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SCATTERED DATA APPROXIMATION WITH A HYBRID SCHEME

Abstract. A local hybrid radial–polynomial approximation scheme is introduced here to modify the scattered data fitting method presented in [6], generating C^1 or C^2 approximating spline surfaces. As for the original method, neither triangulation nor derivative estimate is needed, and the computational complexity is linear. The reported numerical experiments relate to two well known test functions and confirm both the accuracy and the shape recovery capability of the proposed hybrid scheme.

1. Introduction

In this paper we investigate the benefit obtainable using local hybrid radial–polynomial approximations to modify the scattered data fitting method introduced in [6] which is based on direct extension of local polynomials to bivariate splines. The hybrid approach here considered is motivated by the well known excellent quality of scattered data radial basis function approximation [2]. Polynomial terms are also admitted in order to improve the approximation in the subdomains with polynomial-like behaviour of the data (e.g. almost flat areas).

Both the original and our hybrid scheme do not need data triangulation because the standard four directional mesh covering the domain is used. In addition, they do not require derivative estimates because only functional values are required. Clearly, the hybrid approximations must be converted in local polynomials for making possible their extension to splines. However, the additional computational cost of the conversion phase is negligible with respect to the whole cost of the method. In addition, as well as for the original method, the computational complexity of the scheme is linear, and this is obviously a very important feature particularly when large data sets have to be handled. A C^1 cubic or a C^2 sextic final spline approximation is produced, which is advantageous for CAGD applications since splines are a standard tool for that purpose [7].

In our modified approach, thanks to the usage of radial terms only in the local setting, the related local hybrid approximations can be computed without using special numerical techniques because the subset of data used for each of them is small and its size is assumed a priori bounded, which results in avoiding large matrices completely. In addition, for the same reason a simple and no–cost adaptation of the scaling parameter characterizing the radial terms of the hybrid approximations is possible. We note that local scaling adaptation is a nice feature of the scheme because, as proved by the researches reported by various authors (e.g. [1, 13, 14]), the use of different scaling parameters can be very proficuous in particular relating to shape recovery, but it is not easy when global radial schemes are used.

In this paper, in order to investigate the accuracy and the shape recovery capability of the method, we have experimented its performances by means of two reference mathematical test functions, that is the well known Franke [9] and Nielson [15] functions. For both the reported test functions the results highlight the good behaviour of the proposed hybrid scheme.

The paper is organized as follows. In Section 2 the original bivariate spline approximation method is summarized and in Section 3 the local hybrid approximation scheme is introduced. Finally in Section 4 the numerical results related to the two considered test functions are presented.

2. The original method

In this section we give some basic information about the original scattered data approximation scheme introduced in [6, 12] which is a two-stage method extending local approximations to the final global spline approximating surface. In fact, our scheme is obtained acting on the first stage of the original method, that is modifying the local approximations. On the other hand, the philosophy of the method and its second stage, devoted to the spline computation, are unchanged.

First, let us introduce some fundamental definitions (see for details [8]).

The **Bernstein-Bézier representation** of a bivariate polynomial p of total degree $\leq d$ is

$$p = \sum_{i+j+k=d} c_{ijk}\, B^d_{ijk}, \tag{1}$$

where B^d_{ijk}, $i+j+k=d$, $i, j, k \in \mathcal{N}$ are the Bernstein polynomials of degree d related to the reference triangle T with vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

Each coefficient $c_{ijk}, i+j+k=d$ in (1) is associated with the **domain point** $\eta_{ijk} \in T$,

$$\eta_{ijk} := \frac{i}{d}\mathbf{a} + \frac{j}{d}\mathbf{b} + \frac{k}{d}\mathbf{c}.$$

The set of all the domain points associated with T is denoted by $\mathcal{D}_{d,T}$ and the set of all the domain points related to the triangles of the considered triangulation Δ is denoted by $\mathcal{D}_{d,\Delta}$.

A set $\mathcal{M} \subset \mathcal{D}_{d,\Delta}$ is called a **minimal determining set** for the linear subspace $\mathcal{S}$ of the spline space $\mathcal{S}^0_d(\Delta)$ if, setting the coefficients of $s \in \mathcal{S}$ associated with the domain points in $\mathcal{M}$ to zero implies that all the coefficients of s vanish and no proper subset of $\mathcal{M}$ exists with the same property.

We now summarize the original method we refer to, relating to [6] for a complete description. In this approach local polynomials are extended to bivariate splines producing a C^1 or C^2 approximating surface using cubic or sextic splines respectively. The extension to bivariate splines is done in the second stage by using the smoothness conditions between adjacent Bézier triangular patches [8]. A uniform four directional mesh Δ covering the domain $\Omega \subset \mathbb{R}^2$ is used and local polynomials are computed by discrete least squares using the stable Bernstein-Bézier representation form. The computational complexity of the method grows linearly with the number N of data points $\{ (\mathbf{X}_i, f_i), \ i = 1, \dots, N, \mathbf{X}_i \in \Omega \subset \mathbb{R}^2 \}$. Thus, large data and many different data distributions can be efficiently handled, as shown in [6]. The efficiency of the method mainly depends on the theoretical determination of minimal determining sets $\mathcal{M}$ for the spline approximating spaces which consist of all domain points belonging to a set $\mathcal{T}$ of uniformly distributed triangles of Δ. In fact, using this result, local polynomial Bèzier patches can be separately computed for each triangle belonging to $\mathcal{T}$ and then univocally extended to the final spline approximation.

Concerning the local polynomial approximations, it is clear that their accuracy and shape quality heavily influences the corresponding attributes of the spline approximation. As a consequence, an important point is the selection of the data used for defining through the least squares procedure each local polynomial p_T of total degree $\leq d$ ($d = 3$ for cubics and 6 for sextics) on each triangle $T \in \mathcal{T}$. So, they initially correspond to locations $\mathbf{X}_i$ inside a circle Ω_T centered at the barycenter of T and with radius equal to the grid size. However, if they are few, the radius is suitably increased and if they are too many, in order to accelerate the computational process, their number N_T is decreased using a grid-type thinning algorithm. A lower and an upper bound M_{Min} and M_{Max} for N_T are assumed as input parameters provided by the user. Another important input parameter of the method is the tolerance κ_P used to control the inverse of the minimal singular value $\sigma_{min,d,T}$ of the collocation matrix $M_{d,T}$ related to the least-squares local polynomial approximation defined on each $T \in \mathcal{T}$. In fact, as proved in [4], imposing an upper bound for $\sigma^{-1}_{min,d,T}$ allows a direct control on the approximation power of the least-squares scheme, besides guaranteeing its numerical stability. An adaptive degree reduction procedure for guaranteeing this bound is used, producing constant approximations in the worst case.

3. The local hybrid scheme

As we already said in the introduction, the idea of our hybrid method is to enhance the approximation quality of the local approximations by using linear combinations of polynomials and radial basis functions. Once a local hybrid approximation g_T is computed on a triangle $T \in \mathcal{T}$, it is trasformed into a polynomial approximation of degree d computing the discrete least squares polynomial approximation of degree d with respect to the evaluations of g_T at all the $\binom{D+2}{2}$ domain points on T, where it is assumed $D = 2d$. On this concern, we remark that the additional cost related to this conversion phase is negligible with respect to the whole cost of the method mainly for two reasons. First, the collocation matrix associated with each local conversion hybrid–to–polynomial is the same for all triangles $T \in \mathcal{T}$. Second, it has a small

$\sigma^{-1}_{min,d,T}$ (2.87 for $D = 6$ and 21.74 for $D = 12$), so guaranteeing that the least squares polynomial of degree d is a good approximation of g_T [4].

Let $\Xi_T = \{\mathbf{X}_1, \ldots, \mathbf{X}_{N_T}\}$ denote the set of locations related to the triangle T (its definition is based on the same strategy used in the original method described in the previous section). The local mixed approximation g_T has the form

$$g_T(\cdot) = \sum_{j=1}^{m} a_j^T\, p_j^T(\cdot) + \sum_{j=1}^{n_T} b_j^T\, \phi_T(\| \cdot - \mathbf{Y}_j^T \|_2) \tag{2}$$

where $\mathrm{span}\{p_1^T, \ldots, p_m^T\}$ is the space Π_q^2 of bivariate polynomials of degree $q \geq 0$ and $m = \binom{q+2}{2} \leq N_T$. The function $\phi_T : \mathbb{R}_{\geq 0} \to \mathbb{R}$ can be any suitably smooth positive definite function or a conditionally positive definite function of order at most $q + 1$ on $\mathbb{R}^2$ (see [2]). The approximation g_T is constructed minimizing the ℓ_2-norm of the residual on Ξ_T,

$$\Big(\sum_{i=1}^{N_T} (f_i - g_T(\mathbf{X}_i))^2 \Big)^{1/2}, \tag{3}$$

where $0 \leq n_T \leq N_T - m$, and the set of knots $Y_T = \{\mathbf{Y}_j\,,\, j = 1, \ldots, n_T\}$ is a subset of Ξ_T.

We do not consider the additional orthogonality constraints

$$\sum_{j=1}^{n_T} b_j^T\, p(\mathbf{Y}_j^T) = 0, \qquad \text{all } p \in \Pi_q^2, \tag{4}$$

usually required in radial approximation ([2]), because we want to exploit in full the approximation power of the linear space

$$\mathcal{H}_T := \mathrm{span}\left\{ p_1^T, \ldots, p_m^T, \phi_T(\| \cdot - \mathbf{Y}_1^T \|_2), \ldots, \phi_T(\| \cdot - \mathbf{Y}_{n_T}^T \|_2) \right\}.$$

So we have to check the uniqueness of the solution of our least squares problem and this is done requiring that

$$\sigma^{-1}_{min}(C_T) \leq \kappa_H, \tag{5}$$

where κ_H is a user specified tolerance and $\sigma_{min}(C_T)$ is the minimal singular value of the collocation matrix C_T defined by

$$\begin{bmatrix} p_1^T(\mathbf{X}_1) & \ldots & p_m^T(\mathbf{X}_1) & \phi_T(\|\mathbf{X}_1 - \mathbf{Y}_1^T\|_2) & \ldots & \phi_T(\|\mathbf{X}_1 - \mathbf{Y}_{n_T}^T\|_2 \\ \vdots & & \vdots & \vdots & & \vdots \\ p_1^T(\mathbf{X}_{N_T}) & \ldots & p_m^T(\mathbf{X}_{N_T}) & \phi_T(\|\mathbf{X}_{N_T} - \mathbf{Y}_1^T\|_2) & \ldots & \phi_T(\|\mathbf{X}_{N_T} - \mathbf{Y}_{n_T}^T\|_2 \end{bmatrix}.$$

An adaptive ascending iterative strategy is used for defining n_T and the related set of knots Y_T. For the description of the details of such a strategy, the reader is referred to the forthcoming paper [5]. However here we just mention that this strategy is based on the inequality (5). The reason why we control the unique solvability of our least squares problem using (5) instead of a cheaper criterion avoiding the computation of $\sigma_{min}(C_T)$ ([16]) is because it allows us to control also the approximation error $\|f - g_T\|_{C(T)}$, where we are here assuming that $f_i = f(\mathbf{X}_i)$, $i = 1, \ldots, N_T$, being f a continuous function. In fact, if (5) holds and N_T is upper bounded, assuming that the polynomial basis $\{p_1^T, \ldots, p_m^T\}$ and ϕ_T are properly scaled, it can be proved that ([4, 5]) there exists a constant c_T such that

$$\|f - g_T\|_{C(T)} \le c_T \; E(f, \mathcal{H}_T)_{C(T)}, \tag{6}$$

where $E(f, \mathcal{H}_T)_{C(T)}$ is the error of the best approximation of f from $\mathcal{H}_T$,

$$E(f, \mathcal{H}_T)_{C(T)} := \inf_{g \in \mathcal{H}_T} \|f - g\|_{C(T)}.$$

4. Numerical results

The features of our local hybrid approximation scheme are investigated incorporating it into the two-stage scattered data fitting algorithm of [6]. More precisely, the method RQ_2^{av} of [6, Section 5] has been always used in the reported experiments, producing a C^2 piecewise polynomial spline of degree $d = 6$ with respect to the four-directional mesh. For our experiments in (2) we have always considered

$$\phi_T(r) \quad = \quad -\delta d_T \, \phi_{MQ}\left(\frac{r}{\delta d_T}\right) = \sqrt{(\delta d_T)^2 + r^2}, \tag{7}$$

where

$$d_T := \max_{1 \le i, j \le N_T} \|\mathbf{X}_i - \mathbf{X}_j\|_2$$

is the diameter of Ξ_T and δ is a scaling parameter. As this radial basis function is conditionally positive definite of order 1, we take $q = 0$, and thus the polynomial part in (2) is just a constant.

The input parameters to the method are the grid size $n_x \times n_y$ on a rectangular domain, the inverse minimal singular value tolerance κ_H, the minimum and maximum numbers M_{min}, M_{max} of data points belonging to each Ξ_T, the scaling coefficient δ used in (7), the upper bound n_{max} on the knot number n_T used in (2).

We consider here two tests, relating to the Franke (Test 1) and Nielson (Test 2) reference functions reported in Figure 1. Each displayed approximation is depicted together with the related data sample. For both considered tests a uniform 101×101 grid is used for the visualization and for the computation of the maximum (*maxg*) and root mean square (*rmsg*) errors. In all experiments below $n_{max} = 2\binom{d+2}{2} - 1$ and no upper bound for N_T is assigned, that is $M_{max} = N$. The lower bound M_{min} is always 20 and the scaling parameter δ in (7) is 0.4. The tolerance κ_H in (5) is taken to be equal to 10^5.

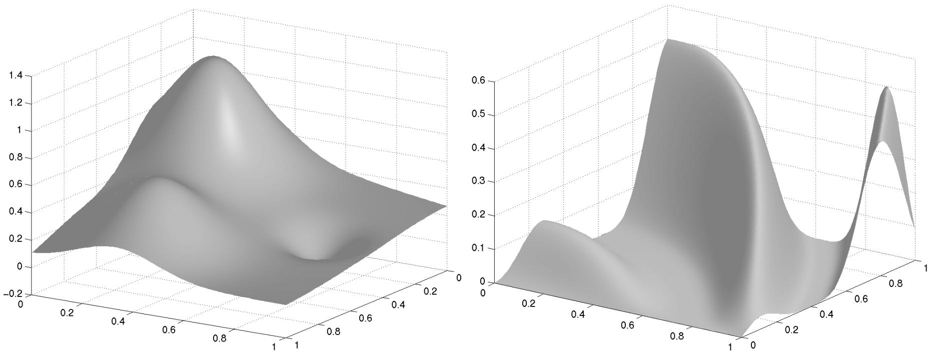

Figure 1: Franke and Nielson parent surfaces on the left and on the right, respectively.

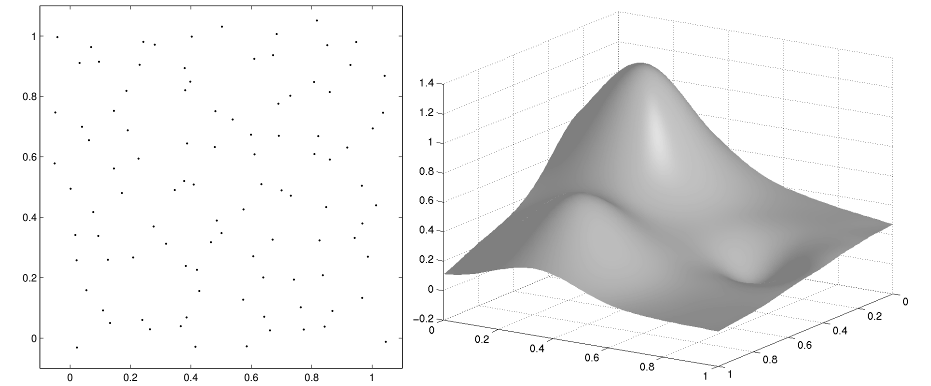

Figure 2: On the left the locations of the 100 data points for Test 1. On the rigth the related approximation.

In our first test, related to the Franke function, a small set of $N = 100$ data points is used. It is available from [10] as `ds3` and is shown on the left of Figure 2. The approximation depicted on the right of Figure 2 has been obtained using a uniform grid of size $n_x = n_y = 5$. The average number of knots used for the local hybrid approximations is 23.9. The related grid errors are $maxg = 1.5 \cdot 10^{-2}$ and $rmsg = 2.7 \cdot 10^{-3}$. For comparison, using the same grid size the errors obtained with the original method and reported in [6] are $maxg = 3.8 \cdot 10^{-2}$ and $rmsg = 7.6 \cdot 10^{-3}$ (see Table 3 of that paper). In addition, we found in the literature the following errors for the interpolation of this data with the global multiquadric method: $maxg = 2.3 \cdot 10^{-2}$ and $rmsg = 3.6 \cdot 10^{-3}$ in the famous Franke's report [9], and $rmsg = 2.6 \cdot 10^{-3}$ in [3]. (In both cases a uniform 33×33 grid was used to compute the error.) Note that the above error from [3] corresponds to the case when a parameter value for multiquadric was found by optimization.

Our second test relates to the Nielson function. First we have considered a small

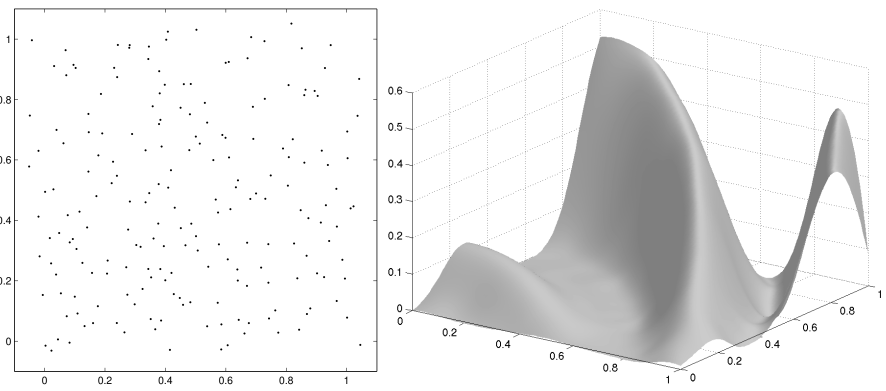

Figure 3: On the left the locations of the 200 data points for Test 2. On the rigth the related approximation.

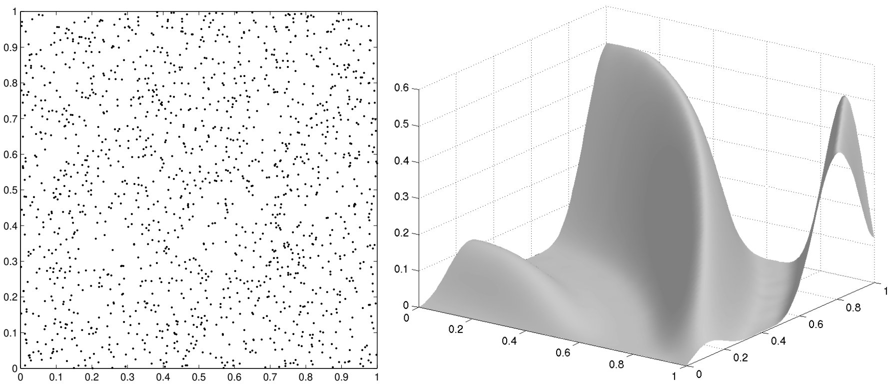

Figure 4: On the left the locations of the 1500 data points for Test 2. On the rigth the related approximation.

set of 200 data points obtained evaluating this function on the locations corresponding to the data points available from [10] as `ds4`. These locations are shown on the left of Figure 3. Again a uniform grid of size $n_x = n_y = 5$ is used. In this case the approximation shown on the right of Figure 3 is obtained using an average knot number equal to 22 and the related grid errors are $maxg = 6.9 \cdot 10^{-2}$ and $rmsg = 1.4 \cdot 10^{-2}$. For comparison, we mention that the same data set is used in [11] to test a least squares approximation method based on multiquadrics and parameter domain distortion. The best root mean square error (computed using a uniform 33×33 grid) reported in [11] is $1.3 \cdot 10^{-2}$ (see Table 1 and Figure 6 of that paper). Even if Figure 3 clearly shows some artifacts, we evaluate positively the results related to this first experiment for Test 2. In fact the accuracy and the shape recovery capability of our scheme are both comparable with those obtained in the best case reported in [11]. We would like also to say on this concern that, even if the results given in [11] have been obtained with remarkably few degrees of freedom, it should be taken into account that the parametric domain distortion method may encounter difficulties when applied to real data, as the authors admit [11, Section 4]. Finally, we get full shape recovery also for this challenging test function when we consider a denser set of 1500 scattered data depicted on the left of Figure 4 and use a finer spline grid by taking $n_x = n_y = 8$. The shape of the corresponding approximation depicted on the right of Figure 4 is almost perfect now and the related grid errors are $maxg = 3.8 \cdot 10^{-2}$ and $rmsg = 1.3 \cdot 10^{-3}$. The mean number of knots used in this case is 22.4.

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