentum operator only in the combinations $a_i^\dagger a_j$, $b_i b_j^\dagger$, $a_i^\dagger b_j^\dagger$ and $b_i a_j$. To analyze the representation structure of such an algebra it is convenient to temporarily assume the indices are discrete and range over N values. However, in the following sections the indices will not be restricted to finite values.

To see how the four bilinears are related to a unitary algebra, consider the following correspondence with fermionic operators A_α, $A_\beta^\dagger$ ranging over $2N$ values: $a_i^\dagger \to A_i^\dagger$, $a_i \to A_i$, $b_i \to A_{i+N}^\dagger$, $b_i^\dagger \to A_{i+N}$, where bilinears in the A's and $A^\dagger$'s satisfy the unitary Lie algebra (over $2N$ values) commutation relations

$$[A_\alpha^\dagger A_\beta, A_\mu^\dagger A_\nu] = A_\alpha^\dagger A_\nu \delta_{\beta,\mu} - A_\mu^\dagger A_\beta \delta_{\alpha,\nu}.$$

For example

$$[b_i a_j, a_k^\dagger b_l^\dagger] = [A_{i+N}^\dagger A_j, A_k^\dagger A_{l+N}] = A_{i+N}^\dagger A_{l+N} \delta_{j,k} - A_k^\dagger A_j \delta_{i,l} = b_i b_l^\dagger \delta_{j,k} - a_k^\dagger a_j \delta_{i,l},$$

as required. It should be noted that if the antifermion creation and annihilation operators are normal ordered, an extra term (central extension) appears in the commutation relations. To avoid such terms and keep the correspondence with the unitary algebra, the antifermion creation and annihilation terms are not normal ordered.

Define

$$\mathcal{A}(X) := A_\alpha^\dagger X_{\alpha\beta} A_\beta, \qquad 0 \le \alpha, \beta \le 2N, \tag{10}$$

then all fermionic terms in P^μ are of this form. For example, the free fermionic four-momentum operator can be written, up to a constant, as

$$P_F^\mu(\mathrm{fr}) = m \sum \int dv\, v^\mu (a_i^\dagger a_i - b_i b_i^\dagger) = m \sum \int \mathcal{A}(E^\mu), \qquad E^\mu = \mathrm{diag}\,(v^\mu, -v^\mu), \tag{11}$$

where in this expression continuous variables have been included.

Given the unitary algebra of $U(2N)$, representations can be given in a variety of ways. In the Gelfand–Zetlin scheme[3], in which irreducible representations of the unitary groups are labelled by nonnegative arrays of integers, the relevant fermionic-antifermionic representations are all the antisymmetric representations, written $(1, \ldots, 1, 0, \ldots, 0)$, of length $2N$, with the identity representations given by all zeroes or all ones. The antisymmetric Fock space is the direct sum of all these irreducible representation spaces, and is of dimension 2^{2N}.

When there are no internal symmetries, a more convenient way of labelling the irreducible representations is with a "baryon number" (or charge) operator. Define $\mathcal{B} := \sum \int (a_i^\dagger a_i + b_i b_i^\dagger) = \sum \int \mathcal{A}(I)$, a first order Casimir operator with eigenvalues from $-N$ to $+N$. Each integer eigenvalue corresponds to a given irreducible representation. For example the $\mathcal{B} = 0$ sector corresponds in the Gelfand–Zetlin notation to arrays with equal numbers of zeroes and ones.

For each baryon number sector there is a cyclic vector which is annihilated by the lowering operators ba, and the raising operators then generate all the possible states, up to a maximum value, at which point the raising operator annihilates. Such a construction begins with the fermion-antifermion Fock vacuum, $|0_F\rangle = |0\rangle|0\rangle$, the tensor product of fermion and anti-fermion Fock vacuum.

Starting with baryon number zero, the fermion-antifermion cyclic vector is defined by $a_i|0_F\rangle = b_i|0_F\rangle = 0$ and hence satisfies $b_i a_j|0_F\rangle = 0$, $\mathcal{B}|0_F\rangle = 0$. States are generated from the vacuum

[3] A good reference for the $U(N)$ groups, including the Gelfand–Zetlin labeling scheme, is given in [6].

by products of raising operators $|i_1 j_1 \cdots\rangle := a^\dagger_{i_1} b^\dagger_{j_1} \cdots |0_F\rangle$. The vacuum carries the identity representation for each $U(N)$ subalgebra, whereas the states $|i,j\rangle := a^\dagger_i b^\dagger_j |0_F\rangle$ carry the tensor product of the fundamental representation of the $U(N)$ algebras.

Similarly for baryon number one, a cyclic vector is given by $|0\rangle_1 := a^\dagger_i |0_F\rangle$, with $b_i a_j |0\rangle_1 = 0$, $\mathcal{B}|0\rangle_1 = |0\rangle_1$. Such a cyclic vector carries the fundamental representation of the $a^\dagger a$ algebra, and the identity representation of the $b^\dagger b$ algebra. A basis in the baryon number one irrep space is again given by products of "bare fermion-antifermion" pairs $|i_1 j_1 \cdots\rangle_1 = a^\dagger_{i_1} b^\dagger_{j_1} \cdots |0\rangle_1$. The dimension of this irrep space is smaller than the baryon zero irrep space, because there are fewer "fermion-antifermion" pairs that are available to span the space.

Continuing in this way there is finally the baryon N irrep space, a one dimensional space whose only state is the cyclic vector itself, $|0\rangle_N := a^\dagger_1 \cdots a^\dagger_N |0_F\rangle$, $\mathcal{B}|0\rangle_N = N|0\rangle_N$. The other one dimensional irrep space is the baryon number $-N$ space, defined by replacing the fermion creation operators with anti-fermion creation operators. These representations do not exist when $N \to \infty$, which means they do not occur when continuous variables are included.

4 Eigenvalue structure of the four-momentum operator

If the definition, equation (10) is substituted into equation (7), the interacting four-momentum operator can be written as

$$P^\mu(I) = g \sum \int \left(\mathcal{A}(X^\mu_k) c_k + \mathcal{A}(X^\mu_k)^\dagger c^\dagger_k\right), \tag{12}$$

$$[X^\mu_k, X^\nu_l] = 0, \tag{13}$$

$$[X^\mu_k, (X^\mu_l)^\dagger] = 0. \tag{14}$$

Equations (13) and (14) follow from the fact that the components of the interacting four-momentum operator commute, and that the four-momentum operator is hermitian. Equation (14) implies that the X's are normal operators, while equation (13) implies that they can all be simultaneously diagonalized.

The $(X^\mu_k)_{\alpha\beta}$ are obtained from a given vertex; for example for the pseudoscalar coupling of a meson to nucleons and antinucleons, two of the four terms in equation (12) are

$$\begin{aligned}
(X^\mu_k)_{ij} &= F^\mu(v_1 - v_2 - v)\bar{u}(v_1, \sigma_1)\gamma_5 u(v_2), \\
(X^\mu_k)_{i+N,j} &= F^\mu(-v_1 - v_2 - v)\bar{v}(v_1, \sigma_1)\gamma_5 u(v_2), \\
F^\mu(u) &:= \int d^4y\, \delta(y \cdot y - 1)\theta(y_0) y^\mu e^{iu\cdot y}.
\end{aligned}$$

Since all the quantities in equation (12) except the coupling constant g are dimensionless, g has the dimensions of mass. If such a coupling constant is written as $g = m\alpha$, where m is the mass factor multiplying the free four-momentum operator, equation (6), then α is a dimensionless coupling constant. In the following the mass factor will be dropped, so that the four-momentum operator is dimensionless.

Now the reason for writing the interacting four-momentum operator in the form (12), is to solve the eigenvalue problem (3). If the one-particle eigenvalue problem can be solved, the mass value of the eigenvalue will be dimensionless. To connect it with a physical mass, the mass factor must be chosen so that the physical mass equals the factor m times the dimensionless eigenvalue mass; in this way the mass factor multiplying both the free and interacting four-momentum operators is fixed.

Combining the expressions for the terms in the four-momentum operator given in equations (6), (12) the eigenvalue problem (3) to be solved can be written as

$$\left(P^\mu_F(\mathrm{fr}) + \sum \int dv \big(\kappa v^\mu c^\dagger_k c_k + \alpha \mathcal{A}(X^\mu_k) c_k + \alpha \mathcal{A}(X^\mu_k)^\dagger c^\dagger_k\big)\right) |\Psi_\lambda\rangle = \lambda^\mu |\Psi_\lambda\rangle, \tag{15}$$

where the first term in equation (15) is the free fermion-antifermion four-momentum operator (11), the second term is the boson free four-momentum operator, with the factor κ the dimensionless ratio of physical meson mass to physical fermion mass, and the third term is the interaction coupling bilinear fermions to the mesons, with a strength given by the dimensionless coupling constant α. The point eigenvalues will give the masses of the stable particles, while the continuous part of the spectrum gives the scattering states.

Since the four-momentum operator transforms as a four-vector under Lorentz transformations, Lorentz covariance can be used to write $\lambda^\mu = (\lambda, 0, 0, 0)$. Then the eigenvector equation can be rewritten as

$$P^0_F(\mathrm{fr}) + \sum \int dv \big(\kappa v^0 c^\dagger_k c_k + \alpha \mathcal{A}(X^0_k) c_k + \alpha \mathcal{A}(X^0_k)^\dagger c^\dagger_k\big) |\Psi_\lambda\rangle = \lambda |\Psi_\lambda\rangle.$$

There is also a corresponding eigenvector equation for the space components of the four-momentum operator, for which the eigenvalue is zero.

Next consider an automorphism on the Heisenberg Lie algebra, with transformation

$$c_k = C_k - \frac{\alpha \mathcal{A}(X^0_k)^\dagger}{\kappa v^0}, \qquad c^\dagger_k = C^\dagger_k - \frac{\alpha \mathcal{A}(X^0_k)}{\kappa v^0}.$$

This will be an automorphism if the C's satisfy the correct commutation relations,

$$[C_k, C^\dagger_{k'}] = \left[c_k + \frac{\alpha \mathcal{A}(X^0_k)^\dagger}{\kappa v^0}, c_{k'} + \frac{\alpha \mathcal{A}(X^0_{k'})}{\kappa v^0}\right] = \delta_{k,k'},$$
$$[C^\dagger_k, C^\dagger_{k'}] = \left[c^\dagger_k + \frac{\alpha \mathcal{A}(X^0_k)}{\kappa v^0}, c^\dagger_{k'} + \frac{\alpha \mathcal{A}(X^0_{k'})}{\kappa v^0}\right] = 0.$$

These commutation relations follow since the $\mathcal{A}(X)$'s commute among themselves, equations (13).

Since the C's also satisfy boson commutation relations, the eigenvalue equation can be written in terms of them:

$$\left(P^0_F(\mathrm{fr}) - \alpha^2 \sum \int dv \, \frac{1}{\kappa v^0} \mathcal{A}(X^0_k) \mathcal{A}(X^0_k)^\dagger + \sum \int dv \, \kappa v^0 C^\dagger_k C_k\right) |\Psi_\lambda\rangle = \lambda |\Psi_\lambda\rangle. \tag{16}$$

If $\alpha = 0$ then the ground state solution is the usual Fock space solution; but the above operators are all positive, so there can never be a Fock space solution for $\alpha \neq 0$, that is, when the physical vacuum is in the Fock space. The goal is to look for solutions of equation (16) that lead to a suitable choice for the physical vacuum.

5 A simple model

To motivate the kinds of considerations that go into getting a physical vacuum, consider a simple model in which N is finite and the baryon number is N. Then it is straightforward to get exact solutions to the eigenvalue problem because the fermion-antifermion sector is one-dimensional.

To get these solutions, it is simplest to realize the boson sector in Bargmann variables, namely, $c^\dagger_k \to z_k$, $c_k \to \frac{\partial}{\partial z_k}$ on a holomorphic Hilbert space [7]. Then the Hamiltonian and eigenfunctions, eigenvalues, are

$$H = \sum e_i \left(a^\dagger_i a_i + b^\dagger_i b_i + z_i \frac{\partial}{\partial z_i}\right) + \alpha \sum \mathcal{A}(X^0_i) \frac{\partial}{\partial z_i} + \mathcal{A}(X^0_i)^\dagger z_i$$

$$= \sum e_i + \sum \mathcal{A}(E) + \sum e_i \left(z_i + \frac{\alpha \mathcal{A}(X_i^0)}{e_i} \right) \left(\frac{\partial}{\partial z_i} + \frac{\alpha \mathcal{A}(X_i^0)^\dagger}{e_i} \right)$$
$$- \alpha^2 \sum \mathcal{A}(X_i^0) \mathcal{A}(X_i^0)^\dagger,$$
$$|\Psi_n\rangle = \prod \left(e_k z_k + \frac{\alpha \mathcal{A}(X_k^0)}{e_k} \right)^{n_k} e^{-\alpha \sum \frac{\mathcal{A}(X_i^0) z_i}{e_i}} a_1^\dagger \cdots a_N^\dagger |0\rangle, \tag{17}$$
$$\lambda_n = \sum \left(e_i + n_i e_i - \frac{\alpha^2 |Y_i^0|^2}{e_i} \right),$$
$$\lambda_{\min} = \sum e_i - \alpha^2 \sum \frac{|Y_i^0|^2}{e_i} \neq 0. \tag{18}$$

Here e_i is the discretized energy, $(Y_i^0) = \sum((X_i^0)_{jj} + (X_i^0)_{j+N,j+N})$, and κ has been set to one. The minimum eigenvalue has a positive term coming from the free "relativistic energy", balanced by a negative term, coming from the interaction between fermions and bosons.

This simple model shows that the minimum eigenvalue expresed by (18) is not zero. But the vacuum by definition has zero energy. As seen in (18) only a definite value of the coupling constant will satisfy such a requirement; that is, $H|\Psi_{\min}\rangle = 0$ determines the value of the coupling constant.

Now remove the N finite restriction, replace the above Hamiltonian with the electromagnetic Hamiltonian and replace the baryon number operator with the charge operator in the charge zero sector. Since the free photon four-momentum operator has no κ factor (since photons are massless), the vacuum solution of H, namely $H|\Omega\rangle = 0$, should fix the value of α, which gives the fine structure constant. Note that unlike the case of a Hamiltonian operator only, where it is always possible to add an arbitrary constant to the Hamiltonian and hence shift the energy by an arbitrary amount, in the case of the four-momentum operator this is not possible without destroying the Lorentz covariance of the operator.

A more interesting example is the above model in the $N = 1$, baryon number 0 sector. Then the Hamiltonian and eigenfunctions for a simple choice of X can be written as

$$H = e(a^\dagger a + b^\dagger b) + z\frac{\partial}{\partial z} + \alpha(a^\dagger b^\dagger + ba) \left(\frac{\partial}{\partial z} + z \right),$$
$$|\Psi_n\rangle = f_1(z)|0\rangle + f_2(z) a^\dagger b^\dagger |0\rangle.$$

The eigenvalue problem generates a matrix differential equation

$$z\frac{\partial f_1}{\partial z} + \alpha \frac{\partial f_2}{\partial z} = \lambda f_1 - \alpha z f_2,$$
$$z\frac{\partial f_2}{\partial z} + \alpha \frac{\partial f_1}{\partial z} = (\lambda - 2e) f_2 - \alpha z f_1$$

which can be rewritten as

$$(z + \alpha) f_+' = (\lambda - e - \alpha z) f_+ + e f_-,$$
$$(z - \alpha) f_-' = (\lambda - e + \alpha z) f_- + e f_+$$

with $f_\pm := f_1 \pm f_2$, $f_\pm$ holomorphic. If e is zero the solution has a structure given by expression (17). However, as far as the "vacuum" structure is concerned, e not zero is the interesting case.

6 Conclusion

In the point form of relativistic quantum mechanics all interactions are in the four-momentum operator, and Lorentz transformations are kinematic (that is, Lorentz generators are free of interactions). Since the four-momentum operator can be written as bilinear in fermion, antifermion creation and annihilation operators, a Lie algebra is generated, which is isomorphic to the Lie algebra of the unitary group. Further, the representations of interest are the antisymmetric representations, and can be labelled by integers, the eigenvalues of the baryon number operator. Similarly the bosonic parts of the four-momentum operator are elements of a Lie algebra, related to the semidirect sum of a unitary algebra with the Heisenberg algebra.

Via the automorphism defined in Section 4, the four-momentum operator can be rewritten in such a way as to exhibit properties of its eigenvalue structure. Though it is not discussed in this paper, a strong coupling approximation of the four-momentum operator, in which the fermion free relativistic energy is neglected, can be solved exactly and has a structure similar to the solutions given in equation (17).

Several simple models show a balance of the minimum eigenvalue between fermionic relativistic energy and fermion-boson coupling terms; this leads to the possibility that the fine structure constant may be determined from requirements on the physical vacuum. However much remains to be done to connect the simple models discussed here with genuine infinite degree of freedom systems.

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