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ADMISSIBILITY AND EXPONENTIAL DICHOTOMY OF EVOLUTIONARY PROCESSES ON HALF-LINE

Abstract. In the present paper we give a new way to characterize the exponential dichotomy of evolutionary processes in terms of ”Perron-type” theorems, without the so-called evolution semigroup.

Also, there are obtained another proofs of some results gives by Van Minh, Räbiger and Schnaubelt.

1. Introduction

Exponential dichotomy have their origins in the work of O.Perron [13]. It has been studied for the case of differential equations by several authors, whose results can be found in the monographs due to Massera-Schäffer [9], Hartman [4], Daleckij-Krein [3], Coppel [2], Chicone-Latushkin [1].

The case of general evolutionary-processes has been studied in [15] by P.Preda for exponential stability and in [14] by P.Preda and M.Megan for exponential dichotomy.

Recently, several results about exponential stability and exponential dichotomy which extend the result of O.Perron were obtained by N. van Minh [11], [12], F. Rabiger [11], Y. Latushkin [6], [7], [8], T. Randolph, R. Schnaubelt [8], [18]. Arguments in these papers again illustrate the general philosophy of “autonomization” of nonautonomous problems by passing from evolution families to associated evolution semigroups. In contrast to this “philosophy” the present paper shows that we can characterize the exponential dichotomy in terms of the admissibility of some suitable pairs of spaces in a direct way, without the so-called evolution semigroup.

So the aim of this paper is to establish the connection between admissibility and exponential dichotomy in a new way, more directly, without using the evolution semigroup .

2. Preliminaries

Let X be a Banach space, $B(x)$ the Banach algebra of all bounded linear operators acting on X and $\mathbb{R}_+ = [0, +\infty)$.

The classical result of O.Perron stands that the differential system

$$(A) \qquad \dot{x}(t) = A(t)x(t), \ t \geq 0.$$

is exponential dichotomic if and only if for all continuous and bounded $f : \mathbb{R}_+ \to X$ there exists a bounded solution of the equation

$$(A, f) \qquad \dot{x}(t) = A(t)x(t) + f(t), \ t \geq 0,$$

where A is an operator valued function, locally Bochner integrable and X a finite dimensional space.

This result was extended to the case of infinite dimensional Banach spaces in a natural way.

The Cauchy problem associated to the equation (A, f) has a solution given by

$$x(t) = U(t, t_0)x(t_0) + \int_{t_0}^{t} U(t, \tau) f(\tau) d\tau.$$

$\mathcal{U}$ is the evolutionary process generated by the equation (A), $U(t, t_0) = \Phi(t)\Phi^{-1}(t_0)$, where Φ is the unique solution of the Cauchy Problem

$$\begin{cases} \Phi'(t) = A(t)\Phi(t) \\ \Phi(0) = I \end{cases}$$

The case where $\{A(t)\}_{t\geq 0}$ is a family of unbounded linear operators, impose another "kind" of solution for (A, f). So we have to deal with the so-called mild solution for (A, f) given by

$$x(t) = U(t, t_0)x_0 + \int_{t_0}^{t} U(t, \tau) f(\tau) d\tau.$$

DEFINITION 1. *A family of bounded linear operators on X, $\mathcal{U} = \{U(t, s)\}_{t\geq s\geq 0}$ is called an evolutionary process if*

1) $U(t, t) = I$ (the identity operator on X), for all $t \geq 0$;

2) $U(t, s)U(s, r) = U(t, r)$, for all $t \geq s \geq r \geq 0$;

3) $U(\cdot, s)x$ is continuous on $[s, \infty)$ for all $s \geq 0$, $x \in X$;
$U(t, \cdot)x$ is continuous on $[0, t]$ for all $t \geq 0$, $x \in X$;

4) there exist $M, \omega > 0$ such that

$$||U(t, s)|| \leq M e^{\omega(t-s)}, \text{ for all } t \geq s \geq 0.$$

We use the following notations:

$$C = \{f : \mathbb{R}_+ \to X : f - \text{continuous and bounded}\}$$

$$C_0 = \{f \in C : f(0) = \lim_{t\to\infty} f(t) = 0\}.$$

We note that C and C_0 are Banach spaces endowed with the norm

$$||f|| = \sup_{t\geq 0} ||f(t)||.$$

DEFINITION 2. *An application $P : \mathbb{R}_+ \to B(X)$ is said to be a dichotomy projection family if*

i) $P^2(t) = P(t)$, *for all* $t \geq 0$;

ii) $P(\cdot)x \in C$, *for all* $x \in X$.

We set $Q(t) = I - P(t),\ t \geq 0$.

DEFINITION 3. *An evolutionary process $\mathcal{U}$ is said to be uniformly exponentially dichotomic (u.e.d) if there exist P a dichotomy projection family and $N, \gamma > 0$ such that*

$d_1)$ $U(t,s)P(s) = P(t)U(t,s)$, *for all* $t \geq s \geq 0$;

$d_2)$ $U(t,s) : Ker\ P(s) \to Ker\ P(t)$ *is an isomorphism for all* $t \geq s \geq 0$;

$d_3)$ $||U(t,s)x|| \leq Ne^{-\gamma(t-s)}||x||$, *for all* $x \in Im\ P(s)$ *and all* $t \geq s \geq 0$.

$d_4)$ $||U(t,s)x|| \geq \dfrac{1}{N}e^{\gamma(t-s)}||x||$, *for all* $x \in Ker\ P(s)$ *and all* $t \geq s \geq 0$.

In what follows we will consider evolutionary processes $\mathcal{U}$ for which exists P a dichotomy projection family such that $d_1)$ and $d_2)$ are satisfied. In that case we will denote by

$$U_1(t,s) = U(t,s)_{|Im\ P(s)},\ U_2(t,s) = U(t,s)_{|Ker\ P(s)}.$$

Let E and F be two closed subspaces of C.

DEFINITION 4. *The pair (E, F) is said to be admissible for $\mathcal{U}$ if for all $f \in E$ the following statements hold*

i) $U_2^{-1}(\cdot,t)Q(\cdot)f(\cdot) \in L^1_{[t,\infty)}(X)$, *for all* $t \geq 0$;

ii) $x_f : \mathbb{R}_+ \to X,\ x_f(t) = \displaystyle\int_0^t U_1(t,s)P(s)f(s)ds - \int_t^\infty U_2^{-1}(s,t)Q(s)f(s)ds$, *lies in* F.

LEMMA 1. *With our assumption we have that $U_2^{-1}(\cdot,t_0)Q(\cdot)x$ is continuous on $[t_0,\infty)$, for all $(t_0,x) \in \mathbb{R}_+ \times X$.*

Proof. Let $t \geq t_0 \geq 0$, $h \in (0,1)$, $x \in X$. Then
$U_2(t+1,t_0) = U_2(t+1,r)U_2(r,t_0)$, for all $r \in [t,t+1]$, and so
$U_2^{-1}(t+h,t_0) = U_2^{-1}(t+1,t_0)U_2(t+1,t+h)$
$U_2^{-1}(t,t_0) = U_2^{-1}(t+1,t_0)U_2(t+1,t)$.

It results that

$$||U_2^{-1}(t+h,t_0)Q(t+h)x - U_2^{-1}(t,t_0)Q(t)x|| =$$

$$= ||U_2^{-1}(t+1,t_0)[U_2(t+1,t+h)Q(t+h)x - U_2(t+1,t)Q(t)x]||$$

$$\leq ||U_2^{-1}(t+1,t_0)||\ ||U_2(t+1,t+h)Q(t+h)x - U_2(t+1,t)Q(t)x||$$

$$= ||U_2^{-1}(t+1,t_0)||\ ||U(t+1,t+h)Q(t+h)x - U(t+1,t)Q(t)x||$$

$$\leq ||U_2^{-1}(t+1,t_0)||\ [||U(t+1,t+h)(Q(t+h)x - Q(t)x)||$$

$$+||U(t+1,t+h)Q(t)x - U(t+1,t)Q(t)x||]$$

$$\leq ||U_2^{-1}(t+1,t_0)||[Me^{\omega(1-h)}||Q(t+h)x - Q(t)x||$$

$$+||U(t+1,t+h)Q(t)x - U(t+1,t)Q(t)x||]$$

It's easy to see that $U_2^{-1}(\cdot,t_0)Q(\cdot)x$ is right-handed continuous on $[t_0,\infty)$.
Consider now $t > t_0 \geq 0,\ h \in (0, t-t_0),\ x \in X$. Then

$$U_2(t,t_0) = U_2(t,t-h)U_2(t-h,t_0)$$

and so

$$U_2^{-1}(t-h,t_0) = U_2^{-1}(t,t_0)U_2(t,t-h)$$

It results that

$$||U_2^{-1}(t-h,t_0)Q(t-h)x - U_2^{-1}(t,t_0)Q(t)x||$$

$$= ||U_2^{-1}(t,t_0)U_2(t,t-h)Q(t-h)x - U_2^{-1}(t,t_0)Q(t)x||$$

$$\leq ||U_2^{-1}(t,t_0)||\ ||U_2(t,t-h)Q(t-h)x - Q(t)x||$$

$$= ||U_2^{-1}(t,t_0)||\ ||U(t,t-h)Q(t-h)x - Q(t)x||$$

$$\leq ||U_2^{-1}(t,t_0)||\ ||U(t,t-h)(Q(t-h)x - Q(t)x)||$$

$$+||U(t,t-h)Q(t)x - Q(t)x||]$$

$$\leq ||U_2^{-1}(t,t_0)||[Me^{\omega h}||Q(t-h)x - Q(t)x||$$

$$+||U(t,t-h)Q(t)x - Q(t)x||]$$

It's clear that $U_2^{-1}(\cdot,t_0)Q(\cdot)x$ is left-handed continuous on $[t_0,\infty)$ and so continuous on $[t_0,\infty)$.

$\square$

3. The main result

LEMMA 2. *If the pair (C_0, C) is admissible to $\mathcal{U}$ then there exists $K > 0$ such that*

$$||x_f|| \leq K||f||, \qquad \textit{for all} \quad f \in C_0.$$

Proof. Let us define $\wedge_t : C_0 \to L^1_{[t,\infty)}(X)$,

$$\wedge_t f = U_2^{-1}(\cdot, t)Q(\cdot)f(\cdot)$$

for any $t \geq 0$. It is obvious that $\wedge_t$ is a linear operator for all $t \geq 0$.

Consider $t \geq 0,\ \{f_n\}_{n\geq 1} \subset C_0,\ f \in C_0,\ g \in L^1_{[t,\infty)}(X)$ such that

$$f_n \xrightarrow{C_0} f \quad , \qquad \wedge_t f_n \xrightarrow{L^1} g.$$

Then there exist a subsequence $\{f_{n_k}\}_{k\geq 1}$ of $\{f_n\}_{n\geq 1}$ such that

$$\wedge_t f_{n_k} \xrightarrow[k\to\infty]{} g \qquad a.e.$$

But

$$||(\wedge_t\ f_{n_k})(s) - (\wedge_t\ f)(s)|| \leq ||U_2^{-1}(s,t)Q(s)||\ ||f_{n_k} - f||$$

for all $k \geq 1$ and all $s \geq t$, and so

$$\wedge_t\ f_{n_k} \longrightarrow \wedge_t\ f \qquad a.e.$$

It follows easily that $\wedge_t$ is a closed operator for all $t \geq 0$ and hence using the Closed-Graph principle it is bounded.

Let $T : C_0 \to C$ be the linear operator defined by

$$(Tf)(t) = \int_0^t U_1(t,s)P(s)f(s)ds - \int_t^\infty U_2^{-1}(s,t)Q(s)f(s)ds$$

If

$$\{g_n\}_{n\geq 1} \subset C_0,\ g \in C_0,\ h \in C,\ g_n \to g \text{ in } C_0,\ Tg_n \to h \text{ in } C$$

then

$$\begin{aligned}
||(Tg_n)(t) - (Tg)(t)|| &\leq ||\int_0^t U_1(t,s)P(s)(g_n(s) - g(s))ds|| \\
&+ ||\int_t^\infty U_t^{-1}(s,t)Q(s)(g_n(s) - g(s))ds|| \\
&\leq \int_0^t ||U_1(t,s)||\ ||P(s)||\ ||g_n(s) - g(s)||ds \\
&+ ||\wedge_t (g_n - g)|| \\
&\leq tMe^{\omega t} \sup_{s\geq 0} ||P(s)||\ ||g_n - g|| + ||\wedge_t (g_n - g)||,
\end{aligned}$$

for all $t \geq 0$ and all $n \in \mathbb{N}^*$.
It follows that $Tg = h$, and hence T is closed, so by closed-graph principle it is also bounded.
So

$$||x_f|| = ||Tf|| \leq ||T||\,||f||, \quad \text{for all } f \in C_0.$$

□

LEMMA 3. *Let $f : \{(t, t_0) \in \mathbb{R}^2 : t \geq t_0 \geq 0\} \to \mathbb{R}_+$, $a > 0$ such that*

i) $f(t, t_0) \leq f(t, s) f(s, t_0)$ for all $t \geq s \geq t_0 \geq 0$;

ii) $f(t, t_0) \leq L$, for all $t_0 \geq 0$ and all $t \in [t_0, t_0 + a]$;

iii) $f(t_0 + a, t_0) \leq \dfrac{1}{e}$ for all $t_0 \geq 0$,

then there exist $N, \gamma > 0$ such that

$$f(t, t_0) \leq Ne^{-\gamma(t-t_0)} \text{ for all } t \geq t_0 \geq 0.$$

Proof. Let $t \geq t_0 \geq 0$ and $n = \left[\dfrac{t - t_0}{a}\right]$. Then

$$\begin{aligned} f(t, t_0) &\leq f(t, na + t_0) f(na + t_0, t_0) \\ &\leq Le^{-n} \\ &\leq Lee^{-\frac{t-t_0}{a}} \end{aligned}$$

For $N = Le$, and $\gamma = \dfrac{1}{a}$ it follows that

$$f(t, t_0) \leq Ne^{-\gamma(t-t_0)}, \quad \text{for all } t \geq t_0 \geq 0.$$

□

LEMMA 4. *If there exists $L > 0$ such that*

$$\int_t^\infty \frac{ds}{||U_2(s, t_0)x||} \leq \frac{L}{||U_2(t, t_0)x||},$$

for all $t \geq t_0 \geq 0$, $x \in Ker\ P(t_0) \setminus \{0\}$ then the condition d_4) is satisfied.

Proof. Let us fix $t_0 \geq 0$ and $x \in Ker\ P(t_0) \setminus \{0\}$ and to define

$$\varphi : [t_0, \infty) \to \mathbb{R}_+,\ \varphi(t) = \int_t^\infty \frac{ds}{||U_2(s, t_0)x||}.$$

It is easy to see that φ is differentiable and

$$\frac{1}{L} \leq -\frac{\varphi'(t)}{\varphi(t)}, \text{ for all } t \geq t_0.$$

By a simple integration we obtain that

$$\varphi(t)e^{\frac{1}{L}(t-r)} \leq \varphi(r), \text{ for all } t \geq r \geq t_0.$$

Hence

$$\int_t^\infty \frac{ds}{||U_2(s,t_0)x||} e^{\frac{1}{L}(t-r)} \leq \frac{L}{||U_2(r,t_0)x||}, \text{ for all } t \geq r \geq t_0.$$

Using that

$$||U_2(s,t_0)x|| \leq Me^{\omega}||U_2(t,t_0)x||,$$

for all $t \geq t_0 \geq 0$ and all $s \in [t, t+1]$ we obtain that

$$\begin{aligned}\frac{e^{\frac{1}{L}(t-r)}}{Me^{\omega}||U_2(t,t_0)x||} &\leq \int_t^{t+1} \frac{ds}{||U_2(s,t_0)x||} e^{\frac{1}{L}(t-r)} \\ &\leq \frac{L}{||U_2(r,t_0)x||}, \quad \text{for all } t \geq r \geq t_0\end{aligned}$$

□

THEOREM 1. *The evolutionary process $\mathcal{U}$ is uniformly exponentially dichotomic if and only if $\mathcal{U}$ is (C_0, C) admissible.*

Proof. Necessity. It follows easily from Definition 1 and Lemma 1 taking into account that the condition d_4) is in fact equivalent with

$$||U_2^{-1}(s,t)Q(s)|| \leq Ne^{\gamma(t-s)}$$

Sufficiency. Let $t_0 > 0$, $\delta \in (0, t_0)$, $x \in Im\ P(t_0)$ and $f : \mathbb{R}_+ \to X$ defined by

$$f(t) = \begin{cases} 0, & 0 \leq t < t_0 - \delta \\ \frac{1}{\delta}(t - t_0 + \delta)x, & t_0 - \delta \leq t < t_0 \\ e^{-2\omega(t-t_0)}U_1(t,t_0)x, & t \geq t_0 \end{cases}$$

It's easy to see that $f \in C_0$, $||f|| \leq M||x||$, $f(t) \in Im\ P(t_0)$, for all $t \in [t_0 - \delta, t_0]$, $f(t) \in Im\ P(t)$, for all $t \geq t_0$.
Then

$$x_f(t) = \int_{t_0-\delta}^{t_0} \frac{1}{\delta}(s - t_0 + \delta)U_1(t,s)P(s)x\,ds$$

$$+\int_{t_0}^{t} e^{-2\omega(s-t_0)}U_1(t,s)P(s)U_1(s,t_0)xds$$
$$+\int_{t}^{\infty} U_2^{-1}(s,t)Q(s)f(s)ds$$
$$=\frac{1}{\delta}\int_{t_0-\delta}^{t_0}(s-t_0+\delta)U(t,s)P(s)xds$$
$$+\int_{t_0}^{t} e^{-2\omega(s-t_0)}dsU(t,t_0)x$$
$$=\frac{1}{\delta}\int_{t_0-\delta}^{t_0}(s-t_0+\delta)U(t,s)P(s)xds$$
$$+\frac{1}{2\omega}[1-e^{-2\omega(t-t_0)}]U_1(t,t_0)x,$$

for all $t \geq t_0$.

By Lemma 2 it results that

$$||x_f(t)|| \leq ||x_f|| \leq K||f|| \leq MK||x||, \text{ for all } t \geq t_0.$$

We observe that

$$\frac{1}{\delta}\int_{t_0-\delta}^{t_0}(s-t_0+\delta)U(t,s)P(s)ds \to 0, \quad \text{for } \delta \to 0$$

which implies that

$$||U_1(t,t_0)x||\frac{1}{2\omega}[1-e^{-2\omega(t-t_0)}] \leq MK||x||,$$

for all $t \geq t_0 \geq 0$ and $x \in Im\ P(t_0)$.
It's now easy to see that there is $K_1 = M(1+2\omega K) > 0$ such that

$$||U_1(t,t_0)|| \leq K_1, \text{ for all } t \geq t_0 \geq 0.$$

If $t_0 > 0, \delta \in (0,t_0), x \in Im\ P(t_0), m \in \mathbb{N}$ and $g : \mathbb{R}_+ \to X$ given by

$$g(t) = \begin{cases} 0, & 0 \leq t < t_0-\delta \\ \frac{1}{\delta}(t-t_0+\delta)x, & t_0-\delta \leq t < t_0 \\ U_1(t,t_0)x, & t_0 \leq t < t_0+n \\ (t_0+n+1-t)U_1(t,t_0)x, & t_0+n \leq t < t_0+n+1 \\ 0, & t \geq t_0+n+1 \end{cases}$$

then $g \in C_0,\ ||g|| \le K_1||x||$ and

$g(t) \in Im\ P(t_0)$, for all $t \in [t_0 - \delta, t_0]$,

$g(t) \in Im\ P(t)$, for all $t \in [t_0, t_0 + n]$,

It follows that

$$x_g(t) = \int_{t_0-\delta}^{t_0} \frac{1}{\delta}(s - t_0 + \delta)U_1(t, s)P(s)ds$$

$$+ \int_{t_0}^{t} U_1(t, s)P(s)U_1(s, t_0)xds + \int_{t}^{\infty} U_2^{-1}(s, t)Q(s)g(s)ds$$

$$= \frac{1}{\delta}\int_{t_0-\delta}^{t_0}(s - t_0 + \delta)U(t, s)P(s)xds + (t - t_0)U_1(t, t_0)x,$$

for all $t \in [t_0, t_0 + n]$.

By Lemma 2 it results that

$$||x_g(t)|| \le ||x_g|| \le K||g|| \le KK_1||x||,\ \text{for all } t \in [t_0, t_0 + n]$$

As we previously noticed

$$\frac{1}{\delta}\int_{t_0-\delta}^{t_0}(s - t_0 + \delta)U(t, s)P(s)xds \to 0, \quad \text{for all } \delta \to 0$$

which implies that

$$(t - t_0)||U_1(t, t_0)x|| \le KK_1||x||,$$

for all $t_0 > 0, n \in \mathbb{N},\ t \in [t_0, t_0+n], x \in Im\ P(t_0)$. Hence $(t-t_0)||U_1(t, t_0)|| \le KK_1$, for all $t \ge t_0 \ge 0$. By Lemma 3 it results d_3).

Let us consider again $t_0 > 0, \delta \in (0, t_0), x \in Ker\ P(t_0) \setminus \{0\}, h : \mathbb{R}_+ \to X$ given by

$$h(t) = \begin{cases} 0, & 0 \le t \le t_0 - \delta \\ \dfrac{(t - t_0 + \delta)}{\delta||x||}x, & t_0 - \delta < t < t_0 \\ \dfrac{1}{||U_2(t, t_0)x||}U_2(t, t_0)x, & t_0 \le t \le t_0 + n \\ \dfrac{(t_0 + n + 1 - t)}{||U_2(t, t_0)x||}U_2(t, t_0)x, & t_0 + n \le t \le t_0 + n + 1 \\ 0, & t \ge t_0 + n + 1 \end{cases}$$

Then $h \in C_0,\ ||h|| \le 1$ and

$h(t) \in Ker\ P(t_0)$, for all $t \in [0, t_0]$,

$h(t) \in Ker\ P(t)$, for all $t \in [t_0, \infty)$.

It follows that

$$x_h(t) = \int_0^t U_1(t,s)P(s)h(s)ds - \int_t^\infty U_2^{-1}(s,t)Q(s)h(s)ds$$

$$= \int_{t_0-\delta}^{t_0} U(t,s)P(s)\left(\frac{s-t_0+\delta}{\delta||x||}x\right)ds$$

$$- \int_t^{t_0+n} U_2^{-1}(s,t)\left(\frac{1}{||U_2(s,t_0)x||}U_2(s,t_0)x\right)ds$$

$$- \int_{t_0+n}^{t_0+n+1} U_2^{-1}(s,t)\left(\frac{t_0+n+1-s}{||U_2(s,t_0)x||}U_2(s,t_0)x\right)ds$$

$$= \frac{1}{\delta||x||}\int_{t_0-\delta}^{t_0}(s-t_0+\delta)U(t,s)P(s)xds$$

$$-\left(\int_t^{t_0+n}\frac{ds}{||U_2(s,t_0)x||} + \int_{t_0+n}^{t_0+n+1}\frac{(t_0+n+1-s)ds}{||U_2(s,t_0)x||}\right)U_2(t,t_0)x,$$

for all $t \in [t_0, t_0+n]$

By Lemma 2 it results that

$$||x_h(t)|| \leq ||x_h|| \leq K||h|| \leq K, \text{ for all } t \in [t_0, t_0+n].$$

Using again the fact that

$$\frac{1}{\delta}\int_{t_0-\delta}^{t_0}(s-t_0+\delta)U(t,s)P(s)xds \to 0 \text{ for } \delta \to 0$$

we obtain that

$$\int_t^{t_0+n}\frac{ds}{||U_2(s,t_0)x||} + \int_{t_0+n}^{t_0+n+1}\frac{(t_0+n+1-s)}{||U_2(s,t_0)x||}ds||U_2(t,t_0)x|| \leq K,$$

for all $t_0 > 0, n \in \mathbb{N}, x \in Ker\ P(t_0) \setminus \{0\}, t \in [t_0, t_0+n]$ which implies that

$$\int_t^\infty \frac{ds}{||U_2(s,t_0)x||} \leq \frac{K}{||U_2(t,t_0)x||},$$

for all $t \geq t_0 \geq 0$, and all $x \in Ker\ P(t_0) \setminus \{0\}$. By Lemma 4. we have that condition d_4) is satisfied.

$\square$

COROLLARY 1. *The following assertions are equivalent*

i) $\mathcal{U}$ *is u.e.d.*

ii) $\mathcal{U}$ *is* (C, C) *admissible* *iii)* $\mathcal{U}$ *is* (C_0, C) *admissible.*

Proof. i) $\Rightarrow$ ii)It follows from Definition 4 and Lemma 1.
ii) $\Rightarrow$ iii) It is obvious.
iii) $\Rightarrow$ iv) It is the sufficiency of Theorem 1. □

COROLLARY 2. *The following results hold:*

i) If $\mathcal{U}$ *is* (C_0, C_0) *admissible then* $\mathcal{U}$ *is u.e.d.*

ii) If $\mathcal{U}$ *is* (C, C_0) *admissible then* $\mathcal{U}$ *is u.e.d.*

The following example shows that there exists an evolutionary process which is exponentially dichotomic but it is not (C_0, C_0) or (C, C_0) admissible.

EXAMPLE 1. Consider

$$X = \mathbb{R}^2,\ U(t,s)(x_1,x_2) = (e^{-(t-s)}x_1, e^{(t-s)}x_2),\ P(t)(x_1,x_2) = (x_1, 0)$$

Then for $f = (f_1, f_2)$ where

$$f_1(t) = f_2(t) = \begin{cases} 0, & t \in [0,1) \\ t-1, & t \in [1,2) \\ 1, & t \in [2,3) \\ 4-t, & t \in [3,4) \\ 0, & t \geq 4 \end{cases}$$

we have that

$(x_f)(0) = (0, -\int_0^\infty e^{-s} f_2(s)ds)$ and

$\int_0^\infty e^{-s} f_2(s)ds \geq \int_2^3 e^{-s}ds > 0.$

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