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Representations of the Quantum Algebra $\mathrm{su}_q(1,1)$ and Discrete q-Ultraspherical Polynomials

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Abstract. We derive orthogonality relations for discrete q-ultraspherical polynomials and their duals by means of operators of representations of the quantum algebra $\mathrm{su}_q(1,1)$. Spectra and eigenfunctions of these operators are found explicitly. These eigenfunctions, when normalized, form an orthonormal basis in the representation space.

Key words: Quantum algebra $su_q(1,1)$; representations; discrete q-ultraspherical polynomials

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1 Representations of $\mathrm{su}_q(1,1)$ with lowest weights

The aim of this paper is to study orthogonality relations for the discrete q-ultraspherical polynomials and their duals by means of operators of representations of the quantum algebra $\mathrm{su}_q(1,1)$.

Throughout the sequel we always assume that q is a fixed positive number such that $q<1$. We use (without additional explanation) notations of the theory of special functions and the standard q-analysis (see, for example, [1]).

The quantum algebra $\mathrm{su}_q(1,1)$ is defined as an associative algebra, generated by the elements J_+, J_-, q^{J_0} and q^{-J_0}, subject to the defining relations

$$q^{J_0}q^{-J_0}=q^{-J_0}q^{J_0}=1,$$
$$q^{J_0}J_\pm q^{-J_0}=q^{\pm1}J_\pm,$$
$$[J_-,J_+]=\frac{q^{J_0}-q^{-J_0}}{q^{1/2}-q^{-1/2}},$$

and the involution relations $(q^{J_0})^*=q^{J_0}$ and $J_+^*=J_-$. (We have replaced J_- by $-J_-$ in the common definition of the algebra $U_q(\mathrm{sl}_2)$; see [2, Chapter 3].)

We are interested in representations of $\mathrm{su}_q(1,1)$ with lowest weights. These irreducible representations are denoted by T_l^+, where l is a lowest weight, which can be *any complex number* (see, for example, [3]). They act on the Hilbert space $\mathcal{H}$ with the orthonormal basis $|n\rangle$, $n=0,1,2,\ldots$. The representation T_l^+ can be given in the basis $|n\rangle$, $n=0,1,2,\ldots$, by the formulas

$$q^{\pm J_0}\,|n\rangle=q^{\pm(l+n)}\,|n\rangle,$$
$$J_+\,|n\rangle=\frac{q^{-(n+l-1/2)/2}}{1-q}\sqrt{(1-q^{n+1})(1-q^{2l+n})}|n+1\rangle,$$
$$J_-\,|n\rangle=\frac{q^{-(n+l-3/2)/2}}{1-q}\sqrt{(1-q^{n})(1-q^{2l+n-1})}|n-1\rangle.$$

For positive values of l the representations T_l^+ are $*$-representations. For studying discrete q-ultraspherical polynomials we use the representations T_l^+ for which $q^{2l-1} = -a$, $a > 0$. They are not $*$-representations. But we shall use operators of these representations which are symmetric or self-adjoint. Note that q^l is a pure imaginary number.

2 Discrete q-ultraspherical polynomials and their duals

There are two types of discrete q-ultraspherical polynomials [4]. The first type, denoted as $C_n^{(a)}(x; q)$, $a > 0$, is a particular case of the well-known big q-Jacobi polynomials. For this reason, we do not consider them in this paper. The second type of discrete q-ultraspherical polynomials, denoted as $\tilde{C}_n^{(a)}(x; q)$, $a > 0$, is given by the formula

$$\tilde{C}_n^{(a)}(x; q) = (-\mathrm{i})^n C_n^{(-a)}(\mathrm{i}x; q) = (-\mathrm{i})^n {}_3\phi_2\left(q^{-n}, -aq^{n+1}, \mathrm{i}x;\ \mathrm{i}\sqrt{a}q, -\mathrm{i}\sqrt{a}q;\ q, q\right). \tag{1}$$

(Here and everywhere below under $\sqrt{a}$, $a > 0$, we understand a positive value of the root.)

The polynomials $\tilde{C}_n^{(a)}(x; q)$ satisfy the recurrence relation

$$x\tilde{C}_n^{(a)}(x; q) = A_n \tilde{C}_{n+1}^{(a)}(x; q) + C_n \tilde{C}_{n-1}^{(a)}(x; q), \tag{2}$$

where

$$A_n = \frac{1 + aq^{n+1}}{1 + aq^{2n+1}}, \qquad C_n = A_n - 1 = \frac{aq^{n+1}(1 - q^n)}{1 + aq^{2n+1}}.$$

Note that $A_n \geq 1$ and, hence, coefficients in the recurrence relation (2) satisfy the conditions $A_n C_{n+1} > 0$ of Favard's characterization theorem for $n = 0, 1, 2, \ldots$. This means that these polynomials are orthogonal with respect to a positive measure. Orthogonality relation for them is derived in [4]. We give here an approach to this orthogonality by means of operators of representations T_l^+ of $\mathrm{su}_q(1, 1)$.

Dual to the polynomials $C_n^{(a)}(x; q)$ are the polynomials $D_n^{(a)}(\mu(x; a)|q)$, where $\mu(x; a) = q^{-x} + aq^{x+1}$. These polynomials are a particular case of the dual big q-Jacobi polynomials, studied in [5], and we do not consider them. Dual to the polynomials $\tilde{C}_n^{(a)}(x; q)$ are the polynomials

$$\tilde{D}_n^{(a)}(\mu(x; -a)|q) := {}_3\phi_2\left(q^{-x}, -aq^{x+1}, q^{-n};\ \mathrm{i}\sqrt{a}q, -\mathrm{i}\sqrt{a}q;\ q, -q^{n+1}\right). \tag{3}$$

For $a > 0$ these polynomials satisfy the conditions of Favard's theorem and, therefore, they are orthogonal. We derive an orthogonality relation for them by means of operators of representations T_l^+ of $\mathrm{su}_q(1, 1)$.

3 Representation operators I and J

Let T_l^+ be the irreducible representation of $\mathrm{su}_q(1, 1)$ with lowest weight l such that $q^{2l-1} = -a$, $a > 0$ (note that a can take any positive value). We consider the operator

$$I := \alpha\, q^{J_0/4}(J_+ A + A\, J_-)\, q^{J_0/4} \tag{4}$$

of the representation T_l^+, where $\alpha = (a^2/q)^{1/2}(1 - q)$ and

$$A = \frac{q^{(J_0-l+2)/2}\sqrt{(1 - a^2q^{J_0-l+1})(1 + aq^{J_0-l+1})}}{\sqrt{(1 + a^2q^{2J_0-2l+1})(1 + a^2q^{2J_0-2l+2})(1 + a^2q^{2J_0-2l+3})}}.$$

We have the following formula for the symmetric operator I:

$$I\,|n\rangle = a_n|n+1\rangle + a_{n-1}|n-1\rangle, \tag{5}$$
$$a_{n-1} = \left(a^2q^{n+1}\right)^{1/2}\left(\frac{(1-q^n)(1+a^2q^n)}{(1+a^2q^{2n-1})(1+a^2q^{2n+1})}\right)^{1/2}.$$

The operator I is bounded. We assume that it is defined on the whole representation space $\mathcal{H}$. This means that I is a self-adjoint operator. Actually, I is a Hilbert–Schmidt operator since $a_{n+1}/a_n \to q^{1/2}$ when $n \to \infty$. Thus, a spectrum of I is simple (since it is representable by a Jacobi matrix with $a_n \neq 0$), discrete and have a single accumulation point at 0 (see [6, Chapter VII]).

To find eigenvectors ψ_λ of the operator I, $I\psi_\lambda = \lambda\psi_\lambda$, we set

$$\psi_\lambda = \sum_n \beta_n(\lambda)|n\rangle.$$

Acting by I upon both sides of this relation, one derives

$$\sum_n \beta_n(\lambda)(a_n|n+1\rangle + a_{n-1}|n-1\rangle) = \lambda\sum \beta_n(\lambda)|n\rangle,$$

where a_n are the same as in (5). Collecting in this identity factors at $|n\rangle$ with fixed n, we obtain the recurrence relation for the coefficients $\beta_n(\lambda)$: $a_n\beta_{n+1}(\lambda) + a_{n-1}\beta_{n-1}(\lambda) = \lambda\beta_n(\lambda)$. Making the substitution

$$\beta_n(\lambda) = \left[\left(a^2q;q\right)_n\left(1+a^2q^{2n+1}\right)/(q;q)_n\left(1+a^2q\right)a^{2n}\right]^{1/2} q^{-n(n+3)/4}\beta_n'(\lambda)$$

we reduce this relation to the following one

$$A_n\beta_{n+1}'(\lambda) + C_n\beta_{n-1}'(\lambda) = \lambda\beta_n'(\lambda),$$

where

$$A_n = \frac{1+a^2q^{n+1}}{1+a^2q^{2n+1}}, \qquad C_n = \frac{a^2q^{n+1}(1-q^n)}{1+a^2q^{2n+1}}.$$

It is the recurrence relation (2) for the discrete q-ultraspherical polynomials $\tilde{C}_n^{(a^2)}(\lambda;q)$. Therefore, $\beta_n'(\lambda) = \tilde{C}_n^{(a^2)}(\lambda;q)$ and

$$\beta_n(\lambda) = \left(\frac{(-a^2q;q)_n\,(1+a^2q^{2n+1})}{(q;q)_n\,(1+a^2q)a^{2n}}\right)^{1/2} q^{-n(n+3)/4}\,\tilde{C}_n^{(a^2)}(\lambda;q). \tag{6}$$

For the eigenfunctions $\psi_\lambda(x)$ we have the expansion

$$\psi_\lambda(x) = \sum_{n=0}^{\infty}\left(\frac{(-a^2q;q)_n\,(1+a^2q^{2n+1})}{(q;q)_n\,(1+a^2q)a^{2n}}\right)^{1/2} q^{-n(n+3)/4}\tilde{C}_n^{(a^2)}(\lambda;q)|n\rangle. \tag{7}$$

Since a spectrum of the operator I is discrete, only a discrete set of these functions belongs to the Hilbert space $\mathcal{H}$ and this discrete set determines the spectrum of I.

We intend to study the spectrum of I. It can be done by using the operator

$$J := q^{-J_0+l} - a^2\,q^{J_0-l+1}.$$

In order to determine how this operator acts upon the eigenvectors ψ_λ, one can use the q-difference equation

$$\begin{aligned}\left(q^{-n}-a^2q^{n+1}\right)\tilde C_n^{(a^2)}(\lambda;q)&=-a^2q\lambda^{-2}\left(\lambda^2+1\right)\tilde C_n^{(a^2)}(q\lambda;q)\\&\quad+\lambda^{-2}a^2q(1+q)\,\tilde C_n^{(a^2)}(\lambda;q)+\lambda^{-2}\left(\lambda^2-a^2q^2\right)\tilde C_n^{(a^2)}\left(q^{-1}\lambda;q\right)\end{aligned}\tag{8}$$

for the discrete q-polynomials polynomials. Multiply both sides of (8) by $d_n\,|n\rangle$, where d_n are the coefficients of $\tilde C_n^{(a^2)}(\lambda;q)$ in the expression (6) for the coefficients $\beta_n(\lambda)$, and sum up over n. Taking into account formula (7) and the fact that $J\,|n\rangle=(q^{-n}-a^2\,q^{n+1})\,|n\rangle$, one obtains the relation

$$J\,\psi_\lambda=-a^2q\lambda^{-2}\left(\lambda^2+1\right)\psi_{q\lambda}+\lambda^{-2}a^2q(1+q)\psi_\lambda+\lambda^{-2}\left(\lambda^2-a^2q^2\right)\psi_{q^{-1}\lambda}\tag{9}$$

which is used below.

4 Spectrum of I and orthogonality of discrete q-ultraspherical polynomials

Let us analyse a form of the spectrum of I by using the representations T_l^+ of the algebra $\mathrm{su}_q(1,1)$ and the method of paper [7]. If λ is a spectral point of I, then (as it is easy to see from (9)) a successive action by the operator J upon the eigenvector ψ_λ leads to the vectors $\psi_{q^m\lambda}$, $m=0,\pm1,\pm2,\ldots$. However, since I is a Hilbert–Schmidt operator, not all these points can belong to the spectrum of I, since $q^{-m}\lambda\to\infty$ when $m\to\infty$ if $\lambda\neq0$. This means that the coefficient $\lambda^{-2}(\lambda^2-a^2q^2)$ at $\psi_{q^{-1}\lambda}$ in (9) must vanish for some eigenvalue λ. There are two such values of λ: $\lambda=aq$ and $\lambda=-aq$. Let us show that both of these points are spectral points of I. We have

$$\tilde C_n^{(a^2)}(aq;q)={}_2\phi_1\left(q^{-n},a^2q^{n+1};\,-aq;\,q,q\right)=a^2q^{n(n+1)}.$$

Likewise,

$$\tilde C_n^{(a^2)}(-aq;q)=a^2q^{n(n+1)}.$$

Hence, for the scalar product $\langle\psi_{aq},\psi_{aq}\rangle$ in $\mathcal H$ we have the expression

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(1+a^2q^{2n+1})\,(-a^2q;q)_n}{(1+a^2q)(q;q)_n\,a^{2n}q^{n(n+3)/2}}\tilde C_n^{(a^2)}(aq;q)^2&=\frac{(-a^2q^2,-q;q)_\infty}{(-a^2q^2;q^2)_\infty}\\&=\left(\frac{(-a^2q^3;q^2)_\infty}{(q;q^2)_\infty}\right)<\infty.\end{aligned}\tag{10}$$

Similarly,

$$\langle\psi_{-aq},\psi_{-aq}\rangle=(-a^2q^2,-1;q)_\infty/(-a^2q^2;q^2)_\infty<\infty.$$

Thus, the values $\lambda=aq$ and $\lambda=-aq$ are the spectral points of I.

Let us find other spectral points of the operator I. Setting $\lambda=aq$ in (9), we see that the operator J transforms ψ_{aq} into a linear combination of the vectors ψ_{aq^2} and ψ_{aq}. We have to show that ψ_{aq^2} belongs to the Hilbert space $\mathcal H$, that is, that

$$\langle\psi_{aq^2},\psi_{aq^2}\rangle=\sum_{n=0}^{\infty}\frac{(1+a^2q^{2n+1})\,(-a^2q;q)_n}{(1+a^2q)(q;q)_n\,a^{2n}}\,q^{-n(n+3)/2}\,\tilde C_n^{(a^2)}(aq^2;q)^2<\infty.$$

It is made in the same way as in the case of big q-Jacobi polynomials in paper [7]. The above inequality shows that ψ_{aq^2} is an eigenvector of I and the point aq^2 belongs to the spectrum of I. Setting $\lambda = aq^2$ in (9) and acting similarly, one obtains that ψ_{aq^3} is an eigenvector of I and the point aq^3 belongs to the spectrum of I. Repeating this procedure, one sees that ψ_{aq^n}, $n = 1, 2, \ldots$, are eigenvectors of I and the set aq^n, $n = 1, 2, \ldots$, belongs to the spectrum of I. Likewise, one concludes that ψ_{-aq^n}, $n = 1, 2, \ldots$, are eigenvectors of I and the set $-aq^n$, $n = 1, 2, \ldots$, belongs to the spectrum of I. Let us show that the operator I has no other spectral points.

The vectors ψ_{aq^n} and ψ_{-aq^n}, $n = 1, 2, \ldots$, are linearly independent elements of the representation space $\mathcal{H}$. Suppose that aq^n and $-aq^n$, $n = 1, 2, \ldots$, constitute the whole spectrum of I. Then the set of vectors ψ_{aq^n} and ψ_{-aq^n}, $n = 1, 2, \ldots$, is a basis of $\mathcal{H}$. Introducing the notations $\Xi_n := \psi_{aq^{n+1}}$ and $\Xi'_n := \psi_{-aq^{n+1}}$, $n = 0, 1, 2, \ldots$, we find from (9) that

$$J\,\Xi_n = -q^{-2n-1}\big(1 + a^2q^{2(n+1)}\big)\Xi_{n+1} + d_n\,\Xi_n - q^{-2n}\big(1 - q^{2n}\big)\,\Xi_{n-1},$$
$$J\,\Xi'_n = -q^{-2n-1}\big(1 + a^2q^{2(n+1)}\big)\,\Xi'_{n+1} + d_n\,\Xi'_n - q^{-2n}\big(1 - q^{2n}\big)\,\Xi'_{n-1},$$

where $d_n = q^{-2n-1}(1+q)$.

As we see, the matrix of the operator J in the basis Ξ_n, Ξ'_n, $n = 0, 1, 2, \ldots$, is not symmetric, although in the initial basis $|n\rangle$, $n = 0, 1, 2, \ldots$, it was symmetric. The reason is that the matrix $M := (a_{mn}\ a'_{m'n'})$ with entries

$$a_{mn} := \beta_m\big(aq^{n+1}\big), \qquad a'_{m'n'} := \beta_{m'}\big(-aq^{n'+1}\big), \qquad m, n, m', n' = 0, 1, 2, \ldots,$$

where $\beta_m(dq^{n+1})$, $d = \pm a$, are coefficients (6) in the expansion

$$\psi_{dq^{n+1}} = \sum_m \beta_m\big(dq^{n+1}\big)\,|n\rangle,$$

is not unitary. (This matrix M is formed by adding the columns of the matrix $(a'_{m'n'})$ to the columns of the matrix (a_{mn}) from the right.) It maps the basis $\{|n\rangle\}$ into the basis $\{\psi_{aq^{n+1}}, \psi_{-aq^{n+1}}\}$ in the representation space. The nonunitarity of the matrix M is equivalent to the statement that the basis Ξ_n, Ξ_n, $n = 0, 1, 2, \ldots$, is not normalized. In order to normalize it we have to multiply Ξ_n by appropriate numbers c_n and Ξ'_n by numbers c'_n. Let

$$\hat{\Xi}_n = c_n\Xi_n, \qquad \hat{\Xi}'_n = c'_n\Xi_n, \qquad n = 0, 1, 2, \ldots,$$

be a normalized basis. Then the operator J is symmetric in this basis and has the form

$$J\,\hat{\Xi}_n = -c_{n+1}^{-1}c_nq^{-2n-1}\big(1 + a^2q^{2(n+1)}\big)\,\hat{\Xi}_{n+1} + d_n\,\hat{\Xi}_n - c_{n-1}^{-1}c_nq^{-2n}\big(1 - q^{2n}\big)\,\hat{\Xi}_{n-1},$$
$$J\,\hat{\Xi}'_n = -c'^{-1}_{n+1}c'_nq^{-2n-1}\big(1 + a^2q^{2(n+1)}\big)\,\hat{\Xi}'_{n+1} + d_n\,\hat{\Xi}'_n - c'^{-1}_{n-1}c'_nq^{-2n}\big(1 - q^{2n}\big)\,\hat{\Xi}'_{n-1}.$$

The symmetricity of the matrix of the operator J in the basis $\{\hat{\Xi}_n, \hat{\Xi}'_n\}$ means that for coefficients c_n we have the relation

$$c_{n+1}^{-1}c_nq^{-2n-1}\big(1 + a^2q^{2(n+1)}\big) = c_n^{-1}c_{n+1}q^{-2n-2}\big(1 - q^{2(n+1)}\big).$$

The relation for c'_n coincides with this relation. Thus,

$$\frac{c_n}{c_{n-1}} = \frac{c'_n}{c'_{n-1}} = \sqrt{\frac{q(1 + a^2q^{2n})}{1 - q^{2n}}}.$$

This means that

$$c_n = C\left(\frac{q^n(-a^2q^2; q^2)_n}{(q^2; q^2)_n}\right)^{1/2}, \qquad c'_n = C'\left(\frac{q^n(-a^2q^2; q^2)_n}{(q^2; q^2)_n}\right)^{1/2},$$

where C and C' are some constants.

Therefore, in the expansions

$$\hat{\Xi}_n \equiv \sum_m \hat{a}_{mn}\,|m\rangle, \qquad \hat{\Xi}_n(x) \equiv \sum_m \hat{a}'_{mn}\,|m\rangle$$

the matrix $\hat{M} := (\hat{a}_{mn}\ \hat{a}'_{mn})$ with entries $\hat{a}_{mn} = c_n\,\beta_m(aq^n)$ and $\hat{a}'_{mn} = c_n\,\beta_m(cq^n)$ is unitary, provided that the constants C and C' are appropriately chosen. In order to calculate these constants, one can use the relations $\sum\limits_{m=0}^{\infty}|\hat{a}_{mn}|^2 = 1$ and $\sum\limits_{m=0}^{\infty}|\hat{a}'_{mn}|^2 = 1$ for $n = 0$. Then these sums are multiples of the sum in (10), so we find that

$$C = C' = \left(\frac{(-a^2q^2;q^2)_\infty}{(-a^2q^2,-q;q)_\infty}\right)^{1/2} = \left(\frac{(q;q^2)_\infty}{(-a^2q^3;q^2)_\infty}\right)^{1/2}.$$

The orthogonality of the matrix $\hat{M} \equiv (\hat{a}_{mn}\ \hat{a}'_{mn})$ means that

$$\sum_m \hat{a}_{mn}\hat{a}_{mn'} = \delta_{nn'}, \qquad \sum_m \hat{a}'_{mn}\hat{a}'_{mn'} = \delta_{nn'}, \qquad \sum_m \hat{a}_{mn}\hat{a}'_{mn'} = 0, \tag{11}$$

$$\sum_n (\hat{a}_{mn}\hat{a}_{m'n} + \hat{a}'_{mn}\hat{a}'_{m'n}) = \delta_{mm'}. \tag{12}$$

Substituting the expressions for $\hat{a}_{mn}$ and $\hat{a}'_{mn}$ into (12), one obtains the relation

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-a^2q^2;q^2)_nq^n}{(q^2;q^2)_n}&\left[\tilde{C}_m^{(a^2)}(aq^{n+1})\tilde{C}_{m'}^{(a^2)}(aq^{n+1}) + \tilde{C}_m^{(a^2)}(-aq^{n+1})\tilde{C}_{m'}^{(a^2)}(-aq^{n+1})\right]\\ &= \frac{(-a^2q^3;q^2)_\infty}{(q;q^2)_\infty}\frac{(1+a^2q)(q;q)_m a^{2m}q^{m(m+3)/2}}{(1+a^2q^{2m+1})(-a^2q;q)_m}\delta_{mm'}.\end{aligned} \tag{13}$$

This identity must give an orthogonality relation for the discrete q-ultraspherical polynomials $\tilde{C}_m^{(a^2)}(y) \equiv \tilde{C}_m^{(a^2)}(y;q)$. An only gap, which appears here, is the following. We have assumed that the points aq^{n+1} and $-aq^{n+1}$, $n = 0,1,2,\ldots$, exhaust the whole spectrum of the operator I. If the operator I would have other spectral points x_k, then on the left-hand side of (13) would appear other summands $\mu_{x_k}\tilde{C}_m^{(a^2)}(x_k;q)\tilde{C}_{m'}^{(a^2)}(x_k;q)$, which correspond to these additional points. Let us show that these additional summands do not appear. For this we set $m = m' = 0$ in the relation (13) with the additional summands. This results in the equality

$$\begin{aligned}&\frac{(-a^2q^2;q^2)_\infty}{(-a^2q^2,-q;q)_\infty}\sum_{n=0}^{\infty}\frac{(aq,-aq;q)_nq^n}{(q,-q;q)_n}\\ &\quad + \frac{(-a^2q^2;q)_\infty}{(-a^2q^2,-q;q)_\infty}\sum_{n=0}^{\infty}\frac{(aq,-aq;q)_nq^n}{(q,-q;q)_n} + \sum_k \mu_{x_k} = 1.\end{aligned} \tag{14}$$

In order to show that $\sum\limits_k \mu_{x_k} = 0$, take into account formula (2.10.13) in [1]. By means of this formula it is easy to show that the relation (14) without the summand $\sum\limits_k \mu_{x_k}$ is true. Therefore, in (14) the sum $\sum\limits_k \mu_{x_k}$ does really vanish and formula (13) gives an orthogonality relation for the discrete q-ultraspherical polynomials.

The relation (13) and the results of Chapter VII in [6] shows that *the spectrum of the operator I coincides with the set of points aq^{n+1}, $-aq^{n+1}$, $n = 0,1,2,\ldots$.*

5 Dual discrete q-ultraspherical polynomials

Now we use the relations (11). They give the orthogonality relation for the set of matrix elements $\hat{a}_{mn}$ and $\hat{a}'_{mn}$, considered as functions of m. Up to multiplicative factors, they coincide with the functions

$$F_n(x;a^2) := {}_3\phi_2\big(x, a^2q/x, aq^{n+1};\, \mathrm{i}aq, -\mathrm{i}aq;\, q, q\big), \qquad n = 0,1,2,\ldots,$$
$$F'_n(x;a^2) := {}_3\phi_2\big(x, a^2q/x, -aq^{n+1};\, \mathrm{i}aq, -\mathrm{i}aq;\, q, q\big), \qquad n = 0,1,2,\ldots,$$

considered on the corresponding sets of points.

Applying the relation (III.12) of Appendix III in [1] we express these functions in terms of dual q-ultraspherical polynomials. Thus we obtain expressions for $\hat{a}_{mn}$ and $\hat{a}'_{mn}$ in terms of these polynomials. Substituting these expressions into the relations (11) we obtain the following orthogonality relation for the polynomials $\tilde{D}_n^{(a^2)}(\mu(m;a^2)|q)$:

$$\sum_{m=0}^{\infty} \frac{(1+a^2q^{2m+1})(-a^2q;q)_m}{(1+a^2q)(q;q)_m} q^{m(m-1)/2} \tilde{D}_n^{(a^2)}\big(\mu(m;-a^2)|q\big)\tilde{D}_{n'}^{(a^2)}\big(\mu(m;-a^2)|q\big)$$
$$= \frac{2(-a^2q^3;q^2)_\infty}{(q;q^2)_\infty} \frac{(q^2;q^2)_n q^{-n}}{(-a^2q^2;q^2)_n}\delta_{nn'}. \qquad (15)$$

This orthogonality relation coincides with the sum of two orthogonality relations (9) and (10) in [4]. The orthogonality measure in (15) is not extremal since it is a sum of two extremal measures.

In order to obtain orthogonality relations (9) and (10) of [4], from the very beginning, instead of operator I, we have to consider operators I_1 and I_2 of the representation T_l^+ of the algebra $U_{q^2}(\mathrm{su}_{1,1})$, which are appropriate for obtaining the orthogonality relations for the sets of polynomials $\tilde{C}_{2k}^{(a)}(\sqrt{a}q^{s+1};q)$, $k = 0,1,2,\ldots$, and $\tilde{C}_{2k+1}^{(a)}(\sqrt{a}q^{s+1};q)$, $k = 0,1,2,\ldots$, from Section 2 in [4]. Then going to the orthogonality relations for dual sets of polynomials in the same way as above, we obtain the extremal orthogonality relations (9) and (10) of [4].

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