Symmetry, Integrability and Geometry: Methods and Applications Vol. 1 (2005), Paper 026, 6 pages

Conservation Laws of Discrete Korteweg–de Vries Equation

Olexandr G. RASIN and Peter E. HYDON

Department of Mathematics and Statistics, University of Surrey, Guildford, Surrey GU2 7XH, UK
E-mail: *o.rasin@surrey.ac.uk*, *p.hydon@surrey.ac.uk*

Received October 21, 2005, in final form December 06, 2005; Published online December 09, 2005
Original article is available at http://www.emis.de/journals/SIGMA/2005/Paper026/

Abstract. All three-point and five-point conservation laws for the discrete Korteweg–de Vries equations are found. These conservation laws satisfy a functional equation, which we solve by reducing it to a system of partial differential equations. Our method uses computer algebra intensively, because the determining functional equation is quite complicated.

Key words: conservation laws; discrete equations; quad-graph

2000 Mathematics Subject Classification: 70H33; 37K10; 39A05

1 Introduction

A direct method for calculation of conservation laws for partial difference equations (PΔE's) was recently introduced by Hydon [3]. This method, which does not use Noether's Theorem, has been used to calculate some low-order conservation laws of various integrable difference equations that are defined on the quad-graph shown in Fig. 1. (For a classification of integrable quad-graph equations, see [1].)

In the current paper, we present a modified version of Hydon's direct method, and use it to derive conservation laws of the discrete Korteweg–de Vries equation [2]:

$$\left(p+q+v_{k+1}^{l+1}-v_k^l\right)\left(q-p+v_{k+1}^l-v_k^{l+1}\right)=q^2-p^2, \tag{1}$$

which is an integrable quad-graph equation. Here p, q are parameters and $p^2 \neq q^2$. To simply matters, we use the transformation

$$v_k^l = u_k^l\sqrt{q^2-p^2}-qk-pl$$

to reduce (1) to

$$\left(u_{k+1}^{l+1}-u_k^l\right)\left(u_{k+1}^l-u_k^{l+1}\right)=1.$$

We shall call this equation dKdV. As with all integrable quad-graph equations, dKdV may be solved to write any one of u_k^l, u_{k+1}^l, u_k^{l+1}, u_{k+1}^{l+1} in terms of the other three. In particular, we will write dKdV as either

$$u_{k+1}^{l+1}=\omega, \qquad \text{where} \quad \omega=\frac{1}{u_{k+1}^l-u_k^{l+1}}+u_k^l,$$

or

$$u_{k+1}^l=\Omega, \qquad \text{where} \quad \Omega=\frac{1}{u_{k+1}^{l+1}-u_k^l}+u_k^{l+1}.$$

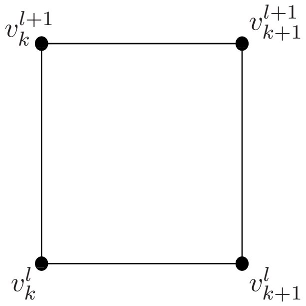

Figure 1. Quad-graph.

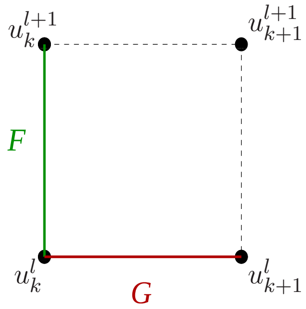

Figure 2. Three-point conservation law.

A conservation law for a partial difference equation (PΔE) on $\mathbb{Z}^2$ is an expression of the form

$$(\mathbf{S}_k - \mathbf{id})F + (\mathbf{S}_l - \mathbf{id})G = 0$$

that is satisfied on all solutions of the equation. Here **id** is the identity mapping and $\mathbf{S}_k$, $\mathbf{S}_l$ are forward shifts of the coordinates k and l respectively:

$$\mathbf{S}_k : \big(k, l, u_k^l\big) \to \big(k+1, l, u_{k+1}^l\big), \qquad \mathbf{S}_l : \big(k, l, u_k^l\big) \to \big(k, l+1, u_k^{l+1}\big).$$

A conservation law is trivial if it holds identically (not just on solutions of the PΔE), or if F and G both vanish on all solutions of the equation. We search for nontrivial conservation laws.

2 Three-point conservation laws

In this section we consider conservation laws that lie on the quad-graph. This means that the functions F, G, $\mathbf{S}_kF$ and $\mathbf{S}_lG$ must depend upon only k, l, u_k^l, u_{k+1}^l, u_k^{l+1} and u_{k+1}^{l+1}. Consequently the most general form of F and G is:

$$F = F\big(k, l, u_k^l, u_k^{l+1}\big), \qquad G = G\big(k, l, u_k^l, u_{k+1}^l\big).$$

The dependence of F and G upon the continuous variables u_i^j is illustrated in Fig. 2; together, these functions lie on three points of the quad-graph. For this reason, we call such conservation laws *three-point conservation laws*.
The three-point conservation laws can be determined directly by substituting dKdV into

$$\begin{aligned} &F\big(k+1, l, u_{k+1}^l, u_{k+1}^{l+1}\big) - F\big(k, l, u_k^l, u_k^{l+1}\big) \\ &\quad + G\big(k, l+1, u_k^{l+1}, u_{k+1}^{l+1}\big) - G\big(k, l, u_k^l, u_{k+1}^l\big) = 0, \end{aligned} \tag{2}$$

and solving the resulting functional equation. The substitution $u_{k+1}^{l+1} = \omega$ yields

$$F\big(k+1, l, u_{k+1}^l, \omega\big) - F\big(k, l, u_k^l, u_k^{l+1}\big) + G\big(k, l+1, u_k^{l+1}, \omega\big) - G\big(k, l, u_k^l, u_{k+1}^l\big) = 0. \tag{3}$$

In order to solve this functional equation we have to reduce it to a system of partial differential equations. To do this, first eliminate terms that contain ω, by applying each of the following (commuting) differential operators to (3):

$$L_1 = \frac{\partial}{\partial u_k^{l+1}} - \frac{\omega_{u_k^{l+1}}}{\omega_{u_k^l}} \frac{\partial}{\partial u_k^l}, \qquad L_2 = \frac{\partial}{\partial u_{k+1}^l} - \frac{\omega_{u_{k+1}^l}}{\omega_{u_k^l}} \frac{\partial}{\partial u_k^l}.$$

The operators L_1, L_2 differentiate with respect to u_{k+1}^l, u_k^{l+1} respectively, keeping ω fixed. This procedure does not depend upon the form of ω; it can be applied equally to any quad-graph equation. In particular, for dKdV, (3) is reduced to

$$F_{u_k^l u_k^l} + G_{u_k^l u_k^l} - \big(u_{k+1}^l - u_k^{l+1}\big)^2 \big(F_{u_k^l u_k^{l+1}} - G_{u_k^l u_{k+1}^l}\big)$$

$$- 2\big(u_{k+1}^l - u_k^{l+1}\big)\big(F_{u_k^l} + G_{u_k^l}\big) = 0, \tag{4}$$

where $F = F(k, l, u_k^l, u_k^{l+1})$ and $G = G(k, l, u_k^l, u_{k+1}^l)$. Differentiating (4) three times with respect to u_k^{l+1} eliminates G and its derivatives, leaving the necessary condition

$$F_{{u_k^l}^2 {u_k^{l+1}}^3} - \big(u_{k+1}^l - u_k^{l+1}\big)^2 F_{u_k^l {u_k^{l+1}}^4} + 4\big(u_{k+1}^l - u_k^{l+1}\big) F_{u_k^l {u_k^{l+1}}^3} = 0. \tag{5}$$

This equation can be split into an overdetermined system by equating powers of u_{k+1}^l. Further information about F may be found by substituting $u_{k+1}^l = \Omega$ into (2). Differentiating

$$F\big(k+1, l, \Omega, u_{k+1}^{l+1}\big) - F\big(k, l, u_k^l, u_k^{l+1}\big) + G\big(k, l+1, u_k^{l+1}, u_{k+1}^{l+1}\big) - G\big(k, l, u_k^l, \Omega\big) = 0$$

with respect to u_k^l, u_{k+1}^{l+1} and keeping Ω fixed yields

$$\begin{aligned} &F_{u_k^{l+1} u_k^{l+1}} - \tilde{G}_{u_k^{l+1} u_k^{l+1}} - \big(u_{k+1}^{l+1} - u_k^l\big)^2 \big(F_{u_k^l u_k^{l+1}} + \tilde{G}_{u_k^{l+1} u_{k+1}^{l+1}}\big) \\ &\qquad - 2\big(u_{k+1}^{l+1} - u_k^l\big)\big(F_{u_k^l} - \tilde{G}_{u_k^{l+1}}\big) = 0, \end{aligned}$$

where $\tilde{G} = G(k, l+1, u_k^l, u_{k+1}^{l+1})$. The function $\tilde{G}$ and its derivatives are eliminated by differentiating three times with respect to u_k^{l+1}, which yields

$$F_{{u_k^l}^3 {u_k^{l+1}}^2} - \big(u_{k+1}^{l+1} - u_k^l\big)^2 F_{{u_k^l}^4 u_k^{l+1}} + 4\big(u_{k+1}^{l+1} - u_k^l\big) F_{{u_k^l}^3 u_k^{l+1}} = 0. \tag{6}$$

The overdetermined system of partial differential equations (5), (6) is easily solved to obtain

$$F = C_1 u_k^l u_k^{l+1} + C_2 {u_k^l}^2 u_k^{l+1} + C_3 u_k^l {u_k^{l+1}}^2 + C_4 {u_k^l}^2 {u_k^{l+1}}^2 + F_1 + F_2,$$

where each C_i is an arbitrary function of k, l, and $F_1 = F_1(k, l, u_k^{l+1})$, $F_2 = F_2(k, l, u_k^l)$ are arbitrary functions. The term $F_2(k, l, u_k^l)$ can be removed (without loss of generality) by adding the trivial conservation law

$$F_T = (\mathbf{S}_l - \mathbf{id})F_2, \qquad G_T = -(\mathbf{S}_k - \mathbf{id})F_2,$$

to F and G respectively.

So far, we have differentiated the determining equations for a conservation law five times; this has created a hierarchy of functional differential equations that every three-point conservation law must satisfy. The unknown functions C_i, F_1 and G are found by going up the hierarchy, a step at a time, to determine the constraints these equations place on the unknown functions. As the constraints are solved sequentially, more and more information is gained about the functions. At the highest stage, the determining equations are satisfied, and the only remaining unknowns are the constants that multiply each conservation law. This is a simple but lengthy process; for brevity, we omit the details.

By this technique we have found all independent nontrivial three-point conservation laws for the dKdV equation; they are as follows[1]:

1. $F = u_k^l \big(u_k^{l+1}\big)^2 - \big(u_k^l\big)^2 u_k^{l+1} + u_k^l - u_k^{l+1}$,
   $G = \big(u_k^l\big)^2 u_{k+1}^l - u_k^l \big(u_{k+1}^l\big)^2$,
2. $F = (-1)^{k+l+1} \left\{ u_k^l \big(u_k^{l+1}\big)^2 + \big(u_k^l\big)^2 u_k^{l+1} - u_k^l - u_k^{l+1} \right\}$,

[1] Note that the conservation laws 1 and 2 are connected to each other by a discrete symmetry of the form $u_k^l \mapsto (-1)^{k+l} i u_k^l$; we are grateful to a referee for this observation.

$$G=(-1)^{k+l}\left\{\left(u_k^l\right)^2u_{k+1}^l+u_k^l\left(u_{k+1}^l\right)^2\right\},$$

3. $$F=(-1)^{k+l+1}\left\{\left(u_k^lu_k^{l+1}\right)^2-2u_k^lu_k^{l+1}+\frac{1}{2}\right\},$$
$$G=(-1)^{k+l}\left\{\left(u_k^lu_{k+1}^l\right)^2\right\},$$

4. $$F=(-1)^{k+l+1}\left\{u_k^lu_k^{l+1}-\frac{1}{2}\right\},$$
$$G=(-1)^{k+l}\left\{u_k^lu_{k+1}^l\right\}.$$

Three of these conservation laws depend on k and l explicitly. If we had chosen functions F and G that depended only upon u_i^j on the quad-graph (and not also upon k and l), we would have found only the first of four the three-point conservation laws.

3 Five-point conservation laws

Higher conservation laws can be found by the approach described above, but the complexity of the calculations increases rapidly with the number of variables u_i^j on which F and G depend. The simplest higher conservation laws are defined on five points, as shown in Fig. 3. The functions F and G are of the form

$$F=F\left(k,l,u_{k-1}^{l+1},u_{k-1}^l,u_k^l,u_k^{l-1}\right),\qquad G=G\left(k,l,u_{k-1}^l,u_k^l,u_k^{l-1},u_{k+1}^{l-1}\right).$$

The points lie in a pair of quad-graphs with a single common point. We seek conservation laws that contain points from both quad-graphs and cannot be reduced to the conservation laws from the previous section (modulo trivial conservation laws).

The determining equation for the five-point conservation laws is

$$\begin{aligned}&F(k+1,l,u_k^{l+1},u_k^l,u_{k+1}^l,u_{k+1}^{l-1})-F(k,l,u_{k-1}^{l+1},u_{k-1}^l,u_k^l,u_k^{l-1})\\&\quad+G(k,l+1,u_{k-1}^{l+1},u_k^{l+1},u_k^l,u_{k+1}^l)-G(k,l,u_{k-1}^l,u_k^l,u_k^{l-1},u_{k+1}^{l-1})=0.\end{aligned}$$

Shifted versions of the dKdV equation are used to eliminate u_{k+1}^l, u_k^{l+1} in favour of the variables u_i^j at the five points that lie on the 'step' shown in bold in Fig. 3. The determining equation can be solved by the same technique which is described in the previous section. This is a very lengthy calculation, so we state the results only.

The extra five-point conservation laws of dKdV equation are

5. $$F=\ln\left(u_k^l-u_{k-1}^{l+1}\right),$$
$$G=\ln\left(\frac{1}{u_{k+1}^{l-1}-u_k^l}-u_{k-1}^l+u_k^{l-1}\right),$$

6. $$F=\ln\left(\frac{1}{u_k^{l-1}-u_{k-1}^l}-u_k^l+u_{k-1}^{l+1}\right),$$
$$G=\ln\left(u_{k-1}^l-u_k^{l-1}\right),$$

7. $$F=l\ln\left(u_k^l-u_{k-1}^{l+1}+\frac{1}{u_{k-1}^l-u_k^{l-1}}\right)-(k-1)\ln\left(u_{k-1}^{l+1}-u_k^l\right),$$
$$G=l\ln\left(u_{k-1}^l-u_k^{l-1}\right)-k\ln\left(\frac{1}{u_{k+1}^{l-1}-u_k^l}-u_{k-1}^l+u_k^{l-1}\right).$$

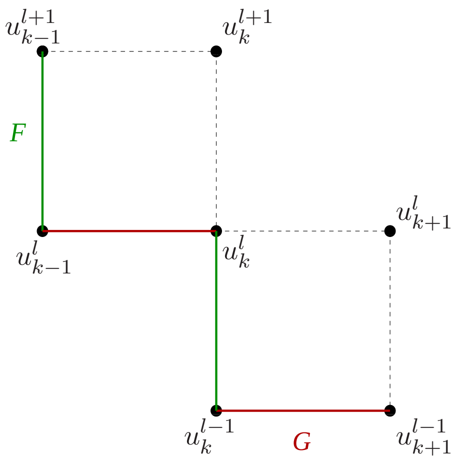

Figure 3. Five-point CL.

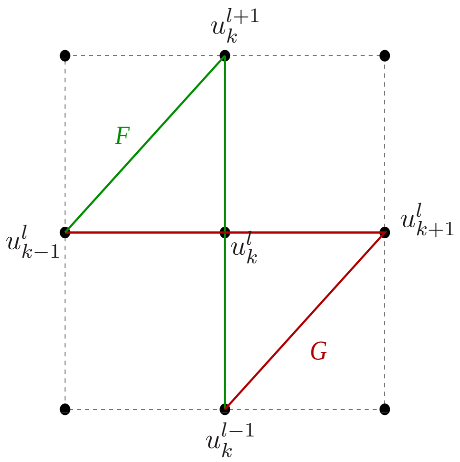

Figure 4. Five-point CL.

These conservation laws are irrational, so it is clear that they cannot be reduced to the polynomial conservation laws in the previous section. The five-point conservation laws can be simplified somewhat with the substitutions

$$u_{k-1}^{l+1} = u_k^l + \frac{1}{u_{k-1}^l - u_k^{l+1}}, \qquad u_{k+1}^{l-1} = u_k^l + \frac{1}{u_k^{l-1} - u_{k+1}^l}.$$

These substitutions move the five-point conservation laws onto the cross shown in Fig. 4. The functions F and G are now of the form

$$F = F'\big(k, l, u_{k-1}^l, u_k^{l-1}, u_k^l, u_k^{l+1}\big), \qquad G = G'\big(k, l, u_{k-1}^l, u_k^{l-1}, u_k^l, u_{k+1}^l\big).$$

Specifically, the five-point conservation laws have the following components:

5. $F' = -\ln\big(u_k^{l+1} - u_{k-1}^l\big),$
   $G' = \ln\big(u_{k+1}^l - u_{k-1}^l\big),$
6. $F' = \ln\left(\dfrac{u_k^{l-1} - u_k^{l+1}}{\big(u_k^{l+1} - u_{k-1}^l\big)\big(u_{k-1}^l - u_k^{l-1}\big)}\right),$
   $G' = \ln\big(u_{k-1}^l - u_k^{l-1}\big),$
7. $F' = l\ln\left(\dfrac{u_k^{l+1} - u_k^{l-1}}{\big(u_k^{l+1} - u_{k-1}^l\big)\big(u_{k-1}^l - u_k^{l-1}\big)}\right) + (k-1)\ln\big(u_{k-1}^l - u_k^{l+1}\big),$
   $G' = l\ln\big(u_{k-1}^l - u_k^{l-1}\big) - k\ln\big(u_{k+1}^l - u_{k-1}^l\big).$

Surprisingly, none of them do depend upon u_k^l. We will describe the circumstances under which this occurs for other quad-graph equations in a separate paper.

4 Conclusion and outlook

The main results of this paper are as follows.

- New conservation laws for the dKdV equation have been found.
- These include higher-order conservation laws, which are irrational.
- We have improved the effectiveness of Hydon's direct method for constructing conservation laws [3].

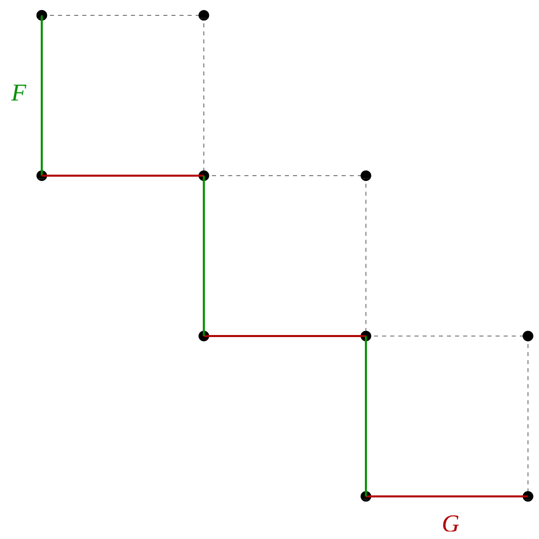

Figure 5. Seven-point CL.

In principle, the same method can be used to construct conservation laws with seven or more points (Fig. 5). However, the calculations become extremely complex, placing heavy demands on even the most sophisticated computer algebra systems. We are currently working to improve the efficiency of the method still further, but it is unlikely that it will be possible to determine conservation laws of very high order directly. However, we are developing ways of combining direct and indirect methods to achieve this aim, as will be reported elsewhere.

[1] Adler V.E., Bobenko A.I., Suris Yu.B., Classification of integrable equations on quad-graphs. The consistency approach, *Comm. Math. Phys.*, 2003, V.233, 513–543; nlin.SI/0202024.

[2] Hirota R., Nonlinear partial difference equations. I. A difference analog of the Korteweg–de Vries equation, *J. Phys. Soc. Japan*, 1977, V.43, 1423–1433.

[3] Hydon P.E., Conservation laws of partial difference equations with two independent variables, *J. Phys. A: Math. Gen.*, 2001, V.34, 10347–10355.