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A COLLOCATION METHOD FOR LINEAR FOURTH ORDER BOUNDARY VALUE PROBLEMS

Abstract. We propose and analyze a numerical method for solving fourth order differential equations modelling two point boundary value problems. The scheme is based on B-splines collocation. The error analysis is carried out and convergence rates are derived.

1. Introduction

Fourth order boundary value problems are common in applied sciences, e.g. the mechanics of beams. For instance, the following problem is found in [3], p. 365: The displacement u of a loaded beam of length $2L$ satisfies under certain assumptions the differential equation

$$\begin{aligned} \frac{d^2}{ds^2}\left(EI(s)\frac{d^2u}{ds^2}\right)+Ku &= q(s), \quad -L\le s\le L,\\ u''(-L) &= u'''(-L)=0,\\ u''(L) &= u'''(L)=0. \end{aligned}$$

Here,

$$I(s)=I_0\left(2-\left(\frac{s^{2}}{L}\right)\right), \quad q(s)=q_0\left(2-\left(\frac{s}{L}\right)^2\right), \quad K=\frac{40EI_0}{L^4},$$

where E and I_0 denote constants.

We wish to consider a general linear problem similar to the one just presented, namely

$$LU \equiv U^{(iv)} + a(x)U''(x) + b(x)U(x) = f(x) \tag{1}$$

for $0 < x < 1$, together with some suitable boundary conditions, say

$$U(0)=U_{00}, \quad U'(0)=U_{01}, \quad U'(1)=U_{11}, \quad U(1)=U_{10}. \tag{2}$$

Here we assume that $a, b \in C^0[0,1]$. In principle, the method we present could be applied also for initial value problems, with minor changes. In such case (2) could be replaced by suitable conditions on the function and the first three derivatives of the unknown function at the point $s = 0$.

The technique we propose here is a $B-$spline collocation method, consisting in finding a function $u_N(x)$

$$u_N(x) = \alpha_1 \Phi_1(x) + \alpha_2 \Phi_2(x) + ... + \alpha_N \Phi_N(x)$$

solving the N×N system of linear equations

$$Lu_N(x_i) \equiv \sum_{j=1}^{N} \alpha_j L\Phi_j(x_i) = f(x_i)\ ,\ 1 \leq i \leq N \tag{3}$$

where $x_1,\ x_2,\ \ldots,\ x_N$ are N distinct points of [0,1] at which all the terms of (3) are defined.

In the next Section the specific method is presented. Section 3 contains its error analysis. Finally some numerical examples are given in Section 4.

2. The method

A variety of methods for the solution of the system of differential equations exist, for instance that are based on local Taylor expansions, see e.g. [1], [2], [6], [7], [8], [16]. These in general would however generate the solution and its derivatives only at the nodes. For these methods then, the need would then arise to reconstruct the solution over the whole interval. The collocation method we are about to describe avoids this problem, as it provides immediately a formula which gives an approximation for the solution over the entire interval where the problem is formulated.

Let us fix n, define then $h = 1/n$ and set $N = 4n+4$; we can then consider the grid over [0, 1] given by $x_i = ih$, $i = 0, ..., n$. We approximate the solution of the problem (1) as the sum of B-splines of order 8 as follows

$$u_N\ (x) = \sum_{i=1}^{4n+4} \alpha_i B_i\ (x)\,. \tag{4}$$

Notice that the nodes needed for the construction of the $B-$spline are {0, 0, 0, 0, 0, 0, 0, 0, h, h, h, h, $2h$, $2h$, $2h$, $2h$, $\ldots$, $(n-1)h$, $(n-1)h$, $(n-1)h$, $(n-1)h$, 1, 1, 1, 1, 1, 1, 1, 1}.

Let us now consider $\theta_j,\ \ j = 1, ..., 4$, the zeros of the Legendre polynomial of degree 4. Under the linear map

$$\tau_{ij} = \frac{h}{2}\theta_j + \frac{x_i + x_{i-1}}{2}, \quad i = 1, ..., n,\ \ j = 1, ..., 4$$

we construct their images $\tau_{ij} \in [x_{i-1}, x_i]$. This is the set of collocation nodes required by the numerical scheme. To obtain a square system for the $4n+4$ unknowns α_i, the $4n$ collocation equations need to be supplemented by the discretized boundary conditions (2).

Letting $\alpha \equiv (\alpha_1, \dots, \alpha_{4n+4})^t$, and setting for $i = 1, ..., n,\ \ j = 1, ..., 4$,

$$\mathbf{F} = (U_{00}, U_{01}, f(\tau_{11}), f(\tau_{12}), ..., f(\tau_{ij}), ..., f(\tau_{n3}), f(\tau_{n4}), U_{11}, U_{10})^t,$$

we can write

$$L_h\alpha \equiv [M_4 + h^2 M_2 + h^4 M_0]\alpha = h^4\mathbf{F} \tag{5}$$

with $M_k \in \mathbb{R}^{(n+4)\times(n+4)}, k = 0, 2, 4$, where the index of each matrix is related to the order of the derivative from which it stems. The system thus obtained is highly structured, in block bidiagonal form. Indeed, for $k = 0, 2, 4$, $\tilde{T}_j^{(k)} \in \mathbb{R}^{2\times4},\ j = 0, 1$, $A_j^{(k)} \in \mathbb{R}^{4\times4},\ j = 0, 1$, $B_j^{(k)} \in \mathbb{R}^{4\times4},\ j = 2, \dots, n$, $C_j^{(k)} \in \mathbb{R}^{4\times4},\ j = 1, \dots, n-1$, we have explicitly

$$M_k = \begin{bmatrix} \tilde{T}_0^{(k)} & O_{2,4} & O_{2,4} & & O_{2,4} & O_{2,4} & & O_{2,4} & O_{2,4} & O_{2,4} \\ A_0^{(k)} & C_1^{(k)} & O & & O & O & & O & O & O \\ O & B_2^{(k)} & C_2^{(k)} & \dots & O & O & \dots & O & O & O \\ O & O & \dots & \dots & O & O & \dots & O & O & O \\ O & O & O & \dots & O & O & \dots & O & O & O \\ O & O & O & \dots & B_j^{(k)} & C_j^{(k)} & \dots & O & O & O \\ & & & & \dots & \dots & & & & \\ O & O & O & \dots & O & O & \dots & B_{n-1}^{(k)} & C_{n-1}^{(k)} & O \\ O & O & O & \dots & O & O & \dots & O & B_n^{(k)} & A_1^{(k)} \\ O_{2,4} & O_{2,4} & O_{2,4} & \dots & O_{2,4} & O_{2,4} & \dots & O_{2,4} & O_{2,4} & \tilde{T}_1^{(k)} \end{bmatrix}$$

Unless otherwise stated, or when without a specific size index, each block is understood to be 4 by 4. Also, to emphasize the dimension of the zero matrix we write $O_m \in \mathbb{R}^{m\times m}$ or $O_{m,n} \in \mathbb{R}^{m\times n}$.

Specifically, for M_4 we have for $T_j \in \mathbb{R}^{2\times2},\ j = 0, 1$,

$$\tilde{T}_0 \equiv \tilde{T}_0^{(4)} = \begin{bmatrix} T_0 & O_2 \end{bmatrix} \qquad \tilde{T}_1 \equiv \tilde{T}_1^{(4)} = \begin{bmatrix} O_2 & T_1 \end{bmatrix} \tag{6}$$

with

$$T_0 = \begin{bmatrix} h^4 & 0 \\ -7h^3 & 7h^3 \end{bmatrix}, \qquad T_1 = \begin{bmatrix} -7h^3 & 7h^3 \\ 0 & h^4 \end{bmatrix} \tag{7}$$

Furthermore for the matrix M_4 all blocks with same name are equal to each other and we set

$$C \equiv C_1^{(4)} = C_2^{(4)} = ... = C_{n-1}^{(4)}, \quad B \equiv B_2^{(4)} = B_3^{(4)} = ... = B_n^{(4)}.$$

For the remaining blocks we explicitly find

$$A_0 \equiv A_0^{(4)} = \begin{bmatrix} 676.898959 & -2556.080843 & 3466.638660 & -1843.444245 \\ 252.6301981 & -637.2153922 & 206.4343097 & 524.0024063 \\ 30.1896807 & 63.1159957 & -181.0553956 & -258.101801 \\ 0.281162 & 10.18023913 & 107.9824229 & 137.5436408 \end{bmatrix} \tag{8}$$

$$C = \begin{bmatrix} 194.1150595 & 59.18676730 & 2.650495372 & 0.03514515003 \\ -329.0767906 & -47.64856975 & 27.10012948 & 3.773710141 \\ 499.494486 & -120.6542168 & -64.5675240 & 31.57877478 \\ -664.532755 & 709.1160198 & -385.1831009 & 84.61236994 \end{bmatrix} \tag{9}$$

$$B = \begin{bmatrix} 84.61236994 & -385.1831008 & 709.1160181 & -664.5327536 \\ 31.57877478 & -64.56752375 & -120.6542173 & 499.4944874 \\ 3.773710141 & 27.1001293 & -47.648570 & -329.076791 \\ 0.03514515003 & 2.6504944 & 59.186765 & 194.11506 \end{bmatrix} \tag{10}$$

$$A_1 \equiv A_1^{(4)} = \begin{bmatrix} 137.5436422 & 107.9824252 & 10.1802390 & 0.28116115 \\ -258.1018004 & -181.0553970 & 63.1159965 & 30.18968105 \\ 524.0024040 & 206.4343108 & -637.2153932 & 252.6301982 \\ -1843.444246 & 3466.638661 & -2556.080843 & 676.8989596 \end{bmatrix} \tag{11}$$

Two main changes hold for the matrices M_2 and M_0, with respect to M_4; the first lies in the top and bottom corners, where $\tilde{T}_j^{(0)} = \tilde{T}_j^{(2)} = O_{2,4}, \quad j = 0, 1$. They contain then a premultiplication by diagonal coefficient matrices. Namely letting $A_{0,2}, C_2, B_2, A_{1,2}, D_i \in \mathbb{R}^{4\times4}$, $D_i = diag(a_{i1}, a_{i2}, a_{i3}, a_{i4})$, with $a_{ij} \equiv a(\tau_{ij}),\ j = 1, 2, 3, 4,\ i = 1, 2, ..., n$, we have

$$A_0^{(2)} = D_1 A_{0,2}, \quad A_1^{(2)} = D_n A_{1,2}$$

$$C_i^{(2)} = D_i C_2, \quad i = 1, 2, ..., n-1$$

$$B_i^{(2)} = D_i B_2, \quad i = 2, 3, ..., n$$

where

$$A_{0,2} = \begin{bmatrix} 29.30827273 & -47.68275514 & 9.072282826 & 7.792345494 \\ 5.67012435 & 2.62408902 & -8.50204661 & -6.772789016 \\ 0.16439223 & 1.339974467 & 3.602756629 & 1.87349900 \\ 0.00006780 & 0.004406270526 & 0.1127212947 & 1.392658748 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1.450129518 & 0.05858914701 & 0.001126911401 & 0.8471353553\ 10^{-5} \\ 4.030419911 & 2.533012603 & 0.3966407043 & 0.02054903207 \\ -9.65902448 & -0.812682181 & 2.782318888 & 0.7087655468 \\ 4.07133215 & -8.31463385 & -0.93008649 & 3.663534093 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 3.663534093 & -0.9300864851 & -8.314633880 & 4.071332233 \\ 0.7087655468 & 2.782318883 & -0.812682182 & -9.659024508 \\ 0.02054903207 & 0.396640665 & 2.53301254 & 4.0304199 \\ 0.8471353553\ 10^{-5} & 0.00112689 & 0.0585890 & 1.4501302 \end{bmatrix}$$

$$A_{1,2} = \begin{bmatrix} 1.392658814 & 0.112721477 & 0.00440599 & 0.000067777 \\ 1.873498986 & 3.602756584 & 1.33997443 & 0.164392258 \\ -6.772789012 & -8.502046610 & 2.62408899 & 5.670124369 \\ 7.792345496 & 9.072282833 & -47.68275513 & 29.30827274 \end{bmatrix}$$

Similarly, for $A_{0,0}, C_0, B_0, A_{1,0}, E_i \in \mathbb{R}^{4\times 4}$, $E_i = diag(b_{i1}, b_{i2}, b_{i3}, b_{i4})$, with $b_{ij} \equiv b(\tau_{ij})$, $j = 1, 2, 3, 4$, $i = 1, 2, ..., n$, we have

$$A_0^{(0)} = E_1 A_{0,0}, \quad A_1^{(0)} = E_n A_{1,0}$$

$$C_i^{(0)} = E_i C_0, \quad i = 1, 2, ..., n-1$$

$$B_i^{(0)} = E_i B_0, \quad i = 2, 3, ..., n$$

with

$$A_{0,0} = \begin{bmatrix} 0.604278729 & 0.3156064435 & 0.07064438205 & 0.008784901454 \\ 0.060601115 & 0.2089471273 & 0.3087560066 & 0.2534672883 \\ 0.000426270 & 0.006057945090 & 0.03689680420 & 0.1248474545 \\ 0.10\,10^{-7} & 0.7425933886\,10^{-6} & 0.00002933256459 & 0.0006554638258 \end{bmatrix}$$

$$C_0 = \begin{bmatrix} 0.0006703169101 & 0.00001503946986 & 0.1853647586\,10^{-6} & 0.9723461945\,10^{-9} \\ 0.1448636180 & 0.02163722179 & 0.001674337031 & 0.00005328376522 \\ 0.4676572160 & 0.2815769859 & 0.07496220012 & 0.007575139336 \\ 0.1985435495 & 0.4197299375 & 0.3055061349 & 0.07553484124 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 0.07553484124 & 0.3055061345 & 0.4197299367 & 0.1985435448 \\ 0.007575139336 & 0.07496219992 & 0.2815769862 & 0.4676572138 \\ 0.00005328376522 & 0.00167433770 & 0.0216372202 & 0.144863633 \\ 0.9723461945\,10^{-9} & 0.1836\,10^{-6} & 0.000015047 & 0.00067030 \end{bmatrix}$$

$$A_{1,0} = \begin{bmatrix} 0.00065546187 & 0.00002934342 & 0.72976\,10^{-6} & 0.78\,10^{-8} \\ 0.1248474495 & 0.03689679808 & 0.00605794290 & 0.0004262700 \\ 0.2534672892 & 0.3087560073 & 0.2089471283 & 0.0606011146 \\ 0.00878490146 & 0.07064438202 & 0.3156064438 & 0.6042787300 \end{bmatrix}$$

In the next Section also some more information on some of the above matrices will be needed, specifically we have

$$\begin{aligned} \|A_1\|_2 &\equiv a_1^* = 0.0321095, \\ \|B^{-1}\|_2 &\equiv b_1^* = 0.1022680, \\ \rho(B^{-1}) &\equiv b_2^* = 0.0069201. \end{aligned} \tag{12}$$

3. Error analysis

We begin by stating two Lemmas which will be needed in what follows.

LEMMA 1. *The spectral radius of any permutation matrix* P *is* $\rho(P) = 1$ *and* $\|P\|_2 = 1$.

Proof. Indeed notice that it is a unitary matrix, as it is easily verified that $P^{-1} = P^* = P$, or that $P^*P = I$, giving the second claim. Moreover, since $\rho(P^*) \equiv \rho(P^{-1}) = \rho(P) = \rho(P)^{-1}$, we find $\rho^2(P) = 1$, i.e. the first claim.

□

LEMMA 2. *Let us introduce the auxiliary diagonal matrix of suitable dimension* $\Delta_m = diag\left(1, \delta^{-1}, \delta^{-2}, ..., \delta^{1-m}\right)$ *choosing* $\delta < 1$ *arbitrarily small. We can consider also the vector norm defined by* $\|x\|_* \equiv \|\Delta x\|_2$ *together with the induced matrix norm* $\|A\|_*$. *Then, denoting by* $\rho(A) \equiv \max_{1\le i\le n} |\lambda_i^{(A)}|$ *the spectral radius of the matrix A, where* $\lambda_i^{(A)},\ i = 1(1)n$ *represent its eigenvalues, we have*

$$\|A\|_* \le \rho(A) + O(\delta), \quad \|\Delta^{-1}\|_2 = 1.$$

Proof. The first claim is a restatement of Theorem 3, [9] p. 13. The second one is immediate from the definition of Δ.

□

Let y_N be the unique B-spline of order 8 interpolating to the solution U of problem (1). If $f \in C^4([0, 1])$ then $U \in C^8([0, 1])$ and from standard results, [4], [15] we have

$$\|D^j(U - y_N)\|_\infty \le c_j h^{8-j}, \quad j = 0, \dots, 7. \tag{13}$$

We set

$$y_N(x) = \sum_{j=1}^{4n+4} \beta_j B_j(x). \tag{14}$$

The function u_N has coefficients that are obtained by solving (5); we define the function $\mathbf{G}$ as the function obtained by applying the very same operator of (5) to the spline y_N, namely

$$\mathbf{G} \equiv h^{-4} L_h \beta \equiv h^{-4}[M_4 + h^2 M_2 + h^4 M_0]\beta. \tag{15}$$

Thus $\mathbf{G}$ differs from $\mathbf{F}$ in that it is obtained by a different combination of the very same B-splines.

Let us introduce the discrepancy vector $\sigma_{ij} \equiv \mathbf{G}(\tau_{ij}) - \mathbf{F}(\tau_{ij}),\ i = 1(1)n,\ j = 1(1)4$ and the error vector $\mathbf{e} \equiv \beta - \alpha$, with components $e_i = \beta_i - \alpha_i,\ i = 1, \dots, 4n+4$. Subtraction of (5), from (15) leads to

$$[M_4 + h^2 M_2 + h^4 M_0]\mathbf{e} = h^4 \sigma. \tag{16}$$

We consider at first the dominant systems arising from (5), (15), i.e.

$$M_4 \tilde{\alpha} = h^4 \mathbf{F}, \quad M_4 \tilde{\beta} = h^4 \mathbf{G}. \tag{17}$$

Subtraction of these equations gives the dominant equation corresponding to (16), namely

$$M_4 \tilde{\mathbf{e}} = h^4 \sigma, \quad \tilde{\mathbf{e}} \equiv \tilde{\alpha} - \tilde{\beta}. \tag{18}$$

Notice first of all, that in view of the definition of **G** and of the fact that y_N interpolates on the exact data of the function, the boundary conditions are the same both for (5) and (15). Hence $\sigma_1 = \sigma_2 = \sigma_{4n+3} = \sigma_{4n+4} = 0$. In view of the triangular structure of T_0 and T_1, it follows then that $\tilde{e}_1 = \tilde{e}_2 = \tilde{e}_{4n+3} = \tilde{e}_{4n+4} = 0$, a remark which will be confirmed more formally later.

We define the following block matrix, corresponding to block elimination performed in a peculiar fashion, so as to annihilate all but the first and last element of the second block row of M_4

$$\tilde{R} = \begin{bmatrix} I_2 & O_{2,4} & O_{2,4} & O_{2,4} & \dots & O_{2,4} & \dots & O_{2,4} & O_{2,4} & O_2 \\ O_{4,2} & I_4 & Q & Q^2 & \dots & Q^{j-2} & \dots & Q^{n-2} & Q^{n-1} & O_{4,2} \\ O_{4,2} & O & I_4 & O & \dots & O & \dots & O & O & O_{4,2} \\ & & & & \dots & & & & & \\ O_{4,2} & O & O & O & \dots & O & \dots & I_4 & O & O_{4,2} \\ O_{4,2} & O & O & O & \dots & O & \dots & O & I_4 & O_{4,2} \\ O_2 & O_{2,4} & O_{2,4} & O_{2,4} & \dots & O_{2,4} & \dots & O_{2,4} & O_{2,4} & I_2 \end{bmatrix}$$

where $Q = -CB^{-1}$. Recall once more our convention for which the indices of the identity and of the zero matrix denote their respective dimensions and when omitted each block is understood to be 4 by 4. Introduce the block diagonal matrix $\tilde{A}^{-1} = diag(I_{4n}, A_1^{-1})$. Observe then that $\tilde{R}M_4\tilde{A}^{-1} = \tilde{M}_4$, with

$$\tilde{M}_4 = \begin{bmatrix} \tilde{T}_0 & O_{2,4} & O_{2,4} & O_{2,4} & \dots & O_{2,4} & O_{2,4} & O_{2,4} \\ A_0 & O & O & O & \dots & O & O & Q^{n-1} \\ O & B & C & O & \dots & O & O & O \\ O & O & B & C & \dots & O & O & O \\ & & & & \dots & & & \\ O & O & O & O & \dots & B & C & O \\ O & O & O & O & \dots & O & B & I_4 \\ O_{2,4} & O_{2,4} & O_{2,4} & O_{2,4} & \dots & O_{2,4} & O_{2,4} & \tilde{T}_1 A_1^{-1} \end{bmatrix}$$

Let us consider now the singular value decomposition of the matrix Q, $Q = V\Lambda U^*$, [12]. Here $\Lambda = diag(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is the diagonal matrix of the singular values of Q, *ordered from the largest to the smallest*. Now, premultiplication of $\tilde{M}_4$ by $S = diag(I_2, V^*, I_{4n-2})$ and then by the block permutation matrix

$$\tilde{P} = \begin{bmatrix} I_2 & O_{2,4} & O_{2,4n-4} & O_{2,2} \\ O_{4,2} & I_4 & O_{4,4n-4} & O_{4,2} \\ O_{2,2} & O_{2,4} & O_{2,4n-4} & I_2 \\ O_{4n-4,2} & O_{4n-4,4} & I_{4n-4} & O_{4n-4,2} \end{bmatrix}$$

followed by postmultiplication by $\tilde{S} = diag(I_{4n}, U)$ and then by

$$\hat{P} = \begin{bmatrix} I_4 & O & O_{4,4n-4} \\ O_{4n-4,4} & O_{4n-4,4} & I_{4n-4} \\ O & I_4 & O_{4,4n-4} \end{bmatrix}$$

gives the block matrix

$$\bar{E} = \begin{bmatrix} \tilde{E} & O_{8,4n-4} \\ \tilde{L} & \tilde{B} \end{bmatrix}. \tag{19}$$

Here

$$\tilde{E} = \begin{bmatrix} \tilde{T}_0 & O_{2,4} \\ V^* A_0 & \Lambda^{n-1} \\ O_{2,4} & \tilde{T}_1 A_1^{-1} U \end{bmatrix} \tag{20}$$

and

$$\tilde{L} = \begin{bmatrix} O_{4n-8,4} & O_{4n-8,4} \\ O_4 & U \end{bmatrix} \tag{21}$$

as well as

$$\tilde{B} = \begin{bmatrix} B & C & O & O & \dots & O & O \\ O & B & C & O & \dots & O & O \\ O & O & B & C & \dots & O & O \\ & & & & \dots & & \\ O & O & O & O & \dots & B & C \\ O & O & O & O & \dots & O & B \end{bmatrix}. \tag{22}$$

It is then easily seen that

$$\bar{E}^{-1} = \begin{bmatrix} \tilde{E}^{-1} & O_{8,4n-4} \\ -\tilde{B}^{-1}\tilde{L}\tilde{E}^{-1} & \tilde{B}^{-1} \end{bmatrix}. \tag{23}$$

In summary, we have obtained $\bar{E} = \tilde{P} S \tilde{R} M_4 \tilde{A}^{-1} \tilde{S} \hat{P}$. It then follows $M_4 = \tilde{R}^{-1} S^{-1} \tilde{P} \bar{E} \hat{P} \tilde{S}^{-1} \tilde{A}$, and in view of Lemma 1, system (18) becomes

$$\bar{E} \hat{P} \tilde{S}^{-1} \tilde{A} \tilde{\mathbf{e}} = h^4 \tilde{P} S \tilde{R} \sigma. \tag{24}$$

To estimate the norm of $\bar{E}^{-1}$ exploiting its triangular structure (19), we concentrate at first on (20). Recalling the earlier remark on the boundary data, we can partition the error from (16) and the discrepancy vectors as follows: $\tilde{\mathbf{e}} = (\tilde{e}_1, \tilde{e}_2, \tilde{\mathbf{e_t}}, \tilde{\mathbf{e_c}}, \tilde{\mathbf{e_b}}, \tilde{e}_{4n+3}, \tilde{e}_{4n+4})^T$, $\tilde{\mathbf{e_t}}, \tilde{\mathbf{e_b}}, \in \mathbb{R}^2$, $\tilde{\mathbf{e_c}}, \in \mathbb{R}^{4n-4}$. Define also $\tilde{\mathbf{e_{out}}} = (\tilde{\mathbf{e_t}}, \tilde{\mathbf{e_b}})^T$, $\hat{\mathbf{e}}_{\mathbf{out}} = (0, 0, \tilde{\mathbf{e_{out}}}, 0, 0)^T$, $\hat{\mathbf{e_t}} = (e_1, e_2, \tilde{\mathbf{e_t}})^T$, $\hat{\mathbf{e_b}} = (\tilde{\mathbf{e_b}}, e_{4n+3}, e_{4n+4})^T$.

Now introduce the projections Π_1, Π_2 corresponding to the top and bottom portions of the matrix (19). Explicitly, they are given by the following matrices

$$\Pi_1 = \begin{bmatrix} I_8 & O_{8,4n-4} \end{bmatrix} \quad \Pi_2 = \begin{bmatrix} O_{4n-4,8} & I_{4n-4} \end{bmatrix}. \tag{25}$$

Consider now the left hand side of the system (24). It can be rewritten in the following fashion

$$\Pi_1 \bar{E} \hat{P} \tilde{S}^{-1} \tilde{A} \tilde{\mathbf{e}} = \tilde{E} \hat{P} \tilde{S}^{-1} \tilde{A} \tilde{\mathbf{e}} = \tilde{E} \begin{bmatrix} \hat{\mathbf{e_t}} \\ U^* A_1 \hat{\mathbf{e_b}} \end{bmatrix} \tag{26}$$

The matrix in its right hand side $Z \equiv \Pi_1 \tilde{P} S \tilde{R}$ instead becomes

(27)

$$Z = \begin{bmatrix} I_2 & O_{2,4} & O_{2,4} & \dots & O_{2,4} & \dots & O_{2,4} & O_{2,4} & O_2 \\ O_{4,2} & V^* & V^*Q & \dots & V^*Q^{j-2} & \dots & V^*Q^{n-2} & V^*Q^{n-1} & O_{4,2} \\ O_2 & O_{2,4} & O_{2,4} & \dots & O_{2,4} & \dots & O_{2,4} & O_{2,4} & I_2 \end{bmatrix}.$$

From (26) using (20), we find

(28)
$$\tilde{E} \begin{bmatrix} \hat{\mathbf{e}}_{\mathbf{t}} \\ U^* A_1 \hat{\mathbf{e}}_{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} \tilde{T}_0 \hat{\mathbf{e}}_{\mathbf{t}} \\ V^* A_0 \hat{\mathbf{e}}_{\mathbf{t}} + \Lambda^{n-1} U^* A_1 \hat{\mathbf{e}}_{\mathbf{b}} \\ \tilde{T}_1 \hat{\mathbf{e}}_{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} T_0 \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \\ V^* A_0 \hat{\mathbf{e}}_{\mathbf{t}} + \Lambda^{n-1} U^* A_1 \hat{\mathbf{e}}_{\mathbf{b}} \\ T_1 \begin{pmatrix} e_{4n+3} \\ e_{4n+4} \end{pmatrix} \end{bmatrix}.$$

Introduce now the following matrix

$$H = \begin{bmatrix} -96.42249156 & 409.2312351 & \lambda_1^{n-1} & 0 \\ -162.6192900 & 738.3915192 & 0 & \lambda_2^{n-1} \\ 264.5383512 & -1216.139747 & 0 & 0 \\ 645.9124120 & -2179.906392 & 0 & 0 \end{bmatrix},$$

where the first two columns are the last two columns of $V^* A_0$. The matrix of the system can then be written as

$$\tilde{E} \equiv R_1^{-1} R_1 \begin{bmatrix} T_0 & O_{2,4} & O_2 \\ Y_0 & H & Y_1 \\ O_2 & O_{2,4} & T_1 \end{bmatrix} \begin{bmatrix} I_4 & O_4 \\ O_4 & U^* A_1 \end{bmatrix}$$
$$= R_1^{-1} \begin{bmatrix} T_0 & O_{2,4} & O_2 \\ Y_0 & \tilde{\Lambda}(I + N_1) & Y_1 \\ O_2 & O_{2,4} & T_1 \end{bmatrix} P^\dagger \begin{bmatrix} I_2 & O_{2,4} \\ O_{2,4} & [U^* A_1]_{1,2} \\ I_2 & O_{2,4} \\ O_{2,4} & [U^* A_1]_{3,4} \end{bmatrix} \equiv R_1^{-1} \bar{\Lambda} P^\dagger \bar{P} S,$$

where we introduced the permutation $P^\dagger$ exchanging the first two with the last two columns of the matrix H, its inverse producing a similar operation on the rows of the matrix to its right; we have denoted the first two rows of such matrix by $[U^* A_1]_{1,2}$ and a similar notation has been used on the last two. R_1 denotes the 8 by 8 matrix corresponding to the elementary row operation zeroing out the element $(4, 2)$ of H, i.e. the element $(6, 4)$ of $\tilde{E}$. Thus $R_1 H P_1$ is upper triangular, with main diagonal given by $\tilde{\Lambda} \equiv diag(\lambda_1^{n-1}, \lambda_2^{n-1}, r, s)$, $\lambda_1 = 5179.993642 > 1$, $\lambda_2 = 11.40188637 > 1$. It can then be written then as $R_1 H P_1 = \tilde{\Lambda}(I + N_1)$, with N_1 upper triangular and nilpotent.

The inverse of the above matrix $\bar{\Lambda}$ is then explicitly given by

$$\bar{\Lambda}^{-1} \equiv \begin{bmatrix} T_0^{-1} & O_{2,4} & O_2 \\ -T_0^{-1}(I+N_1)^{-1}\tilde{\Lambda}^{-1}Y_0 & (I+N_1)^{-1}\tilde{\Lambda}^{-1} & -T_1^{-1}(I+N_1)^{-1}\tilde{\Lambda}^{-1}Y_1 \\ O_2 & O_{2,4} & T_1^{-1} \end{bmatrix}$$

where $\tilde{N}_1$ denotes a nilpotent upper triangular matrix.

From (20) and the discussion on the boundary conditions the top portion of this system gives for the right hand side $h^4 Z\sigma = h^4\,[0, 0, \sigma_c, 0, 0]^T$. Thus from $\bar{\Lambda}^{-1}Z\sigma$ gives immediately $e_1 = e_2 = e_{4n+3} = e_{4n+4} = 0$ as claimed less formally earlier. The top part of the dominant system then simplifies by removing the two top and bottom equations, as well as the corresponding null components of the error and right hand side vectors. Introduce also the projection matrix $\Pi_3 = diag(\mathbf{0}_2, I_4, \mathbf{0}_2)$, where $\mathbf{0}_m$ denotes the null vector of dimension m. We then obtain

$$\hat{\mathbf{e}}_{\mathbf{out}} = \Pi_3\hat{\mathbf{e}}_{\mathbf{out}} = h^4\Pi_3\bar{\Lambda}^{-1}Z\sigma_c = h^4\Pi_3 S\bar{P}P^{\dagger}(I+N_1)^{-1}\tilde{\Lambda}^{-1}R_1\Pi_1\tilde{P}S\tilde{R}\sigma_c$$

from which letting $\lambda^{\dagger} \equiv \max(\lambda_1^{1-n}, \lambda_2^{1-n}, r^{-1}, s^{-1}) = \max(r^{-1}, s^{-1})$, the estimate follows using Lemmas 1 and 2

$$\begin{aligned} \|\hat{\mathbf{e}}_{\mathbf{out}}\|_* &\le h^4\|\Pi_3\|_*\|S\|_*\|\bar{P}\|_*\|P^{\dagger}\|_*\|(I+N)^{-1}\|_*\|\tilde{\Lambda}^{-1}\|_* \\ &\qquad \|R_1\|_*\|\Pi_1\tilde{P}S\Delta\Delta^{-1}\tilde{R}\Delta\Delta^{-1}\sigma_c\|_* \\ &\le h^4\lambda^{\dagger}(1+O(\delta))^4[\rho(S)+O(\delta)][\rho(R_1)+O(\delta)]\| \\ &\qquad \Pi_1\tilde{P}S\Delta\|_*\|\Delta^{-1}\tilde{R}\Delta\|_*\|\Delta^{-1}\sigma_c\|_* \\ &\le h^4\lambda^{\dagger}(1+O(\delta))^6\|\Pi_1\tilde{P}S\Delta\|_*\|(I+\tilde{R}_2)\|_*\|\Delta\Delta^{-1}\sigma_c\|_2 \\ &\le h^4\lambda^{\dagger}(1+O(\delta))^7\|\Pi_1\tilde{P}S\Delta\|_*\sqrt{4n-4}\|\sigma_c\|_\infty \end{aligned} \tag{29}$$

as $\tilde{R}_2$ is upper triangular and nilpotent. Now observe that the product $\tilde{P}S\Delta = diag(D_1, V^*D_2, D_3, D_4)$, where each block is as follows

$$\begin{aligned} D_1 &= diag(1, \delta^{-1}),\ D_2 = diag(\delta^{-2}, \delta^{-3}, \delta^{-4}, \delta^{-5}), \\ D_3 &= diag(\delta^{-8}, \delta^{-9}),\ D_4 = diag(\delta^{-10}, \ldots, \delta^{-4n-3}, \delta^{-6}, \delta^{-7}). \end{aligned}$$

It follows that $\Pi_1\tilde{P}S\Delta = diag(D_1, V^*D_2, D_3, \mathbf{0}_{4n-4})$. Hence

$$\begin{aligned} \|\Pi_1\tilde{P}S\Delta_{4n+4}\|_* &= \|\|diag(I_2, V^*, I_2)\ diag(D_1, D_2, D_3)\|_* \\ &\le \|diag(I_2, V^*, I_2)\|_*\|diag(D_1, D_2, D_3)\|_* \\ &\le [\rho(diag(I_2, V^*, I_2)) + O(\delta)](1+O(\delta)) \le (1+O(\delta))^2 \end{aligned} \tag{30}$$

since for the diagonal matrix $\rho[diag(D_1, D_2, D_3)] = 1$ and from Lemma 1 $\rho(V^*) = 1$, the matrix V being unitary. But also,

$$\|\hat{\mathbf{e}}_{\mathbf{out}}\|_*^2 = \|\Delta_4\hat{\mathbf{e}}_{\mathbf{out}}\|_2^2 = \hat{\mathbf{e}}_{\mathbf{out}}^*\Delta_4^2\hat{\mathbf{e}}_{\mathbf{out}} = \sum_{i=1}^{4} e_i^2\delta^{2i-8} \ge \|\hat{\mathbf{e}}_{\mathbf{out}}\|_\infty^2$$

i.e. $\|\hat{\mathbf{e}}_{\mathbf{out}}\|_* \geq \|\hat{\mathbf{e}}_{\mathbf{out}}\|_\infty$. In summary combining (29) with (30) we have

$$\begin{aligned} \|\hat{\mathbf{e}}_{\mathbf{out}}\|_\infty &\leq h^{\frac{7}{2}} 2\lambda^\dagger (1+O(\delta))^9 \|\sigma_c\|_\infty \leq h^{\frac{7}{2}} 2\lambda^\dagger (1+O(\delta))\|\sigma_c\|_\infty \\ &\equiv h^{\frac{7}{2}} \eta \|\sigma\|_\infty \end{aligned} \tag{31}$$

which can be restated also as $h^{-4}\|\tilde{E}\mathbf{e}_{\mathbf{out}}\|_\infty \geq [\eta h^{\frac{7}{2}}]^{-1}\|\mathbf{e}_{\mathbf{out}}\|_\infty$ i.e. from Thm. 4.7 of [10], p. 88, the estimate on the inverse follows

$$\|\tilde{E}^{-1}\|_\infty \leq \eta n^{\frac{1}{2}}.$$

Looking now at the remaining part of (18) with the bottom portion matrix of $\bar{E}$, see (19), we can rewrite it as $\tilde{B}\tilde{\mathbf{e}}_{\mathbf{c}} = \sigma_c - \tilde{L}\hat{\mathbf{e}}_{\mathbf{out}}$. We have $\tilde{B} = E\hat{B}$, with $\hat{B} = diag(B, \ldots, B)$ and

$$E = \begin{bmatrix} I & -Q & O & O & \ldots & O & O \\ O & I & -Q & O & \ldots & O & O \\ O & O & I & -Q & \ldots & O & O \\ & & & & \ldots & & \\ O & O & O & O & \ldots & I & -Q \\ O & O & O & O & \ldots & O & I \end{bmatrix} \tag{32}$$

and thus $\tilde{B}^{-1} = \hat{B}^{-1}E^{-1}$. Notice that E^{-1} is a block upper triangular matrix, with the block main diagonal containing only identity matrices, it can then be written as $E^{-1} = I_{4n-4} + U_0$, U_0 being nilpotent (i.e. block upper triangular with zeros on the main diagonal). Thus Lemma 2 can be applied once more. The system can then be solved to give

$$\tilde{\mathbf{e}}_{\mathbf{c}} = \hat{B}^{-1}E^{-1}[h^4\sigma - \tilde{L}\hat{\mathbf{e}}_{\mathbf{out}}].$$

Premultiplying this system by Δ^{-1} and taking norms, we obtain using (29),

$$\begin{aligned} \|\Delta^{-1}\tilde{\mathbf{e}}_{\mathbf{c}}\|_* &\leq h^4\|\Delta^{-1}\hat{B}^{-1}E^{-1}\sigma\|_* + \|\Delta^{-1}\hat{B}^{-1}E^{-1}\tilde{L}\hat{\mathbf{e}}_{\mathbf{out}}\|_* \\ &\leq h^4\|\Delta\Delta^{-1}\hat{B}^{-1}E^{-1}\sigma\|_2 + \|\Delta^{-1}\|_*\|\hat{B}^{-1}\|_*\|E^{-1}\|_*\|U\hat{\mathbf{e}}_{\mathbf{out}}\|_* \\ &\leq h^4\|\hat{B}^{-1}\|_2\|E^{-1}\sigma\|_2 + [\rho(\hat{B}^{-1}) + O(\delta)][1+O(\delta)]\|U\|_*\|\hat{\mathbf{e}}_{\mathbf{out}}\|_* \\ &\leq h^4\|B^{-1}\|_2\sqrt{4n-4}\|E^{-1}\sigma\|_\infty + \rho(B^{-1})[1+O(\delta)]^3\eta h^{\frac{7}{2}} \\ &\leq h^4 b_1^* 2\sqrt{n}\|E^{-1}\|_\infty\|\sigma\|_\infty + \rho(B^{-1})[1+O(\delta)]\eta h^{\frac{7}{2}}\|\sigma\|_\infty \\ &\leq h^{\frac{7}{2}} 2b_1^* e_\infty^* \|\sigma\|_\infty + b_2^*[1+O(\delta)]\eta h^{\frac{7}{2}}\|\sigma\|_\infty \\ &\leq h^{\frac{7}{2}}\left[2b_1^* e_\infty^* + b_2^*[1+O(\delta)]\eta\right]\|\sigma\|_\infty \equiv h^{\frac{7}{2}}\mu\|\sigma\|_\infty . \end{aligned} \tag{33}$$

On the other hand

$$\|\Delta^{-1}\tilde{\mathbf{e}}_{\mathbf{c}}\|_* = \|\Delta\Delta^{-1}\tilde{\mathbf{e}}_{\mathbf{c}}\|_2 = \|\tilde{\mathbf{e}}_{\mathbf{c}}\|_2 \geq \|\tilde{\mathbf{e}}_{\mathbf{c}}\|_\infty.$$

In summary, by recalling (12) and since $\|E^{-1}\|_\infty \equiv e_\infty^* = 72.4679$

$$\|\tilde{\mathbf{e}}_{\mathbf{c}}\|_\infty \leq \mu n^{-\frac{7}{2}}\|\sigma\|_\infty.$$

Together with the former estimate (31) on $\|\hat{\mathbf{e}}_{\mathbf{out}}\|_\infty$, we then have

$$\|\tilde{\mathbf{e}}\|_\infty \leq \nu n^{-\frac{7}{2}} \|\sigma\|_\infty,$$

which implies, once again from Thm. 4.7 of ([10]), $h^{-4}\|M_4\tilde{\mathbf{e}}\|_\infty \geq \nu^{-1}n^{-\frac{1}{2}}\|\tilde{\mathbf{e}}\|_\infty$, i.e. in summary we can state the result formally as follows.

THEOREM 1. *The matrix M_4 is nonsingular. The norm of its inverse matrix is given by*

$$\|M_4^{-1}\|_\infty \leq \nu n^{\frac{1}{2}}. \tag{34}$$

Now, upon premultiplication of (16) by the inverse of M_4, letting $N \equiv M_4^{-1}(M_2 + h^2 M_0)$, we have

$$\mathbf{e} = h^4(I + h^2 N)^{-1} M_4^{-1}\sigma. \tag{35}$$

As the matrices M_2 and M_0 have entries which are bounded above, since they are built using the coefficients a and b, which are continuous functions on [0, 1], i.e. themselves bounded above, Banach's lemma, [12] p. 431, taking h sufficiently small, allows an estimate of the solution as follows.

$$\|\mathbf{e}\|_\infty \leq h^4\|(I + h^2 N)^{-1}\|_\infty \|M_4^{-1}\|_\infty \|\sigma\|_\infty \leq \frac{h^4\nu\|\sigma\|_\infty n^{\frac{1}{2}}}{1 - h^2\|N\|_\infty} \leq \gamma n^{-\frac{7}{2}}\|\sigma\|_\infty, \tag{36}$$

having applied the previous estimate (34). Observe that

$$\|u_N - y_N\|_\infty \leq \|\mathbf{e}\|_\infty \max_{0\leq x\leq 1} \sum_{i=0}^{4n+4} B_i(x) \leq \theta\|\mathbf{e}\|_\infty.$$

Applying again (13) to σ, using the definition (5) of L_h, we find for $1 \leq k \leq n,\ j = 1(1)4$, by the continuity of the functions $\mathbf{F}, \mathbf{G}$

$$|\sigma_{4k+j}| = h^4|\mathbf{G}(\tau_{k,j}) - \mathbf{F}(\tau_{k,j})| \leq \zeta_{k,j}h^4. \tag{37}$$

It follows then $\|\sigma\|_\infty \leq \zeta h^4$ and from (36), $\|\mathbf{e}\|_\infty \leq \gamma h^{\frac{15}{2}}$. Taking into account this result, use now the triangular inequality as follows

$$\|U - u_N\|_\infty \leq \|U - y_N\|_\infty + \|y_N - u_N\|_\infty \leq c_0 h^8 + \eta\gamma h^{\frac{15}{2}} \leq c^* h^{\frac{15}{2}}$$

in view of (13) and (36). Hence, recalling that $N = 4n + 4$, we complete the error analysis, stating in summary the convergence result as follows

THEOREM 2. *If $f \in \mathrm{C}^4([0, 1])$, so that $U \in \mathrm{C}^8([0, 1])$ then the proposed B-spline collocation method (5) converges to the solution of (1) in the Chebyshev norm; the convergence rate is given by*

$$\|U - u_N\|_\infty \leq c^* N^{-\frac{15}{2}}. \tag{38}$$

REMARK 1. The estimates we have obtained are not sharp and in principle could be improved.

4. Examples

We have tested the proposed method on several problems. In the Figures we provide the results of the following examples. They contain the semilogarithmic plots of the error, in all cases for $n = 4$, i.e. $h = .25$. In other words, they provide the number of correct significant digits in the solution.

EXAMPLE 1. We consider the equation

$$y^{(4)} - 3y^{(2)} - 4y = 4\cosh(1),$$

with solution $y = \cosh(2x - 1) - \cosh(1)$.

EXAMPLE 2. Next we consider the equation with the same operator L but with different, variable right hand side

$$y^{(4)} - 3y^{(2)} - 4y = -6\exp(-x),$$

with solution $y = \exp(-x)$.

EXAMPLE 3. Finally we consider the variable coefficient equation

$$y^{(4)} - xy^{(2)} + y\sin(x) = \frac{24}{(x+3)^5} - \frac{2x}{(x+3)^3} + \frac{\sin(x)}{x+3},$$

with solution $y = \frac{1}{x+3}$.

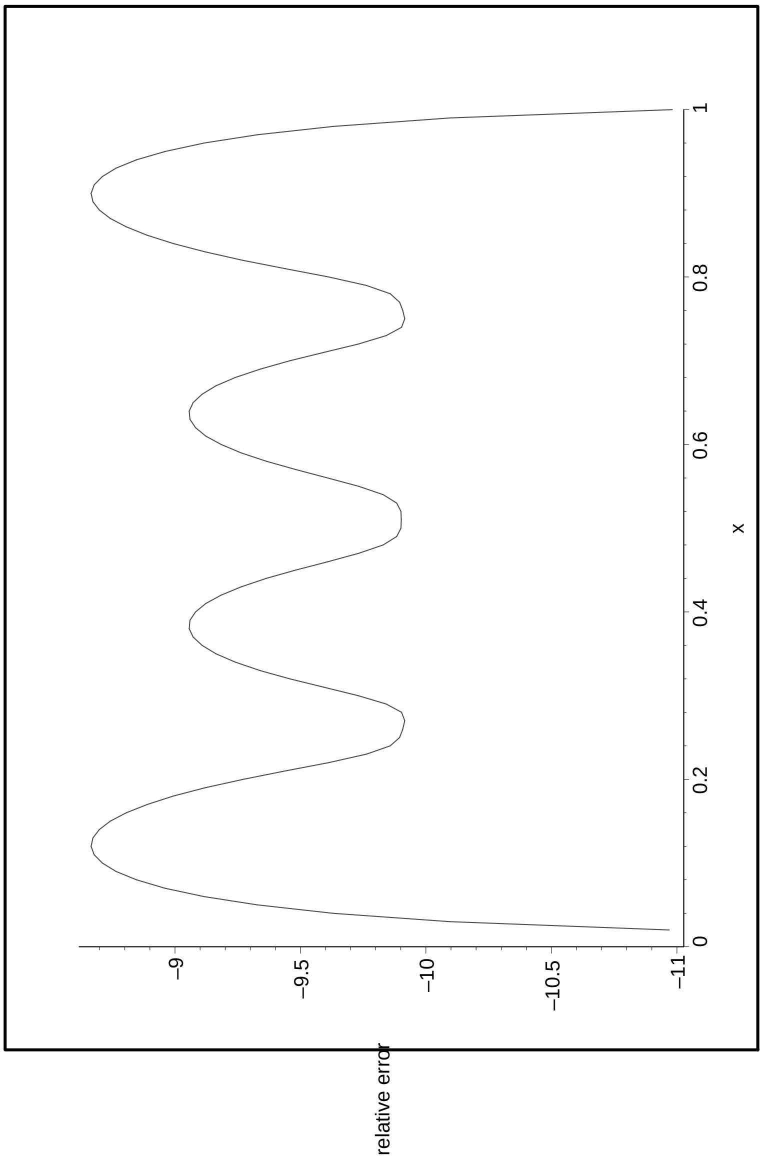

Figure 1: Semilogarithmic graph of the relative error for Example 1.

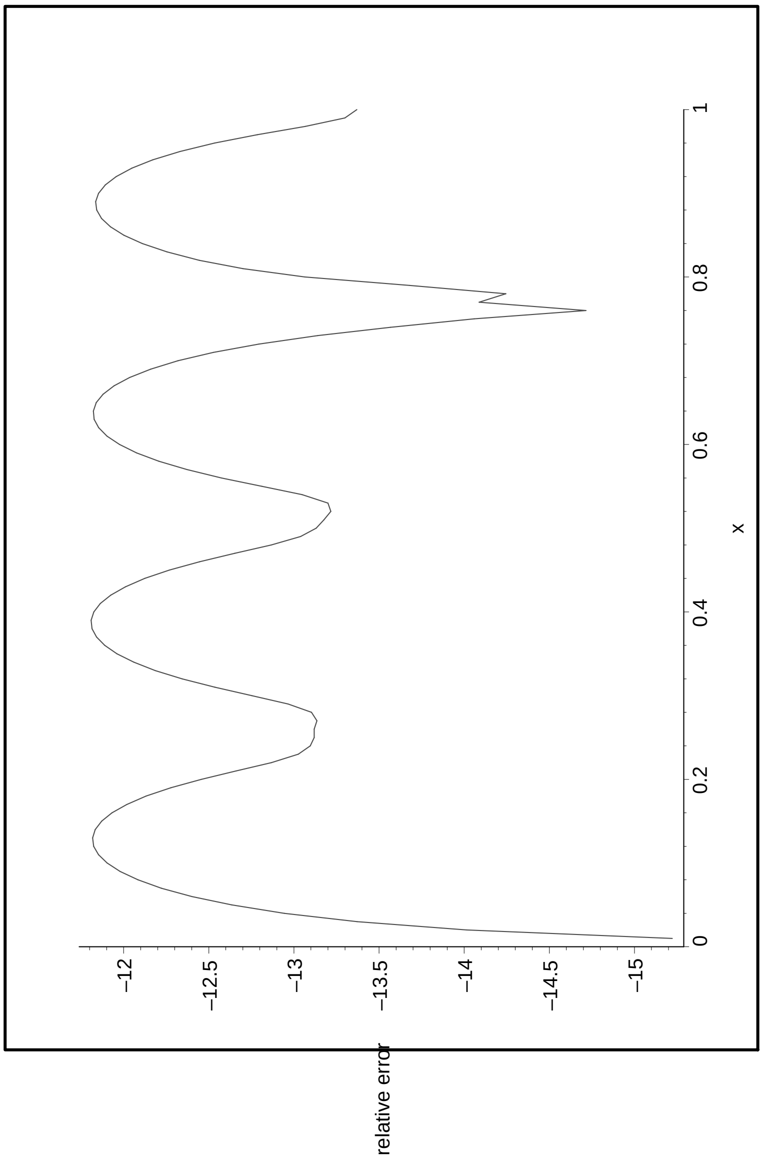

Figure 2: Semilogarithmic graph of the relative error for Example 2.

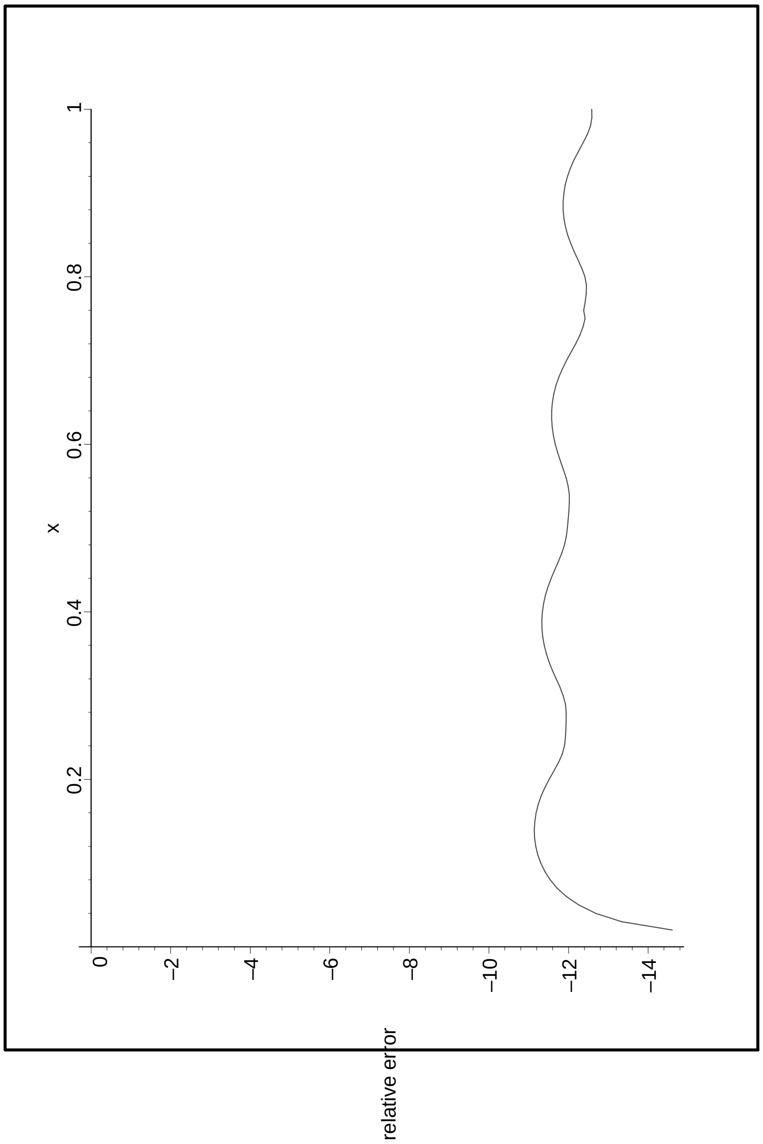

Figure 3: Semilogarithmic graph of the relative error for Example 3.

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