A System of $n = 3$ Coupled Oscillators with Magnetic Terms: Symmetries and Integrals of Motion

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Abstract. The properties of a system of $n = 3$ coupled oscillators with linear terms in the velocities (magnetic terms) depending in two parameters are studied. We proved the existence of a bi-Hamiltonian structure arising from a non-symplectic symmetry, as well the existence of master symmetries and additional integrals of motion (weak superintegrability) for certain particular values of the parameters.

Key words: non-symplectic symmetries; bi-Hamiltonian structures; master symmetries; cubic integrals

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1 Introduction

The main objective of this paper is to present a study of a system that, although arising from a quadratic Hamiltonian of simple form, it is however endowed with several interesting properties related with the theory of nonstandard symmetries and the existence of cubic integrals of motion.

Let us consider the Lagrangian of a system of $n = 3$ coupled oscillators with an additional coupling term linear in the velocities (the so-called magnetic terms)

$$
\begin{aligned}
L &= \frac{1}{2}\left(v_1^2 + v_2^2 + v_3^2\right) - (V_1 + V_0),\\
V_0 &= \frac{1}{2}k\left(q_{21}^2 + q_{32}^2 + q_{13}^2\right),\\
V_1 &= b(q_{32}v_1 + q_{13}v_2 + q_{21}v_3),
\end{aligned} \tag{1}
$$

where b and k are arbitrary constants, q_{ij} denote $q_i - q_j$, and we assume that, for easiness of notation, the mass m of the particle is set equal to unity. See Refs. [6] and [14] for two recent papers dealing with similar systems.

The starting point is to consider $L_0 = T - V_0$, where T denotes the kinetic energy, as the basic original system, and the additional term V_1 (linear in the velocities) as introducing a deformation. We try to formulate the results in explicit dependence of b in order to study the changes of the dynamics when the parameter b varies.

The Lagrangian L has two exact Noether symmetries generated by the vector fields

$$
\begin{aligned}
Y_2 &= \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} + \frac{\partial}{\partial q_3},\\
Y_3 &= q_{32}\frac{\partial}{\partial q_1} + q_{13}\frac{\partial}{\partial q_2} + q_{21}\frac{\partial}{\partial q_3},
\end{aligned}
$$

with associated constants of the motion

$$I_2 = v_1 + v_2 + v_3,$$
$$I_3 = q_{32}v_1 + q_{13}v_2 + q_{21}v_3 - b\big(q_{21}^2 + q_{32}^2 + q_{13}^2\big).$$

The first constant I_1 is the total energy given by

$$I_1 = \Delta(L) - L = \frac{1}{2}\big(v_1^2 + v_2^2 + v_3^2\big) + \frac{1}{2}k\big(q_{21}^2 + q_{32}^2 + q_{13}^2\big).$$

where $\Delta = v_i(\partial/\partial v_i)$ represents the Liouville vector field.

The Legendre transformation

$$p_1 = v_1 - bq_{32}, \qquad p_2 = v_2 - bq_{13}, \qquad p_3 = v_3 - bq_{21}$$

leads to the following Hamiltonian

$$\begin{aligned} H &= \frac{1}{2}\big[(p_1 + bq_{32})^2 + (p_2 + bq_{13})^2 + (p_3 + bq_{21})^2\big] + \frac{1}{2}k\big(q_{21}^2 + q_{32}^2 + q_{13}^2\big) \\ &= \frac{1}{2}\big(p_1^2 + p_2^2 + p_3^2\big) + b(L_1 + L_2 + L_3) + \frac{1}{2}(b^2 + k)\big(q_{21}^2 + q_{32}^2 + q_{13}^2\big) \end{aligned} \tag{2}$$

and the above three constants become

$$J_1 = H, \qquad J_2 = p_1 + p_2 + p_3, \qquad J_3 = L_1 + L_2 + L_3,$$

where L_i, $i = 1, 2, 3$, denote the components of the angular momentum $L_i = q_j p_k - q_k p_j$. This three integrals satisfy $\{J_r, J_s\} = 0$, $r, s = 1, 2, 3$, and make of H an integrable system for all the values of b and k. Thus, we arrive to the following result

Proposition 1. *If we consider the magnetic term V_1 as a continuous deformation of the original system $L_0 = T - V_0$ (the value of b representing the intensity of the deformation) then this deformation preserves both the existence of Noether symmetries and the integrability of the original undeformed system.*

Notice that H can also be written as $H = J_0 + bJ_3$ with J_0 defined as follows

$$J_0 = \frac{1}{2}\big(p_1^2 + p_2^2 + p_3^2\big) + \frac{1}{2}(b^2 + k)\big(q_{21}^2 + q_{32}^2 + q_{13}^2\big).$$

2 Symmetries and integrals of motion

In differential geometric terms, the dynamics of a time-independent Hamiltonian system is determined by a vector field on the $2n$–dimensional cotangent bundle T^*Q of a n-dimensional manifold Q. Cotangent bundles are manifolds endowed, in a natural or canonical way, with a symplectic structure ω_0 that, in coordinates $\{(q_j, p_j);\ j = 1, 2, \ldots, n\}$, is given by

$$\omega_0 = dq_j \wedge dp_j, \qquad \omega_0 = -d\theta_0, \qquad \theta_0 = p_j dq_j$$

(we write all the indices as subscripts and we use the summation convention on the repeated index). Given a differentiable function $F = F(q, p)$, the vector field X_F defined as the solution of the equation

$$i(X_F)\omega_0 = dF$$

is called the Hamiltonian vector field of the function F. There are two important properties:

(i) The Hamiltonian vector field of a given function is well defined without ambiguities. This uniqueness is a consequence of the symplectic character of the two–form ω_0.

(ii) Suppose that we are given a Hamiltonian $H = H(q, p)$. Then the dynamics is given by the Hamiltonian vector field Γ_H of the Hamiltonian function. That is, $i(\Gamma_H)\omega_0 = dH$.

2.1 Bi-Hamiltonian structure

At this point we recall that a (infinitesimal) dynamical symmetry of a Hamiltonian system (T^*Q, ω_0, H) is a vector field Y such that it satisfies $[Y, \Gamma_H] = 0$. On the other hand it is known that, in some very particular cases, the Hamiltonian systems can admit dynamical but non-symplectic symmetries (for a classification of the symmetries in geometric terms see [4] and [15]). In this case we have the following property.

Proposition 2. *Suppose there is a vector field Y that is a dynamical symmetry of Γ_H but does not preserve the symplectic two-form*

$$\mathcal{L}_Y \omega_0 = \omega_Y \neq 0.$$

Then (i) the dynamical vector field Γ_H is bi-Hamiltonian, and (ii) the function $Y(H)$ is the new Hamiltonian, and therefore it is a constant of motion.

Proof. For a proof of this proposition see [1, 2, 18], and references therein. A similar property is studied in [20, 22] for the case of Poisson manifolds. ■

Let us denote by Y_3 the Hamiltonian vector field of the function J_3

$$i(Y_3)\omega_0 = dJ_3. \tag{3}$$

Then we have the following property:

Proposition 3. *The vector field Y_3, defined by* (3) *as the canonical infinitesimal symmetry associated with J_3, can be written as a linear combination of two dynamical but non-symplectic symmetries of Γ_H.*

Proof. We can write Y_3 as follows

$$Y_3 = Y_3^a - Y_3^b,$$
$$Y_3^a = q_3 \frac{\partial}{\partial q_1} + q_1 \frac{\partial}{\partial q_2} + q_2 \frac{\partial}{\partial q_3} + p_3 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial p_3},$$
$$Y_3^b = q_2 \frac{\partial}{\partial q_1} + q_3 \frac{\partial}{\partial q_2} + q_1 \frac{\partial}{\partial q_3} + p_2 \frac{\partial}{\partial p_1} + p_3 \frac{\partial}{\partial p_2} + p_1 \frac{\partial}{\partial p_3}.$$

Then the vector field Y_3^a (or Y_3^b) is neither locally-Hamiltonian with respect to ω_0

$$\mathcal{L}_{Y_3^a} \omega_0 = \omega_a \neq 0,$$

nor a symmetry of the Hamiltonian function, $Y_3^a(H) \neq 0$. Nevertheless it satisfies

$$[Y_3^a, \Gamma_H] = 0.$$

Therefore Y_3^a (or Y_3^b) is a dynamical but non-symplectic symmetry of the Hamiltonian system. ■

Thus the dynamical vector field Γ_H is a bi-Hamiltonian system with respect to (ω_0, ω_a)

$$i(\Gamma_H)\omega_0 = dH, \qquad i(\Gamma_H)\omega_a = dH_a, \tag{4}$$

with ω_a and $H_a = Y_a(H)$ given by

$$\omega_a = dq_1 \wedge (dp_2 + dp_3) + dq_2 \wedge (dp_1 + dp_3) + dq_3 \wedge (dp_1 + dp_3),$$
$$H_a = \frac{1}{2} J_2^2 - H.$$

The function $H_a = Y_a(H)$ can be considered as a new Hamiltonian for Γ_H.

The bi-Hamiltonian structure $(\Gamma_H, \omega_0, \omega_a)$ determines a $(1,1)$-tensor field R by the relation

$$\omega_a(X,Y) = \omega_0(RX,Y), \qquad \forall\, X,Y \in \mathcal{X}(T^*Q),$$

or, equivalently, $R = \widehat{\omega}_0^{-1} \circ \widehat{\omega}_a$ [with $\widehat{\omega}(X) = i(X)\omega \in \wedge^1(T^*Q)$]. We have obtained

$$\begin{aligned} R = {} & \frac{\partial}{\partial q_1} \otimes (dq_2 + dq_3) + \frac{\partial}{\partial q_2} \otimes (dq_1 + dq_3) + \frac{\partial}{\partial q_3} \otimes (dq_1 + dq_2) \\ & + \frac{\partial}{\partial p_1} \otimes (dp_2 + dp_3) + \frac{\partial}{\partial p_2} \otimes (dp_1 + dp_3) + \frac{\partial}{\partial p_3} \otimes (dp_1 + dp_2). \end{aligned} \tag{5}$$

Starting with the basic Hamiltonian system $(\omega_0, \Gamma_0 = \Gamma_H, dH_0 = dH)$, and iterating R, we can construct a sequence of 2-forms ω_k, vector fields Γ_k, and exact 1-forms dH_k, $k = 1,2,\dots$, defined by $\widehat{\omega}_k = \widehat{\omega}_0 \circ R^k$, $\Gamma_k = R^k(\Gamma_0)$, and $dH_k = R^k(dH_0)$, that leads to sequence of bi-Hamiltonian vector fields. The two first steps become

$$i(\Gamma_0)\omega_1 = i(\Gamma_1)\omega_0 = dH_1, \qquad i(\Gamma_0)\omega_2 = i(\Gamma_1)\omega_1 = i(\Gamma_2)\omega_0 = dH_2,$$

where ω_2, Γ_2, and H_2, are given by

$$\omega_2 = 2\omega_0 + \omega_1, \qquad \Gamma_2 = 2\Gamma_0 + \Gamma_1, \qquad H_2 = 2H_0 + H_1.$$

So, in this particular case, the sequence $(\omega_k, \Gamma_k, dH_k)$ closes over itself after two steps.

2.2 Master symmetries

The function $T = T(q,p)$ is said to be a generator of constants of motion of degree m if it is not preserved by the dynamics but it generates an integral of motion by time derivation

$$\frac{d}{dt}T \neq 0, \quad \dots, \quad \frac{d^{m-1}}{dt^{m-1}}T \neq 0, \quad \frac{d^m}{dt^m}T = 0.$$

In differential geometric terms, a vector field X on T^*Q that satisfies

$$[\Gamma_H, X] \neq 0, \qquad [\Gamma_H, \widetilde{X}] = 0, \qquad \widetilde{X} = [\Gamma_H, X],$$

is called a "master symmetry" or a "generator of symmetries" of degree $m = 1$ for Γ_H [3, 5, 7, 17]. If a Hamiltonian H admits two of such generators, T_1, T_2, (master integrals) then the function I_{12} defined by

$$I_{12} = I_2 T_1 - I_1 T_2, \qquad I_r = \frac{d}{dt}T_r, \qquad r = 1,2,$$

is a new constant of motion (independent of I_1 and I_2). In geometric terms, if we denote by X_r, the Hamiltonian vector fields of T_r, then $\widetilde{X}_r$ is Hamiltonian as well, and we have

$$i(X_r)\omega_0 = dT_r, \qquad i(\widetilde{X}_r)\omega_0 = d\widetilde{T}_r, \qquad \widetilde{T}_r = \Gamma_H(T_r), \qquad r = 1,2,$$

so that the vector field X_{12} defined as

$$X_{12} = T_1\widetilde{X}_2 - T_2\widetilde{X}_1 + \widetilde{T}_2 X_1 - \widetilde{T}_1 X_2, \qquad i(X_{12})\omega_0 = dI_{12},$$

is a symmetry of the system (T^*Q, ω_0, H) with the function I_{12} as its associated constant of motion.

The Hamiltonian vector field X_1 of the function $T_1 = q_1 + q_2 + q_3$ satisfies

$$i(X_1)\omega_0 = dT_1, \qquad [\Gamma_H, X_1] = \widetilde{X}_1, \qquad \widetilde{X}_1 = \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} + \frac{\partial}{\partial q_3}, \qquad [\Gamma_H, \widetilde{X}_1] = 0,$$

so X_1 is a "generator of symmetries" of degree $m = 1$ for (2). Similarly if we consider $T_2 = q_1p_1 + q_2p_2 + q_3p_3$ we obtain

$$i(X_2)\omega_0 = dT_2, \qquad [\Gamma_H, X_2] = \widetilde{X}_2, \qquad [\Gamma_H, \widetilde{X}_2] \stackrel{*}{=} 0,$$

where the asterisk means that the Lie bracket vanishes only if $k+b^2 = 0$. Hence in the particular case $k + b^2 = 0$, the system is superintegrable with an additional integral of the form I_{12} that becomes

$$J_4 \equiv I_{12} = L_1(p_2 - p_3) + L_2(p_3 - p_1) + L_3(p_1 - p_2). \tag{6}$$

2.3 Third-order integrals

Linear integrals of motion be obtained from exact Noether symmetries and quadratic integrals from Hamilton–Jacobi (Schrödinger) separability but third-order integrals must be obtaining by other alternative procedures; because of this, the number of known integrable systems admitting cubic in the momenta constants is very limited [9, 10, 13, 21, 23]. For $n = 2$ we can point out the Fokas–Lagerstrom and the Holt potentials for the Euclidean plane [8, 12], and three of the Drach potentials for the pseudoeuclidean plane [16, 19]. For $n = 3$ two very special systems are well known: the three-particle Toda chain and three-particle Calogero–Moser system which are both endowed with a cubic constant of the form

$$J = \frac{1}{3}\left(p_1^3 + p_2^3 + p_3^3\right) + \text{terms of first order}. \tag{7}$$

At this point we recall the following property: polynomial constants of the motion for mechanical systems (Riemmanian metric with a potential) are either even or odd in the momenta (for a discussion of this result see, e.g., [11]). Therefore if J is a cubic integral for a Hamiltonian such as the Calogero–Moser or the Toda systems then it must also contain linear terms but neither quadratic nor independent ones. Nevertheless this property is not true for the more general case of a non-mechanical Lagrangian. In our case, due to the presence of linear terms in the velocities (magnetic terms) when looking for third-order integrals we must assume for the function J_4 a polynomial expression containing also even powers.

We start with a third-order function J of a general form

$$J = \sum_{i+j+k=3} a_{ijk}p_1^i p_2^j p_3^k + \sum_{i+j+k=2} b_{ijk}p_1^i p_2^j p_3^k + \sum_i c_i p_i + d,$$

where a_{ijk}, b_{ijk}, c_i, and d are functions of q_1, q_2, q_3. Then the condition $\{H, J\} = 0$ leads to a system of equations; some of these equations restrict the expressions of the coefficients a_{ijk}, $i + j + k = 3$, of the higher-order terms; the other equations couple the derivatives of V_1 and V_0 with the coefficients a_{ijk}, b_{ijk}, c_i, and d or with the derivatives of these coefficients. Unfortunately, the general case is difficult to be studied but some particular cases can be analyzed assuming simple particular forms for the coefficients a_{ijk}.

We have studied the existence of a cubic integral of such a particular form for this three-particle system and we have arrived (we omit the details) at the existence of the following function

$$\begin{aligned} J_4 = {} & \frac{1}{3}\left(p_1^3 + p_2^3 + p_3^3\right) - b\left(q_{32}p_1^2 + q_{13}p_2^2 + q_{21}p_3^2\right) \\ & + b^2(z_1p_1 + z_2p_2 + z_3p_3) + 9b^3\left(q_{21}^3 + q_{13}^3 + q_{32}^3\right), \end{aligned} \tag{8}$$

where z_1, z_2, and z_3, are given by

$$z_1 = 4\big(q_{21}^2 + q_{13}^2\big) + q_{32}^2, \qquad z_2 = 4\big(q_{32}^2 + q_{21}^2\big) + q_{13}^2, \qquad z_3 = 4\big(q_{13}^2 + q_{32}^2\big) + q_{21}^2.$$

The important point is that the function J_4 is a constant of motion, not for all the values of the parameters (b, k), but only in the particular case of k and b satisfying $k = 8b^2$. We have verified that in this case the four functions H, J_2, J_3, J_4 are independent and that the system (J_2, H, J_4) is involutive (nevertheless $\{J_3, J_4\} \neq 0$). Thus, when k and b are related by $k = 8b^2$ then the system becomes superintegrable with an additional third-order integral.

We finally note that J_2 is linear, H quadratic, and J_4 cubic, and that they have expressions of the form

$$J_2 = p_1 + p_2 + p_3, \qquad H = p_1^2 + p_2^2 + p_3^2 + \cdots, \qquad J_4 = p_1^3 + p_2^3 + p_3^3 + \cdots \tag{9}$$

that closely resemble the expressions of the three first constants of Toda or the Calogero–Moser systems.

3 Final comments

The Lagrangian system (1) (Hamiltonian (2)) is a system endowed with a bi-Hamiltonian structure, non-symplectic symmetries and master symmetries. Moreover for a particular value of the parameter b, it becomes superintegrable with a cubic integral.

The non-symplectic symmetry Y_3^a is b-independent and hence the bi-Hamiltonian structure $(\Gamma_H, \omega_0, \omega_a)$ is the one endowed by the the original oscillators system $L_0 = T - V_0$ that is preserved under the b-deformation introduced by the magnetic term V_1. Concerning the master symmetries, the first one is always present but the second only exists if $k + b^2 = 0$. So in this particular case the system becomes (weakly) superintegrable. This particular case is very peculiar because the Hamiltonian (2) becomes a system of $n = 3$ particles with velocity-linear couplings but without the potential term.

Finally the particular $k = 8b^2$ case is very interesting. First because, as stated above, the number of known integrable systems with third-order integrals is very small. Second because the existence of this property for a very particular case of the parameters resembles situations characterizing other known systems (e.g., the Hénon–Heiles system is only integrable for some very particular values of a parameter). Finally because the expression obtained for J_4 in (8) shows similarity with the third-order integral (7) of the Toda or the Calogero–Moser systems. This leads to the necessity of the study of the more general $n > 3$ case, that poses the question of the nature of the interaction (in the Toda chain the interactions are between neighbours but in the Calogero–Moser case they are between every two particles). On the other hand this also leads to the search of a Lax pair generating the three functions (J_2, H, J_4), as traces of the powers of an appropriate matrix. Nevertheless, notice that this Lax structure (if it exists) must correspond to the L system with the additional condition $k = 8b^2$; so it cannot be obtained as a continuous deformation of a previous Lax pair for the L_0 system. We think that these are open questions that must be investigated.

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