Second Order Superintegrable Systems in Three Dimensions

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Abstract. A classical (or quantum) superintegrable system on an n-dimensional Riemannian manifold is an integrable Hamiltonian system with potential that admits $2n-1$ functionally independent constants of the motion that are polynomial in the momenta, the maximum number possible. If these constants of the motion are all quadratic, the system is second order superintegrable. Such systems have remarkable properties. Typical properties are that 1) they are integrable in multiple ways and comparison of ways of integration leads to new facts about the systems, 2) they are multiseparable, 3) the second order symmetries generate a closed quadratic algebra and in the quantum case the representation theory of the quadratic algebra yields important facts about the spectral resolution of the Schrödinger operator and the other symmetry operators, and 4) there are deep connections with expansion formulas relating classes of special functions and with the theory of Exact and Quasi-exactly Solvable systems. For $n = 2$ the author, E.G. Kalnins and J. Kress, have worked out the structure of these systems and classified all of the possible spaces and potentials. Here I discuss our recent work and announce new results for the much more difficult case $n = 3$. We consider classical superintegrable systems with nondegenerate potentials in three dimensions and on a conformally flat real or complex space. We show that there exists a standard structure for such systems, based on the algebra of 3×3 symmetric matrices, and that the quadratic algebra always closes at order 6. We describe the Stäckel transformation, an invertible conformal mapping between superintegrable structures on distinct spaces, and give evidence indicating that all our superintegrable systems are Stäckel transforms of systems on complex Euclidean space or the complex 3-sphere. We also indicate how to extend the classical 2D and 3D superintegrability theory to include the operator (quantum) case.

Key words: superintegrability; quadratic algebra; conformally flat spaces

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1 Introduction and examples

In this paper I will report on recent and ongoing work with E.G. Kalnins and J. Kress to uncover the structure of second order superintegrable systems, both classical and quantum mechanical. I will concentrate on the basic ideas; the details of the proofs can be found elsewhere. The results on the quadratic algebra structure of 3D conformally flat systems with nondegenerate potential have appeared recently. The results on the 3D Stäckel transform and multiseparability of superintegrable systems with nondegenerate potentials are announced here.

Superintegrable systems can lay claim to be the most symmetric solvable systems in mathematics though, technically, many such systems admit no group symmetry. In this paper I will only consider superintegrable systems on complex conformally flat spaces. This is no restriction at all in two dimensions. An n-dimensional complex Riemannian space is conformally flat if and only if it admits a set of local coordinates $x_1, \ldots, x_n$ such that the contravariant metric tensor takes the form $g^{ij} = \delta^{ij}/\lambda(\boldsymbol{x})$. Thus the metric is $ds^2 = \lambda(\boldsymbol{x})\big(\sum_{i=1}^{n} dx_i^2\big)$. A classical superintegrable system $\mathcal{H} = \sum_{ij} g^{ij}p_ip_j + V(\boldsymbol{x})$ on the phase space of this manifold is one that admits $2n-1$ functionally independent generalized symmetries (or constants of the motion) $\mathcal{S}_k$, $k = 1, \ldots, 2n-1$ with $\mathcal{S}_1 = \mathcal{H}$ where the $\mathcal{S}_k$ are polynomials in the momenta p_j. That is, $\{\mathcal{H}, \mathcal{S}_k\} = 0$ where

$$\{f, g\} = \sum_{j=1}^{n}(\partial_{x_j}f\partial_{p_j}g - \partial_{p_j}f\partial_{x_j}g)$$

is the Poisson bracket for functions $f(\boldsymbol{x},\boldsymbol{p})$, $g(\boldsymbol{x},\boldsymbol{p})$ on phase space [1, 2, 3, 4, 5, 6, 7, 8]. It is easy to see that $2n-1$ is the maximum possible number of functionally independent symmetries and, locally, such (in general nonpolynomial) symmetries always exist. The system is second order superintegrable if the $2n-1$ functionally independent symmetries can be chosen to be quadratic in the momenta. Usually a superintegrable system is also required to be integrable, i.e., it is assumed that n of the constants of the motion are in involution, although I will not make that assumption in this paper. Sophisticated tools such as R-matrix theory can be applied to the general study of superintegrable systems, e.g., [9, 10, 11]. However, the most detailed and complete results are known for second order superintegrable systems because separation of variables methods for the associated Hamilton–Jacobi equations can be applied. Standard orthogonal separation of variables techniques are associated with second-order symmetries, e.g., [12, 13, 14, 15, 16, 17] and multiseparable Hamiltonian systems provide numerous examples of superintegrability. Thus here I concentrate on second-order superintegrable systems, on those in which the symmetries take the form $\mathcal{S} = \sum a^{ij}(\boldsymbol{x})p_ip_j + W(\boldsymbol{x})$, quadratic in the momenta.

There is an analogous definition for second-order quantum superintegrable systems with Schrödinger operator

$$H = \Delta + V(\boldsymbol{x}), \qquad \Delta = \frac{1}{\sqrt{g}}\sum_{ij}\partial_{x_i}\big(\sqrt{g}g^{ij}\big)\partial_{x_j},$$

the Laplace–Beltrami operator plus a potential function [12]. Here there are $2n-1$ second-order symmetry operators

$$S_k = \frac{1}{\sqrt{g}}\sum_{ij}\partial_{x_i}\big(\sqrt{g}a^{ij}_{(k)}\big)\partial_{x_j} + W^{(k)}(\boldsymbol{x}), \qquad k = 1, \ldots, 2n-1$$

with $S_1 = H$ and $[H, S_k] \equiv HS_k - S_kH = 0$. Again multiseparable systems yield many examples of superintegrability, though not all multiseparable systems are superintegrable and not all second-order superintegrable systems are multiseparable.

The basic motivation for studying superintegrable systems is that they can be solved explicitly and in multiple ways. It is the information gleaned from comparing the distinct solutions and expressing one solution set in terms of another that is a primary reason for their interest.

Two dimensional second order superintegrable systems have been studied and classified by the author and his collaborators in a recent series of papers [18, 19, 20, 21]. Here we concentrate on three dimensional (3D) systems where new complications arise. We start with some simple 3D examples to illustrate some of the main features of superintegrable systems. (To make clearer the connection with quantum theory and Hilbert space methods we shall, for these examples alone, adopt standard physical normalizations, such as using the factor $-\frac{1}{2}$ in front of the free Hamiltonian.) Consider the Schrödinger equation $H\Psi = E\Psi$ or ($\hbar = m = 1$, $x_1 = x$, $x_2 = y$,

$x_3 = z$)

$$H\Psi = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Psi + V(x,y,z)\Psi = E\Psi.$$

The generalized anisotropic oscillator corresponds to the 4-parameter potential

$$V(x,y,z) = \frac{\omega^2}{2}\left(x^2 + y^2 + 4(z+\rho)^2\right) + \frac{1}{2}\left[\frac{k_1^2 - \frac{1}{4}}{x^2} + \frac{k_2^2 - \frac{1}{4}}{y^2}\right].$$

(This potential is "nondegenerate" in a precise sense that I will explain later.) The corresponding Schrödinger equation has separable solutions in five coordinate systems: Cartesian coordinates, cylindrical polar coordinates, cylindrical elliptic coordinates, cylindrical parabolic coordinates and parabolic coordinates. The energy eigenstates for this equation are degenerate and important special function identities arise by expanding one basis of separable eigenfunctions in terms of another. A second order symmetry operator for this equation is a second order linear differential operator S such that $[H,S] = 0$, where $[A,B] = AB - BA$. A basis for these operators is

$$\begin{aligned}
&M_1 = \partial_x^2 - \omega^2 x^2 + \frac{k_1^2 - \frac{1}{4}}{x^2}, \qquad M_2 = \partial_y^2 - \omega^2 y^2 - \frac{k_2^2 - \frac{1}{4}}{y^2},\\
&P = \partial_z^2 - 4\omega^2(z+\rho)^2, \qquad L = L_{12}^2 - \left(k_1^2 - \frac{1}{4}\right)\frac{y^2}{x^2} - \left(k_2^2 - \frac{1}{4}\right)\frac{x^2}{y^2} - \frac{1}{2},\\
&S_1 = -\frac{1}{2}(\partial_x L_{13} + L_{13}\partial_x) + \rho\partial_x^2 + (z+\rho)\left(\omega^2 x^2 - \frac{k_1^2 - \frac{1}{4}}{x^2}\right),\\
&S_2 = -\frac{1}{2}(\partial_y L_{23} + L_{23}\partial_y) + \rho\partial_y^2 + (z+\rho)\left(\omega^2 y^2 - \frac{k_2^2 - \frac{1}{4}}{y^2}\right),
\end{aligned}$$

where $L_{ij} = x_i\partial_{x_j} - x_j\partial_{x_i}$. It can be verified that these symmetries generate a "quadratic algebra" that closes at level six. Indeed, the nonzero commutators of the above basis are

$$[M_1, L] = [L, M_2] = Q, \qquad [L, S_1] = [S_2, L] = B, \qquad [M_i, S_i] = A_i, \qquad [P, S_i] = -A_i.$$

Nonzero commutators of the basis symmetries with Q (4th order symmetries) are expressible in terms of the second order symmetries:

$$\begin{aligned}
&[M_i, Q] = [Q, M_2] = 4\{M_1, M_2\} + 16\omega^2 L, \qquad [S_1, Q] = [Q, S_2] = 4\{M_1, M_2\},\\
&[L, Q] = 4\{M_1, L\} - 4\{M_2, L\} + 16\left(1 - k_1^2\right)M_1 - 16\left(1 - k_2^2\right)M_2.
\end{aligned}$$

There are similar expressions for commutators with B and the A_i. Also the squares of Q, B, A_i and products such as $\{Q, B\}$, (all 6th order symmetries) are all expressible in terms of 2nd order symmetries. Indeed

$$\begin{aligned}
Q^2 &= \frac{8}{3}\{L, M_1, M_2\} + 8\omega^2\{L, L\} - 16\left(1 - k_1^2\right)M_1^2 - 16\left(1 - k_2^2\right)M_2^2\\
&\quad + \frac{64}{3}\{M_1, M_2\} - \frac{128}{3}\omega^2 L - 128\omega^2\left(1 - k_1^2\right)\left(1 - k_2^2\right),\\
\{Q, B\} &= -\frac{8}{3}\{M_2, L, S_1\} - \frac{8}{3}\{M_1, L, S_2\} + 16\left(1 - k_1^2\right)\{M_2, S_2\} + 16\left(1 - k_2^2\right)\{M_1, S_1\}\\
&\qquad - \frac{64}{3}\{M_1, S_2\} - \frac{64}{3}\{M_2, S_1\}.
\end{aligned}$$

Here $\{C_1, \dots, C_j\}$ is the completely symmetrized product of operators $C_1, \dots, C_j$. (For complete details see [22].) The point is that the algebra generated by products and commutators of the 2nd order symmetries closes at order 6. This is a remarkable fact, and ordinarily not the case for an integrable system.

A counterexample to the existence of a quadratic algebra in Euclidean space is given by the Schrödinger equation with 3-parameter extended Kepler–Coulomb potential:

$$\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}\right) + \left[2E + \frac{2\alpha}{\sqrt{x^2+y^2+z^2}} - \left(\frac{k_1^2 - \frac{1}{4}}{x^2} + \frac{k_2^2 - \frac{1}{4}}{y^2}\right)\right]\Psi = 0.$$

This equation admits separable solutions in the four coordinates systems: spherical, sphero-conical, prolate spheroidal and parabolic coordinates. Again the bound states are degenerate and important special function identities arise by expanding one basis of separable eigenfunctions in terms of another. However, the space of second order symmetries is only 5 dimensional and, although there are useful identities among the generators and commutators that enable one to derive spectral properties algebraically, there is no finite quadratic algebra structure. The key difference with our first example is, as we shall show later, that the 3-parameter Kepler–Coulomb potential is degenerate and it cannot be extended to a 4-parameter potential.

In [20, 21] there are examples of superintegrable systems on the 3-sphere that admit a quadratic algebra structure. A more general set of examples arises from a space with metric

$$ds^2 = \lambda(A, B, C, D, \boldsymbol{x})\left(dx^2 + dy^2 + dz^2\right),$$

where

$$\lambda = A(x+iy) + B\left(\frac{3}{4}(x+iy)^2 + \frac{z}{4}\right) + C\left((x+iy)^3 + \frac{1}{16}(x-iy) + \frac{3z}{4}(x+iy)\right)$$
$$+ D\left(\frac{5}{16}(x+iy)^4 + \frac{z^2}{16} + \frac{1}{16}\left(x^2+y^2\right) + \frac{3z}{8}(x+iy)^2\right).$$

The nondegenerate classical potential is $V = \lambda(\alpha, \beta, \gamma, \delta, \boldsymbol{x})/\lambda(A, B, C, D, \boldsymbol{x})$. If $A = B = C = D = 0$ this is a nondegenerate metric on complex Euclidean space. The quadratic algebra always closes, and for general values of A, B, C, D the space is not of constant curvature. As will be apparent later. This is an example of a superintegrable system that is Stäckel equivalent to a system on complex Euclidean space.

Observed common features of superintegrable systems are that they are usually multiseparable and that the eigenfunctions of one separable system can be expanded in terms of the eigenfunctions of another. This is the source of nontrivial special function expansion theorems [23]. The symmetry operators are in formal self-adjoint form and suitable for spectral analysis. Also, the quadratic algebra identities allow us to relate eigenbases and eigenvalues of one symmetry operator to those of another. The representation theory of the abstract quadratic algebra can be used to derive spectral properties of the second order generators in a manner analogous to the use of Lie algebra representation theory to derive spectral properties of quantum systems that admit Lie symmetry algebras, [23, 24, 25, 26]. (Note however that for superintegrable systems with nondegenerate potential, there is no first order Lie symmetry.)

Another common feature of quantum superintegrable systems is that they can be modified by a gauge transformation so that the Schrödinger and symmetry operators are acting on a space of polynomials [27]. This is closely related to the theory of exactly and quasi-exactly solvable systems [28, 29]. The characterization of ODE quasi-exactly solvable systems as embedded in PDE superintegrable systems provides considerable insight into the nature of these phenomena [30].

The classical analogs of the above examples are obtained by the replacements $\partial_{x_i} \to p_{x_i}$ and modification of the potential by curvature terms. Commutators go over to Poisson brackets.

The operator symmetries become second order constants of the motion. Symmetrized operators become products of functions. The quadratic algebra relations simplify: the highest order terms agree with the operator case but there are fewer nonzero lower order terms.

Many examples of 3D superintegrable systems are known, although they have not been classified [31, 32, 33, 34, 35, 36]. Here, we employ a theoretical method based on integrability conditions to derive structure common to all such systems, with a view to complete classification, at least for classical systems with nondegenerate potentials. We show that for systems with nondegenerate potentials there exists a standard structure based on the algebra of 3×3 symmetric matrices, and that the quadratic algebra closes at level 6. For 2D nondegenerate superintegrable systems we earlier showed that the $3 = 2(2) - 1$ functionally independent constants of the motion were (with one exception) also linearly independent, so at each regular point we could find a unique constant of the motion that matches a quadratic expression in the momenta at that point. However, for 3D systems we have only $5 = 2(3) - 1$ functionally independent constants of the motion and the quadratic forms span a 6 dimensional space. This is a major problem. However, for nondegenerate potentials we prove the "5 implies 6 Theorem" to show that the space of second order constants of the motion is in fact 6 dimensional: there is a symmetry that is functionally dependent on the symmetries that arise from superintegrability, but linearly independent of them. With that result established, the treatment of the 3D case can proceed in analogy with the nondegenerate 2D case treated in [18]. Though the details are quite complicated, the spaces of truly 2nd, 3rd, 4th and 6th order constants of the motion can be shown to be of dimension 6, 4, 21 and 56, respectively and we can construct explicit bases for the 4th and 6th order constants in terms of products of the 2nd order constants. This means that there is a quadratic algebra structure.

Using this structure we can show that all 3D superintegrable systems with nondegenerate potential are multiseparable. We study the Stäckel transform, or coupling constant metamorphosis [37, 38], for 3D classical superintegrable systems. This is a conformal transformation of a superintegrable system on one space to a superintegrable system on another space. We give evidence that all nondegenerate 3D superintegrable systems are Stäckel transforms of constant curvature systems, just as in the 2D case, though we don't completely settle the issue. This provides the theoretical basis for a complete classification of 3D superintegrable systems with nondegenerate potential, a program that is underway. Finally we indicate the quantum analogs of our results for 3D classical systems.

2 Conformally flat spaces in three dimensions

We assume that there is a coordinate system x, y, z and a nonzero function $\lambda(x,y,z) = \exp G(x,y,z)$ such that the Hamiltonian is

$$\mathcal{H} = \frac{p_1^2 + p_2^2 + p_3^2}{\lambda} + V(x,y,z).$$

A quadratic constant of the motion (or generalized symmetry)

$$\mathcal{S} = \sum_{k,j=1}^{3} a^{kj}(x,y,z)p_k p_j + W(x,y,z) \equiv \mathcal{L} + W, \qquad a^{jk} = a^{kj}$$

must satisfy $\{\mathcal{H}, \mathcal{S}\} = 0$, i.e.,

$$a_i^{ii} = -G_1 a^{1i} - G_2 a^{2i} - G_3 a^{3i},$$
$$2a_i^{ij} + a_j^{ii} = -G_1 a^{1j} - G_2 a^{2j} - G_3 a^{3j}, \qquad i \neq j,$$

$$a_k^{ij} + a_j^{ki} + a_i^{jk} = 0, \qquad i, j, k \text{ distinct}$$

and

$$W_k = \lambda \sum_{s=1}^{3} a^{sk} V_s, \qquad k = 1, 2, 3. \tag{1}$$

(Here a subscript j denotes differentiation with respect to x_j.) The requirement that $\partial_{x_\ell} W_j = \partial_{x_j} W_\ell$, $\ell \neq j$ leads from (1) to the second order Bertrand–Darboux partial differential equations for the potential.

$$\sum_{s=1}^{3} \left[V_{sj} \lambda a^{s\ell} - V_{s\ell} \lambda a^{sj} + V_s \left((\lambda a^{s\ell})_j - (\lambda a^{sj})_\ell \right) \right] = 0. \tag{2}$$

For second order superintegrabilty in 3D there must be five functionally independent constants of the motion (including the Hamiltonian itself). Thus the Hamilton–Jacobi equation admits four additional constants of the motion:

$$\mathcal{S}_h = \sum_{j,k=1}^{3} a_{(h)}^{jk} p_k p_j + W_{(h)} = \mathcal{L}_h + W_{(h)}, \qquad h = 1, \dots, 4.$$

We assume that the four functions $\mathcal{S}_h$ together with $\mathcal{H}$ are functionally independent in the six-dimensional phase space. (Here the possible V will always be assumed to form a vector space and we require functional independence for each such V and the associated $W^{(h)}$. This means that we require that the five quadratic forms $\mathcal{L}_h$, $\mathcal{H}_0$ are functionally independent.) In [20] it is shown that the matrix of the 15 Bertrand–Darboux equations for the potential has rank at least 5, hence we can solve for the second derivatives of the potential in the form

$$\begin{aligned}
V_{22} &= V_{11} + A^{22} V_1 + B^{22} V_2 + C^{22} V_3, \\
V_{33} &= V_{11} + A^{33} V_1 + B^{33} V_2 + C^{33} V_3, \\
V_{12} &= \qquad\ \ A^{12} V_1 + B^{12} V_2 + C^{12} V_3, \\
V_{13} &= \qquad\ \ A^{13} V_1 + B^{13} V_2 + C^{13} V_3, \\
V_{23} &= \qquad\ \ A^{23} V_1 + B^{23} V_2 + C^{23} V_3.
\end{aligned} \tag{3}$$

If the matrix has rank > 5 then there will be additional conditions of the form $D^1_{(s)} V_1 + D^2_{(s)} V_2 + D^3_{(s)} V_3 = 0$. Here the A^{ij}, B^{ij}, C^{ij}, $D^i_{(s)}$ are functions of x that can be calculated explicitly. For convenience we take $A^{ij} \equiv A^{ji}$, $B^{ij} \equiv B^{ji}$, $C^{ij} \equiv C^{ji}$.

Suppose now that the superintegrable system is such that the rank is exactly 5 so that the relations are only (3). Further, suppose the integrability conditions for system (3) are satisfied identically. In this case we say that the potential is *nondegenerate*. Otherwise the potential is *degenerate*. If V is nondegenerate then at any point $\boldsymbol{x}_0$, where the A^{ij}, B^{ij}, C^{ij} are defined and analytic, there is a unique solution $V(\boldsymbol{x})$ with arbitrarily prescribed values of $V_1(\boldsymbol{x}_0)$, $V_2(\boldsymbol{x}_0)$, $V_3(\boldsymbol{x}_0)$, $V_{11}(\boldsymbol{x}_0)$ (as well as the value of $V(\boldsymbol{x}_0)$ itself.) The points $\boldsymbol{x}_0$ are called *regular*. The points of singularity for the A^{ij}, B^{ij}, C^{ij} form a manifold of dimension < 3. Degenerate potentials depend on fewer parameters. For example, it may be that the rank of the Bertrand–Darboux equations is exactly 5 but the integrability conditions are not satisfied identically. This occurs for the generalized Kepler–Coulomb potential.

From this point on we assume that V is nondegenerate. Substituting the requirement for a nondegenerate potential (3) into the Bertrand–Darboux equations (2) we obtain three equations for the derivatives a_i^{jk}, the first of which is

$$\left(a_3^{11} - a_1^{31}\right) V_1 + \left(a_3^{12} - a_1^{32}\right) V_2 + (a_3^{13} - a_1^{33}) V_3$$

$$
\begin{aligned}
&+ a^{12}\left(A^{23}V_1 + B^{23}V_2 + C^{23}V_3\right) - \left(a^{33} - a^{11}\right)\left(A^{13}V_1 + B^{13}V_2 + C^{13}V_3\right)\\
&- a^{23}\left(A^{12}V_1 + B^{12}V_2 + C^{12}V_3\right) + a^{13}\left(A^{33}V_1 + B^{33}V_2 + C^{33}V_3\right)\\
&= \left(-G_3a^{11} + G_1a^{13}\right)V_1 + \left(-G_3a^{12} + G_1a^{23}\right)V_2 + \left(-G_3a^{13} + G_1a^{33}\right)V_3,
\end{aligned}
$$

and the other two are obtained in a similar fashion.

Since V is a nondegenerate potential we can equate coefficients of V_1, V_2, V_3, V_{11} on each side of the conditions $\partial_1 V_{23} = \partial_2 V_{13} = \partial_3 V_{12}$, $\partial_3 V_{23} = \partial_2 V_{33}$, etc., to obtain integrability conditions, the simplest of which include

$$
\begin{aligned}
&A^{23} = B^{13} = C^{12}, \qquad B^{12} - A^{22} = C^{13} - A^{33},\\
&B^{23} = A^{31} + C^{22}, \qquad C^{23} = A^{12} + B^{33},\\
&A^{12}_1 + B^{12}A^{12} + A^{33}_2 + A^{33}A^{12} + B^{33}A^{22} + C^{33}A^{23} = A^{23}_3 + B^{23}A^{23} + C^{23}A^{33},\\
&A^{13}_2 + A^{13}A^{12} + B^{13}A^{22} + C^{13}A^{23} = A^{23}_1 + B^{23}A^{12} + C^{23}A^{13}\\
&\qquad = A^{12}_3 + A^{13}A^{12} + B^{12}A^{23} + C^{12}A^{33}.
\end{aligned}
$$

Using the nondegenerate potential condition and the Bertrand–Darboux equations we can solve for all of the first partial derivatives a^{jk}_i of a quadratic symmetry to obtain

$$
\begin{aligned}
a^{11}_1 &= -G_1a^{11} - G_2a^{12} - G_3a^{13}, \qquad (4)\\
a^{22}_2 &= -G_1a^{12} - G_2a^{22} - G_3a^{23},\\
a^{33}_3 &= -G_1a^{13} - G_2a^{23} - G_3a^{33},\\
3a^{12}_1 &= a^{12}A^{22} - \left(a^{22} - a^{11}\right)A^{12} - a^{23}A^{13} + a^{13}A^{23}\\
&\quad + G_2a^{11} - 2G_1a^{12} - G_2a^{22} - G_3a^{23},\\
3a^{11}_2 &= -2a^{12}A^{22} + 2\left(a^{22} - a^{11}\right)A^{12} + 2a^{23}A^{13} - 2a^{13}A^{23}\\
&\quad - 2G_2a^{11} + G_1a^{12} - G_2a^{22} - G_3a^{23},\\
3a^{13}_3 &= -a^{12}C^{23} + \left(a^{33} - a^{11}\right)C^{13} + a^{23}C^{12} - a^{13}C^{33}\\
&\quad - G_1a^{11} - G_2a^{12} - 2G_3a^{13} + G_1a^{33},\\
3a^{33}_1 &= 2a^{12}C^{23} - 2\left(a^{33} - a^{11}\right)C^{13} - 2a^{23}C^{12} + 2a^{13}C^{33}\\
&\quad - G_1a^{11} - G_2a^{12} + G_3a^{13} - 2G_1a^{33},\\
3a^{23}_2 &= a^{23}(B^{33} - B^{22}) - \left(a^{33} - a^{22}\right)B^{23} - a^{13}B^{12} + a^{12}B^{13}\\
&\quad - G_1a^{13} - 2G_2a^{23} - G_3a^{33} + G_3a^{22},\\
3a^{22}_3 &= -2a^{23}\left(B^{33} - B^{22}\right) + 2(a^{33} - a^{22})B^{23} + 2a^{13}B^{12} - 2a^{12}B^{13}\\
&\quad - G_1a^{13} + G_2a^{23} - G_3a^{33} - 2G_3a^{22},\\
3a^{13}_1 &= -a^{23}A^{12} + \left(a^{11} - a^{33}\right)A^{13} + a^{13}A^{33} + a^{12}A^{23}\\
&\quad - 2G_1a^{13} - G_2a^{23} - G_3a^{33} + G_3a^{11},\\
3a^{11}_3 &= 2a^{23}A^{12} + 2\left(a^{33} - a^{11}\right)A^{13} - 2a^{13}A^{33} - 2a^{12}A^{23}\\
&\quad + G_1a^{13} - G_2a^{23} - G_3a^{33} - 2G_3a^{11},\\
3a^{33}_2 &= -2a^{13}C^{12} + 2\left(a^{22} - a^{33}\right)C^{23} + 2a^{12}C^{13} - 2a^{23}\left(C^{22} - C^{33}\right)\\
&\quad - G_1a^{12} - G_2a^{22} + G_3a^{23} - 2G_2a^{33},\\
3a^{23}_3 &= a^{13}C^{12} - \left(a^{22} - a^{33}\right)C^{23} - a^{12}C^{13} - a^{23}\left(C^{33} - C^{22}\right)\\
&\quad - G_1a^{12} - G_2a^{22} - 2G_3a^{23} + G_2a^{33},\\
3a^{12}_2 &= -a^{13}B^{23} + \left(a^{22} - a^{11}\right)B^{12} - a^{12}B^{22} + a^{23}B^{13}
\end{aligned}
$$

$$
\begin{aligned}
&\quad - G_1a^{11} - 2G_2a^{12} - G_3a^{13} + G_1a^{22},\\
3a^{22}_1 &= 2a^{13}B^{23} - 2\left(a^{22} - a^{11}\right)B^{12} + 2a^{12}B^{22} - 2a^{23}B^{13}\\
&\quad - G_1a^{11} + G_2a^{12} - G_3a^{13} - 2G_1a^{22},\\
3a^{23}_1 &= a^{12}\left(B^{23} + C^{22}\right) + a^{11}\left(B^{13} + C^{12}\right) - a^{22}C^{12} - a^{33}B^{13}\\
&\quad + a^{13}\left(B^{33} + C^{23}\right) - a^{23}\left(C^{13} + B^{12}\right) - 2G_1a^{23} + G_2a^{13} + G_3a^{12}.\\
3a^{12}_3 &= a^{12}\left(-2B^{23} + C^{22}\right) + a^{11}\left(C^{12} - 2B^{13}\right) - a^{22}C^{12} + 2a^{33}B^{13}\\
&\quad + a^{13}\left(-2B^{33} + C^{23}\right) + a^{23}\left(-C^{13} + 2B^{12}\right) - 2G_3a^{12} + G_2a^{13} + G_1a^{23}.\\
3a^{13}_2 &= a^{12}\left(B^{23} - 2C^{22}\right) + a^{11}\left(B^{13} - 2C^{12}\right) + 2a^{22}C^{12} - a^{33}B^{13}\\
&\quad + a^{13}\left(B^{33} - 2C^{23}\right) + a^{23}\left(2C^{13} - B^{12}\right) - 2G_2a^{13} + G_1a^{23} + G_3a^{12},
\end{aligned}
$$

plus the linear relations

$$
\begin{aligned}
&A^{23} = B^{13} = C^{12}, \qquad B^{23} - A^{31} - C^{22} = 0,\\
&B^{12} - A^{22} + A^{33} - C^{13} = 0, \qquad B^{33} + A^{12} - C^{23} = 0.
\end{aligned}
$$

Using the linear relations we can express C^{12}, C^{13}, C^{22}, C^{23} and B^{13} in terms of the remaining 10 functions.

Since the above system of first order partial differential equations is involutive the general solution for the 6 functions a^{jk} can depend on at most 6 parameters, the values $a^{jk}(\boldsymbol{x}_0)$ at a fixed regular point $\boldsymbol{x}_0$. For the integrability conditions we define the vector-valued function

$$
\boldsymbol{h}(x,y,z) = \left(a^{11}\ a^{12}\ a^{13}\ a^{22}\ a^{23}\ a^{33}\right)
$$

and directly compute the 6×6 matrix functions $\mathcal{A}^{(j)}$ to get the first-order system

$$
\partial_{x_j}\boldsymbol{h} = \mathcal{A}^{(j)}\boldsymbol{h}, \qquad j = 1,2,3.
$$

The integrability conditions for this system are are

$$
\mathcal{A}^{(j)}_i\boldsymbol{h} - \mathcal{A}^{(i)}_j\boldsymbol{h} = \mathcal{A}^{(i)}\mathcal{A}^{(j)}\boldsymbol{h} - \mathcal{A}^{(j)}\mathcal{A}^{(i)}\boldsymbol{h} \equiv [\mathcal{A}^{(i)}, \mathcal{A}^{(j)}]\boldsymbol{h}. \tag{5}
$$

In terms of the 6×6 matrices

$$
\begin{aligned}
&\mathcal{S}^{(1)} = \mathcal{A}^{(3)}_2 - \mathcal{A}^{(2)}_3 - [\mathcal{A}^{(2)}, \mathcal{A}^{(3)}], \qquad \mathcal{S}^{(2)} = \mathcal{A}^{(1)}_3 - \mathcal{A}^{(3)}_1 - [\mathcal{A}^{(3)}, \mathcal{A}^{(1)}],\\
&\mathcal{S}^{(3)} = \mathcal{A}^{(2)}_1 - \mathcal{A}^{(1)}_2 - [\mathcal{A}^{(1)}, \mathcal{A}^{(2)}],
\end{aligned}
$$

the integrabilty conditions are

$$
\mathcal{S}^{(1)}\boldsymbol{h} = \mathcal{S}^{(2)}\boldsymbol{h} = \mathcal{S}^{(3)}\boldsymbol{h} = 0. \tag{6}
$$

3 The $5 \Longrightarrow 6$ Theorem

Now assume that the system of equations (4) admits a 6-parameter family of solutions a^{jk}. (The requirement of superintegrability *appears* to guarantee only a 5-parameter family of solutions.) Thus at any regular point we can prescribe the values of the a^{jk} arbitrarily. This means that (5) or (6) holds identically in $\boldsymbol{h}$. Thus $\mathcal{S}^{(1)} = \mathcal{S}^{(2)} = \mathcal{S}^{(3)} = 0$. This would be the analog of what happens in the 2D case where there are 3independent terms in the quadratic form and 3 functionally (and linearly) independent symmetries. However, in the 3D case there are only 5 functionally independent symmetries, so we can't guarantee that the symmetry equations admit a 6-parameter family of solutions. Fortunately, by careful study of the integrability conditions of these equations and use of the requirement that the potential is nondegenerate, we can prove the $5 \Longrightarrow 6$ theorem [20].

Theorem 1 **($5 \Longrightarrow 6$).** *Let V be a nondegenerate potential corresponding to a conformally flat space in 3 dimensions that is superintegrable, i.e., suppose V satisfies the equations (3) whose integrability conditions hold identically, and there are 5 functionally independent constants of the motion. Then the space of second order symmetries for the Hamiltonian $\mathcal{H} = \left(p_x^2+p_y^2+p_z^2\right)/\lambda(x,y,z)+V(x,y,z)$ (excluding multiplication by a constant) is of dimension $D=6$.*

Corollary 1. *If $\mathcal{H}+V$ is a superintegrable conformally flat system with nondegenerate potential, then the dimension of the space of 2nd order symmetries*

$$\mathcal{S} = \sum_{k,j=1}^{3} a^{kj}(x,y,z)p_k p_j + W(x,y,z)$$

is 6. At any regular point (x_0,y_0,z_0), and given constants $\alpha^{kj}=\alpha^{jk}$, there is exactly one symmetry $\mathcal{S}$ (up to an additive constant) such that $a^{kj}(x_0,y_0,z_0)=\alpha^{kj}$. Given a set of 5 functionally independent 2nd order symmetries $\mathcal{L}=\{\mathcal{S}_\ell : \ell = 1,\dots 5\}$ associated with the potential, there is always a 6th second order symmetry $\mathcal{S}_6$ that is functionally dependent on $\mathcal{L}$, but linearly independent.

4 Third order constants of the motion

The key to understanding the structure of the space of constants of the motion for superintegrable systems with nondegenerate potential is an investigation of third order constants of the motion. We have

$$\mathcal{K} = \sum_{k,j,i=1}^{3} a^{kji}(x,y,z)p_k p_j p_i + b^\ell(x,y,z)p_\ell,$$

which must satisfy $\{\mathcal{H},\mathcal{K}\}=0$. Here a^{kji} is symmetric in the indices k, j, i.

The conditions are

$$a_i^{iii} = -\frac{3}{2}\sum_s a^{sii}(\ln\lambda)_s,$$

$$3a_i^{jii} + a_j^{iii} = -3\sum_s a^{sij}(\ln\lambda)_s, \qquad i \neq j$$

$$a_i^{ijj} + a_j^{iij} = -\frac{1}{2}\sum_s a^{sjj}(\ln\lambda)_s - \frac{1}{2}\sum_s a^{sii}(\ln\lambda)_s, \qquad i \neq j,$$

$$2a_i^{ijk} + a_j^{kii} + a_k^{jii} = -\sum_s a^{sjk}(\ln\lambda)_s, \qquad i,\ j,\ k \text{ distinct},$$

$$b_k^j + b_j^k = 3\lambda\sum_s a^{skj}V_s, \qquad j \neq k, \quad j,k = 1,2,3,$$

$$b_j^j = \frac{3}{2}\lambda\sum_s a^{sjj}V_s - \frac{1}{2}\sum_s b^s(\ln\lambda)_s, \qquad j = 1,2,3,$$

and

$$\sum_s b^s V_s = 0.$$

The a^{kji} is just a third order Killing tensor. We are interested in such third order symmetries that could possibly arise as commutators of second order symmetries. Thus we require that the highest order terms, the a^{kji} in the constant of the motion, be independent of the four independent parameters in V. However, the b^ℓ must depend on these parameters. We set

$$b^\ell(x,y,z) = \sum_{j=1}^{3} f^{\ell,j}(x,y,z)V_j(x,y,z).$$

(Here we are excluding the purely first order symmetries.) In [20] the following result is obtained.

Theorem 2. *Let $\mathcal{K}$ be a third order constant of the motion for a conformally flat superintegrable system with nondegenerate potential V:*

$$\mathcal{K} = \sum_{k,j,i=1}^{3} a^{kji}(x,y,z)p_k p_j p_i + \sum_{\ell=1}^{3} b^\ell(x,y,z)p_\ell.$$

Then

$$b^\ell(x,y,z) = \sum_{j=1}^{3} f^{\ell,j}(x,y,z)V_j(x,y,z)$$

with $f^{\ell,j} + f^{j,\ell} = 0$, $1 \le \ell, j \le 3$. The a^{ijk}, b^ℓ are uniquely determined by the four numbers

$$f^{1,2}(x_0,y_0,z_0), \quad f^{1,3}(x_0,y_0,z_0), \quad f^{2,3}(x_0,y_0,z_0), \quad f_3^{1,2}(x_0,y_0,z_0)$$

at any regular point (x_0,y_0,z_0) of V.

Let

$$\mathcal{S}_1 = \sum a^{kj}_{(1)} p_k p_j + W_{(1)}, \qquad \mathcal{S}_2 = \sum a^{kj}_{(2)} p_k p_j + W_{(2)}$$

be second order constants of the the motion for a superintegrable system with nondegenerate potential and let $\mathcal{A}_{(i)}(x,y,z) = \{a^{kj}_{(i)}(x,y,z)\}$, $i = 1,2$ be 3×3 matrix functions. Then the Poisson bracket of these symmetries is given by

$$\{\mathcal{S}_1, \mathcal{S}_2\} = \sum_{k,j,i=1}^{3} a^{kji}(x,y,z)p_k p_j p_i + b^\ell(x,y,z)p_\ell,$$

where

$$f^{k,\ell} = 2\lambda \sum_j \left(a^{kj}_{(2)} a^{j\ell}_{(1)} - a^{kj}_{(1)} a^{j\ell}_{(2)}\right).$$

Differentiating, we find

$$f_i^{k,\ell} = 2\lambda \sum_j \left(\partial_i a^{kj}_{(2)} a^{j\ell}_{(1)} + a^{kj}_{(2)} \partial_i a^{j\ell}_{(1)} - \partial_i a^{kj}_{(1)} a^{j\ell}_{(2)} - a^{kj}_{(1)} \partial_i a^{j\ell}_{(2)}\right) + G_i f^{k,\ell}. \tag{7}$$

Clearly, $\{\mathcal{S}_1, \mathcal{S}_2\}$ is uniquely determined by the skew-symmetric matrix

$$[\mathcal{A}_{(2)}, \mathcal{A}_{(1)}] \equiv \mathcal{A}_{(2)}\mathcal{A}_{(1)} - \mathcal{A}_{(1)}\mathcal{A}_{(2)},$$

hence by the constant matrix $[\mathcal{A}_{(2)}(x_0,y_0,z_0), \mathcal{A}_{(1)}(x_0,y_0,z_0)]$ evaluated at a regular point, and by the number $\mathcal{F}(x_0,y_0,z_)) = f_3^{1,2}(x_0,y_0,z_0)$.

For superintegrable nondegenerate potentials there is a standard structure allowing the identification of the space of second order constants of the motion with the space S_3 of 3×3

symmetric matrices, as well as identification of the space of third order constants of the motion with a subspace of the space $K_3 \times F$ of 3×3 skew-symmetric matrices K_3 crossed with the line $F = \{\mathcal{F}(\boldsymbol{x}_0)\}$. Indeed, if $\boldsymbol{x}_0$ is a regular point then there is a $1-1$ linear correspondence between second order symmetries $\mathcal{S}$ and their associated symmetric matrices $\mathcal{A}(\boldsymbol{x}_0)$. Let $\{\mathcal{S}_1, \mathcal{S}_2\}' = \{\mathcal{S}_2, \mathcal{S}_1\}$ be the reversed Poisson bracket. Then the map

$$\{\mathcal{S}_1, \mathcal{S}_2\}' \Longleftrightarrow [\mathcal{A}_{(1)}(\boldsymbol{x}_0), \mathcal{A}_{(2)}(\boldsymbol{x}_0)]$$

is an algebraic homomorphism. Here, $\mathcal{S}_1$, $\mathcal{S}_2$ are in involution if and only if matrices $\mathcal{A}_{(1)}(\boldsymbol{x}_0)$, $\mathcal{A}_{(2)}(\boldsymbol{x}_0)$ commute and $\mathcal{F}(\boldsymbol{x}_0) = 0$. If $\{\mathcal{S}_1, \mathcal{S}_2\} \neq 0$ then it is a third order symmetry and can be uniquely associated with the skew-symmetric matrix $[\mathcal{A}_{(1)}(\boldsymbol{x}_0), \mathcal{A}_{(2)}(\boldsymbol{x}_0)]$ and the parameter $\mathcal{F}(\boldsymbol{x}_0)$. Let $\mathcal{E}^{ij}$ be the 3×3 matrix with a 1 in row i, column j and 0 for every other matrix element. Then the matrices

$$\mathcal{A}^{(ij)} = \frac{1}{2}(\mathcal{E}^{ij} + \mathcal{E}^{ji}) = \mathcal{A}^{(ji)}, \qquad i, j = 1, 2, 3 \tag{8}$$

form a basis for the 6-dimensional space of symmetric matrices. Moreover,

$$[\mathcal{A}^{(ij)}, \mathcal{A}^{(k\ell)}] = \frac{1}{2}\big(\delta_{jk}\mathcal{B}^{(i\ell)} + \delta_{j\ell}\mathcal{B}^{(ik)} + \delta_{ik}\mathcal{B}^{(j\ell)} + \delta_{i\ell}\mathcal{B}^{(jk)}\big),$$

where

$$\mathcal{B}^{(ij)} = \frac{1}{2}(\mathcal{E}^{ij} - \mathcal{E}^{ji}) = -\mathcal{B}^{(ji)}, \qquad i, j = 1, 2, 3.$$

Here $\mathcal{B}^{(ii)} = 0$ and $\mathcal{B}^{(12)}$, $\mathcal{B}^{(23)}$, $\mathcal{B}^{(31)}$ form a basis for the space of skew-symmetric matrices. To obtain the commutation relations for the second order symmetries we need to use relations (7) to compute the parameter $\mathcal{F}(\boldsymbol{x}_0)$ associated with each commutator $[\mathcal{A}^{(ij)}, \mathcal{A}^{(k\ell)}]$. The results are straightforward to compute, using relations (4).

Commutator	$3\mathcal{F}/\lambda$
$[\mathcal{A}^{(12)}, \mathcal{A}^{(11)}] = \mathcal{B}^{(21)}$	$-3A^{13} - B^{23} - G_3$
$[\mathcal{A}^{(13)}, \mathcal{A}^{(11)}] = \mathcal{B}^{(31)}$	$A^{12} - B^{33} + G_2$
$[\mathcal{A}^{(22)}, \mathcal{A}^{(11)}] = 0$	$-4A^{23}$
$[\mathcal{A}^{(23)}, \mathcal{A}^{(11)}] = 0$	$2(A^{22} - A^{33})$
$[\mathcal{A}^{(33)}, \mathcal{A}^{(11)}] = 0$	$4A^{23}$
$[\mathcal{A}^{(13)}, \mathcal{A}^{(12)}] = \frac{1}{2}\mathcal{B}^{(32)}$	$\frac{1}{2}(3B^{12} - A^{22} + 3A^{33} - G_1)$
$[\mathcal{A}^{(22)}, \mathcal{A}^{(12)}] = \mathcal{B}^{(21)}$	$-3B^{23} - A^{13} - G_3$
$[\mathcal{A}^{(23)}, \mathcal{A}^{(12)}] = \frac{1}{2}\mathcal{B}^{(31)}$	$\frac{1}{2}(-3B^{33} - 3A^{12} + 2B^{22} + G_2)$
$[\mathcal{A}^{(33)}, \mathcal{A}^{(12)}] = 0$	$2(B^{23} - A^{13})$
$[\mathcal{A}^{(22)}, \mathcal{A}^{(13)}] = 0$	$-2B^{33}$
$[\mathcal{A}^{(23)}, \mathcal{A}^{(13)}] = \frac{1}{2}\mathcal{B}^{(21)}$	$-C^{33} + \frac{1}{2}B^{23} - \frac{1}{2}A^{13} - \frac{1}{2}G_3$
$[\mathcal{A}^{(33)}, \mathcal{A}^{(13)}] = \mathcal{B}^{(31)}$	$A^{12} + B^{33} + G_2$
$[\mathcal{A}^{(23)}, \mathcal{A}^{(22)}] = \mathcal{B}^{(32)}$	$A^{33} - A^{22} - B^{12} - G_1$
$[\mathcal{A}^{(33)}, \mathcal{A}^{(22)}] = 0$	$-4A^{23}$
$[\mathcal{A}^{(33)}, \mathcal{A}^{(23)}] = \mathcal{B}^{(32)}$	$A^{22} - A^{33} - B^{12} - G_1$

A consequence of these results is [20]

Corollary 2. *Let V be a superintegrable nondegenerate potential on a conformally flat space, not a Stäckel transform of the isotropic oscillator. Then the space of truly third order constants of the motion is 4-dimensional and is spanned by Poisson brackets of the second order constants of the motion.*

Corollary 3. *We can define a standard set of 6 second order basis symmetries*

$$\mathcal{S}^{(jk)} = \sum a^{hs}_{(jk)}(\boldsymbol{x})p_h p_s + W^{(jk)}(\boldsymbol{x})$$

corresponding to a regular point $\boldsymbol{x}_0$ by $(a_{(jk)})(\boldsymbol{x}_0) = A^{(jk)}$, $W^{(jk)}(\boldsymbol{x}_0) = 0$.

5 Maximum dimensions of the spaces of polynomial constants

In order to demonstrate the existence and structure of quadratic algebras for 3D nondegenerate superintegrable systems on conformally flat spaces, it is important to compute the dimensions of the spaces of symmetries of these systems that are of orders 4 and 6. These symmetries are necessarily of a special type. The highest order terms in the momenta are independent of the parameters in the potential, while the terms of order 2 less in the momenta are linear in these parameters, those of order 4 less are quadratic, and those of order 6 less are cubic. We will obtain these dimensions exactly, but first we need to establish sharp upper bounds.

The following results are obtained by a careful study of the defining conditions and the integrability conditions for higher order constants of the motion [20]:

Theorem 3. *The maximum possible dimension of the space of purely fourth order symmetries for a nondegenerate 3D potential is 21. The maximal possible dimension of the space of truly sixth order symmetries is 56.*

6 Bases for the fourth and sixth order constants of the motion

It follows from Section 5 that, for a superintegrable system with nondegenerate potential, the dimension of the space of truly fourth order constants of the motion is at most 21. Note from Section 4 that at any regular point $\boldsymbol{x}_0$, we can define a standard basis of 6 second order constants of the motion $\mathcal{S}^{(ij)} = A^{(ij)} + W^{(ij)}$ where the quadratic form $A^{(ij)}$ has matrix $\mathcal{A}^{(ij)}$ defined by (8) and $W^{(ij)}$ is the potential term with $W^{(ij)}(\boldsymbol{x}_0) \equiv 0$ identically in the parameters $W^{(\alpha)}$. By taking homogeneous polynomials of order two in the standard basis symmetries we can construct fourth order symmetries.

Theorem 4. *The 21 distinct standard monomials $\mathcal{S}^{(ij)}\mathcal{S}^{(jk)}$, defined with respect to a regular point $\boldsymbol{x}_0$, form a basis for the space of fourth order symmetries.*

Indeed, we can choose the basis symmetries in the form

1. $\left(\mathcal{S}^{(ii)}\right)^2$, $\mathcal{S}^{(ii)}\mathcal{S}^{(ij)}$, $\mathcal{S}^{(ii)}\mathcal{S}^{(jj)}$, $\mathcal{S}^{(ii)}\mathcal{S}^{(jk)}$

for $i, j, k = 1, \dots, 3$, i, j, k pairwise distinct (15 possibilities).

2. $\mathcal{S}^{(ii)}\mathcal{S}^{(jj)} - \left(\mathcal{S}^{(ij)}\right)^2$

for $i, j = 1, \dots, 3$, i, j pairwise distinct (3 possibilities).

3. $\mathcal{S}^{(ij)}\mathcal{S}^{(ik)} - \mathcal{S}^{(ii)}\mathcal{S}^{(jk)}$

for $i, j, k = 1, \dots, 3$, i, j, k pairwise distinct (3 possibilities).

It is a straightforward computation to show that these 21 symmetries are linearly independent. Since the maximum possible dimension of the space of fourth order symmetries is 21, they must form a basis. See [20] for the details of the proof.

Now from Section 5 the dimension of the space of purely sixth order constants of the motion is at most 56. Again we can show that the 56 independent homogeneous third order polynomials in the symmetries $\mathcal{S}^{(ij)}$ form a basis for this space.

At the sixth order level we have the symmetries

1. $\big(\mathcal{S}^{(ii)}\big)^3,\quad \big(\mathcal{S}^{(ii)}\big)^2\mathcal{S}^{(ij)},\quad \big(\mathcal{S}^{(ii)}\big)^2\mathcal{S}^{(jj)},\quad \big(\mathcal{S}^{(ii)}\big)^2\mathcal{S}^{(jk)}$

for $i,j,k=1,\dots,3$, i, j, k pairwise distinct (18 possibilities).

2. $\mathcal{S}^{(ii)}\mathcal{S}^{(ij)}\mathcal{S}^{(jj)},\quad \mathcal{S}^{(ii)}\mathcal{S}^{(ij)}\mathcal{S}^{(jk)},\quad \mathcal{S}^{(ii)}\mathcal{S}^{(jj)}\mathcal{S}^{(kk)},$

for $i,j,k=1,\dots,3$, i, j, k pairwise distinct (10 possibilities).

3. $\mathcal{S}^{(\ell m)}\big(\mathcal{S}^{(ii)}\mathcal{S}^{(jj)}-\big(\mathcal{S}^{(ij)}\big)^2\big)$

for $i,j=1,\dots,3$, i, j pairwise distinct (10 possibilities).

4. $\mathcal{S}^{(\ell m)}\big(\mathcal{S}^{(ij)}\mathcal{S}^{(ik)}-\mathcal{S}^{(ii)}\mathcal{S}^{(jk)}\big)$

for $i,j,k=1,\dots,3$, i, j, k pairwise distinct (18 possibilities).

Theorem 5. *The* 56 *distinct standard monomials $\mathcal{S}^{(hi)}\mathcal{S}^{(jk)}\mathcal{S}^{(\ell m)}$, defined with respect to a regular $\boldsymbol{x}_0$, form a basis for the space of sixth order symmetries.*

See [20] for the details of the proof. We conclude that the quadratic algebra closes.

7 Second order conformal Killing tensors

There is a close relationship between the second-order Killing tensors of a conformally flat space in 3D and the second order conformal Killing tensors of flat space. A second order conformal Killing tensor for a space $\mathcal{V}$ with metric $ds^2=\lambda(\boldsymbol{x})\big(dx_1^2+dx_2^2+dx_3^2\big)$ and free Hamiltonian $\mathcal{H}=\big(p_1^2+p_2^2+p_3^2\big)/\lambda$ is a quadratic form $\mathcal{S}=\sum a^{ij}(x_1,x_2,x_3)p_ip_j$ such that

$$\{\mathcal{H},\mathcal{S}\}=f(x_1,x_2,x_3)\mathcal{H},$$

form some function f. Since f is arbitrary, it is easy to see that $\mathcal{S}$ is a conformal Killing tensor for $\mathcal{V}$ if and only if it is a conformal Killing tensor for flat space $dx_1^2+dx_2^2+dx_3^2$. The conformal Killing tensors for flat space are very well known, e.g., [16]. The space of conformal Killing tensors is infinite dimensional. It is spanned by products of the conformal Killing vectors

$$\begin{gathered}p_1,\quad p_2,\quad p_3,\quad x_3p_2-x_2p_3,\quad x_1p_3-x_3p_1,\quad x_2p_1-x_1p_2,\quad x_1p_1+x_2p_2+x_3p_3,\\ \big(x_1^2-x_2^2-x_3^2\big)p_1+2x_1x_3p_3+2x_1x_2p_2,\quad \big(x_2^2-x_1^2-x_3^2\big)p_2+2x_2x_3p_3+2x_2x_1p_1,\\ \big(x_3^2-x_1^2-x_2^2\big)p_3+2x_3x_1p_1+2x_3x_2p_2,\end{gathered}$$

and terms $g(x_1,x_2,x_3)\big(p_1^2+p_2^2+p_3^2\big)$ where g is an arbitrary function. Since every Killing tensor is also a conformal Killing tensor, we see that every second-order Killing tensor for V_3 can be expressed as a linear combination of these second-order generating elements though, of course, the space of Killing tensors is only finite dimensional. This shows in particular that every a^{ij} and every $a^{ii}-a^{jj}$ with $i\neq j$ is a polynomial of order at most 4 in x_1, x_2, x_3, no matter what is the choice of λ.

It is useful to pass to new variables

$$a^{11},\quad a^{24},\quad a^{34},\quad a^{12},\quad a^{13},\quad a^{23}$$

for the Killing tensor, where $a^{24}=a^{22}-a^{11}$, $a^{34}=a^{33}-a^{11}$. Then we can establish the result

Theorem 6. *Necessary and sufficient conditions that the quadratic form* $\mathcal{S} = \sum_{ij} a^{ij} p_i p_j$ *be a second order Killing tensor for the space with metric* $ds^2 = \lambda\big(dx_1^2 + dx_2^2 + dx_3^2\big)$ *are*

1. $\mathcal{S}$ *is a conformal Killing tensor on the flat space with metric* $dx_1^2 + dx_2^2 + dx_3^2$.
2. *The following integrability conditions hold:*

$$\begin{aligned}
&\big(\lambda_2 a^{12} + \lambda_3 a^{13}\big)_2 = \big(\lambda_1 a^{12} + \big(a^{24}\lambda\big)_2 + \lambda_3 a^{23}\big)_1,\\
&\big(\lambda_2 a^{12} + \lambda_3 a^{13}\big)_3 = \big(\lambda_1 a^{13} + \lambda_2 a^{23} + \big(a^{34}\lambda\big)_3\big)_1,\\
&\big(\lambda_1 a^{12} + \big(a^{24}\lambda\big)_2 + \lambda_3 a^{23}\big)_3 = \big(\lambda_1 a^{13} + \lambda_2 a^{23} + \big(a^{34}\lambda\big)_3\big)_2.
\end{aligned}$$

8 The Stäckel transform for three-dimensional systems

The Stäckel transform [37] or coupling constant metamorphosis [38] plays a fundamental role in relating superintegrable systems on different manifolds. Suppose we have a superintegrable system

$$H = \frac{p_1^2 + p_2^2 + p_3^2}{\lambda(x,y,z)} + V(x,y,z),$$

in local orthogonal coordinates, with nondegenerate potential $V(x,y,z)$:

$$\begin{aligned}
V_{33} &= V_{11} + A^{33}V_1 + B^{33}V_2 + C^{33}V_3,\\
V_{22} &= V_{11} + A^{22}V_1 + B^{22}V_2 + C^{22}V_3,\\
V_{23} &= A^{23}V_1 + B^{23}V_2 + C^{23}V_3,\\
V_{13} &= A^{13}V_1 + B^{13}V_2 + C^{13}V_3,\\
V_{12} &= A^{12}V_1 + B^{12}V_2 + C^{12}V_3
\end{aligned} \tag{9}$$

and suppose $U(x,y,z)$ is a particular solution of equations (9), nonzero in an open set. Then the transformed system

$$\tilde{H} = \frac{p_1^2 + p_2^2 + p_3^2}{\tilde{\lambda}(x,y,z)} + \tilde{V}(x,y,z)$$

with nondegenerate potential $\tilde{V}(x,y,z)$:

$$\begin{aligned}
\tilde{V}_{33} &= \tilde{V}_{11} + \tilde{A}^{33}\tilde{V}_1 + \tilde{B}^{33}\tilde{V}_2 + \tilde{C}^{33}\tilde{V}_3,\\
\tilde{V}_{22} &= \tilde{V}_{11} + \tilde{A}^{22}\tilde{V}_1 + \tilde{B}^{22}\tilde{V}_2 + \tilde{C}^{22}\tilde{V}_3,\\
\tilde{V}_{23} &= \tilde{A}^{23}\tilde{V}_1 + \tilde{B}^{23}\tilde{V}_2 + \tilde{C}^{23}\tilde{V}_3,\\
\tilde{V}_{13} &= \tilde{A}^{13}\tilde{V}_1 + \tilde{B}^{13}\tilde{V}_2 + \tilde{C}^{13}\tilde{V}_3,\\
\tilde{V}_{12} &= \tilde{A}^{12}\tilde{V}_1 + \tilde{B}^{12}\tilde{V}_2 + \tilde{C}^{12}\tilde{V}_3,
\end{aligned}$$

is also superintegrable, where

$$\begin{aligned}
&\tilde{\lambda} = \lambda U, \qquad \tilde{V} = \frac{V}{U},\\
&\tilde{A}^{33} = A^{33} + 2\frac{U_1}{U}, \qquad \tilde{B}^{33} = B^{33}, \qquad \tilde{C}^{33} = C^{33} - 2\frac{U_3}{U},\\
&\tilde{A}^{22} = A^{22} + 2\frac{U_1}{U}, \qquad \tilde{B}^{22} = B^{22} - 2\frac{U_2}{U}, \qquad \tilde{C}^{22} = C^{22},
\end{aligned}$$

$$\tilde{A}^{23} = A^{23}, \qquad \tilde{B}^{23} = B^{23} - \frac{U_3}{U}, \qquad \tilde{C}^{23} = C^{23} - \frac{U_2}{U},$$
$$\tilde{A}^{13} = A^{13} - \frac{U_3}{U}, \qquad \tilde{B}^{13} = B^{13}, \qquad \tilde{C}^{13} = C^{13} - \frac{U_1}{U},$$
$$\tilde{A}^{12} = A^{12} - \frac{U_2}{U}, \qquad \tilde{B}^{12} = B^{12} - \frac{U_1}{U}, \qquad \tilde{C}^{12} = C^{12}.$$

Let $S = \sum a^{ij}p_ip_j + W = S_0 + W$ be a second order symmetry of H and $S_U = \sum a^{ij}p_ip_j + W_U = S_0 + W_U$ be the special case that is in involution with $\left(p_1^2 + p_2^2 + p_3^2\right)/\lambda + U$. Then

$$\tilde{S} = S_0 - \frac{W_U}{U}H + \frac{1}{U}H$$

is the corresponding symmetry of $\tilde{H}$. Since one can always add a constant to a nondegenerate potential, it follows that $1/U$ defines an inverse Stäckel transform of $\tilde{H}$ to H. See [37] for many examples of this transform.

9 Multiseparability and Stäckel equivalence

From the general theory of variable separation for Hamilton–Jacobi equations [16, 17] we know that second order symmetries $\mathcal{S}_1$, $\mathcal{S}_2$ define a separable system for the equation

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{\lambda(x, y, z)} + V(x, y, z) = E$$

if and only if

1. The symmetries $\mathcal{H}$, $\mathcal{S}_1$, $\mathcal{S}_2$ form a linearly independent set as quadratic forms.
2. $\{\mathcal{S}_1, \mathcal{S}_2\} = 0$.
3. The three quadratic forms have a common eigenbasis of differential forms.

This last requirement means that, expressed in coordinates x, y, z, at least one of the matrices $\mathcal{A}_{(j)}(\boldsymbol{x})$ can be diagonalized by conjugacy transforms in a neighborhood of a regular point and that $[\mathcal{A}_{(2)}(\boldsymbol{x}), \mathcal{A}_{(1)}(\boldsymbol{x})] = 0$. However, for nondegenerate superintegrable potentials in a conformally flat space we see from Section 5 that

$$\{\mathcal{S}_1, \mathcal{S}_2\} = 0 \iff [\mathcal{A}_{(2)}(\boldsymbol{x}_0), \mathcal{A}_{(1)}(\boldsymbol{x}_0)] = 0, \qquad \text{and} \qquad \mathcal{F}(\boldsymbol{x}_0) = 0$$

so that the intrinsic conditions for the existence of a separable coordinate system are simplified.

Theorem 7. *Let V be a superintegrable nondegenerate potential in a* 3D *conformally flat space. Then V defines a multiseparable system.*

See [21] for the details of the proof.

In [39] the following result was obtained.

Theorem 8. *Let u_1, u_2, u_3 be an orthogonal separable coordinate system for a* 3D *conformally flat space with metric $d\tilde{s}^2$ Then there is a function f such that*

$$f d\tilde{s}^2 = ds^2$$

where ds^2 is a constant curvature space metric and ds^2 is orthogonally separable in exactly these same coordinates u_1, u_2, u_3. The function f is called a Stäckel multiplier with respect to this coordinate system.

Thus the possible separable coordinate systems for a conformally flat space are all obtained, modulo a Stäckel multiplier, from separable systems on 3D flat space or on the 3-sphere. This result provides evidence that, as in the 2D case, all nondegenerate 3D superintegrable systems on conformally flat spaces are Stäckel equivalent to a superintegrable system on either 3D flat space or the 3-sphere, but we have not yet settled this issue.

10 Discussion and conclusions

We have shown that all classical superintegrable systems with nondegenerate potential on real or complex 3D conformally flat spaces admit 6 linearly independent second order constants of the motion (even though only 5 functionally independent second order constants are assumed) and that the spaces of fourth order and sixth order symmetries are spanned by polynomials in the second order symmetries. (An interesting issue here is the form of the functional dependence relation between the 6 linearly independent symmetries. It appears that the relation is always of order 8 in the momenta, but we have as yet no general proof.) This implies that a quadratic algebra structure always exists for such systems. We worked out their common structure and related it to algebras of 3×3 symmetric matrices. We demonstrated that such systems are always multiseparable, more precisely they permit separation of variables in at least three orthogonal coordinate systems.

We also studied the Stäckel transform, a conformal invertible mapping from a superintegrable system on one space to a system on another space. Using prior results from the theory of separation of variables on conformally flat spaces, we gave evidence that every nondegenerate superintegrable system on such a space is Stäckel equivalent to a superintegrable system on complex Euclidean space or on the complex 3-sphere, though we haven't yet settled the issue. This suggests that to classify all such superintegrable systems we can restrict attention to these two constant curvature spaces, and then obtain all other cases via Stäckel transforms. We are making considerable progress on the classification theory [21], though the problem is complicated.

All of our 2D and 3D classical results can be extended to quantum systems and the Schrödinger equation and we are in the process of writing these up.

Another interesting set of issues comes from the consideration of 3D superintegrable systems with degenerate, but multiparameter, potentials. In some cases such as the extended Kepler–Coulomb potential there is no quadratic algebra, whereas in other cases the quadratic algebra exists. Understanding the underlying structure of these systems is a major challenge. Finally there is the challenge of generalizing the 2D and 3D results to higher dimensions.

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