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C. Conti - L. Gori - F. Pitolli

SOME RECENT RESULTS ON A NEW CLASS OF BIVARIATE REFINABLE FUNCTIONS

Abstract. In this paper a new class of bivariate refinable functions is presented and some of its properties are investigated. The new class is constructed by convolving a tensor product refinable function of special type with $\chi_{[0,1]}$, the characteristic function of the interval [0, 1]. As in the case of box splines, the convolution product here used is the directional convolution product.

1. Introduction

It is well know that refinable functions play a key role in different fields like, just to mention two of the most significative, subdivision algorithms and wavelets. That is why there is an enormous amount of literature analyzing properties and applications of refinable functions in both the univariate and multivariate setting. In spite of their importance in many applications, the explicit form of refinable functions known in the literature reduces, in practice, to the two celebrated cases of B-splines and box-splines on uniform grids and of Daubechies refinable functions (see [2], [3], [6], and [7], for example). This is especially true in the multivariate setting where tensor product of univariate refinable functions are mainly taken into account. The considerations above motivated us in constructing and investigating a new family of bivariate non tensor-product refinable functions. Thus, starting with a bivariate function which is a tensor-product of finitely supported totally positive refinable functions, the new functions are obtained by using the directional convolution product with the characteristic function of the interval [0, 1]. The idea is definitely borrowed from box-splines but the bivariate function we start with is not the characteristic function of $[0, 1]^2$. The univariate functions used to construct the tensor product belong to a large class of refinable functions introduced in [9], [8] by the two last authors so that they will be called GP functions. The class of GP functions contains as a particular case the cardinal B-splines with which they share many useful properties. The differences between the B-splines and the GP functions are mainly due to the fact that the refinement mask is characterized by one or more extra parameters that afford additional degrees of freedom which reveals its effectiveness in several applications.

The outline of the paper is as follows. In Section 2 we first recall the definition of the directional convolution product of a bivariate function and a univariate function. Then, we investigate which properties of the bivariate function are preserved after the directional convolution with $\chi_{[0,1]}$, the characteristic function of $[0, 1]$, is made. In Section 3 the new class of bivariate refinable functions is characterized. In the closing Section 4 a few examples are presented.

2. Directional convolution

We start this Section by recalling the definition of the directional convolution product (see also [7] for a possible use of it).

DEFINITION 1. *Let $F : \mathbb{R}^2 \to \mathbb{R}$, $\quad g : \mathbb{R} \to \mathbb{R}$ be a bivariate and a univariate function, respectively, and let $\mathbf{e} \in \mathbb{Z}^2$ be a direction vector. The* convolution product *between F and g along the direction $\mathbf{e}$ is defined as*

$$(F *_{\mathbf{e}} g)(x) := \int_{\mathbb{R}} F(x - \mathbf{e}t)g(t)\, dt, \qquad x \in \mathbb{R}^2. \tag{1}$$

Next, let Φ be a bivariate refinable function that is a solution of a refinement equation of type

$$\Phi(x) = \sum_{\alpha \in \mathbb{Z}^2} a_\alpha \Phi(2x - \alpha), \qquad x \in \mathbb{R}^2, \tag{2}$$

where the set of coefficients a_α forms the so called refinement mask $\mathbf{a} = \{a_\alpha, \alpha \in \mathbb{Z}^2\}$. The mask $\mathbf{a}$ is supposed to be of compact support and satisfying $\sum_{\alpha \in \mathbb{Z}^2} a_{\alpha+2\gamma} = 1$ for all $\gamma \in \{0, 1\}^2$. Furthermore, we assume that the Fourier transform of Φ satisfies $\widehat{\Phi}(0) = 1$. Here we define the Fourier transform of a given function F as

$$\hat{F}(\omega) := \int_{\mathbb{R}^2} F(x) e^{-i\omega\cdot x}\, dx. \tag{3}$$

Using the above introduced directional convolution product we defined the bivariate function $\Psi : \mathbb{R}^2 \to \mathbb{R}$

$$\Psi(x) := (\Phi *_{\mathbf{e}} \chi_{[0,1]})(x) = \int_0^1 \Phi(x - \mathbf{e}t)\, dt, \tag{4}$$

where $\mathbf{e} \in \{-1, 0, 1\}^2$, and $\chi_{[0,1]}$, in the following for shortness χ, is the characteristic function of the unit interval $[0, 1]$.

PROPOSITION 1. *Let Φ be a refinable function with refinement mask $\mathbf{a}$ such that $\sum_{\alpha \in \mathbb{Z}^2} \Phi(\cdot - \alpha) = 1$. Then, the function Ψ defined in (4) is refinable with refinement mask*

$$\mathbf{b} = \{b_\alpha = \frac{a_\alpha + a_{\alpha-\mathbf{e}}}{2},\ \alpha \in \mathbb{Z}^2\}\,.$$

Furthermore, the integer translates of Ψ *form a partition of unity, namely* $\sum_{\alpha\in\mathbb{Z}^2}\Psi(\cdot-\alpha)=1$.

Proof. By the Ψ definition we get

$$\begin{aligned}
\Psi(x) &= \int_0^1 \Phi(x-\mathbf{e}t)\,dt = \sum_{\alpha\in\mathbb{Z}^2} a_\alpha \int_0^1 \Phi(2(x-\mathbf{e}t)-\alpha)dt \\
&= \tfrac{1}{2}\sum_{\alpha\in\mathbb{Z}^2} a_\alpha \int_0^2 \Phi(2x-\mathbf{e}t-\alpha)dt \\
&= \tfrac{1}{2}\sum_{\alpha\in\mathbb{Z}^2} a_\alpha \left[\int_0^1 \Phi(2x-\mathbf{e}t-\alpha)dt + \int_0^1 \Phi(2x-\mathbf{e}t-\alpha-\mathbf{e})dt\right] \\
&= \tfrac{1}{2}\sum_{\alpha\in\mathbb{Z}^2} a_\alpha[\Psi(2x-\alpha)+\Psi(2x-\alpha-\mathbf{e})] \\
&= \sum_{\alpha\in\mathbb{Z}^2} \frac{1}{2}(a_\alpha + a_{\alpha-\mathbf{e}})\Psi(2x-\alpha)
\end{aligned}$$

so that Ψ is refinable with refinement mask $\mathbf{b} = \{b_\alpha = \frac{a_\alpha+a_{\alpha-\mathbf{e}}}{2},\ \alpha\in\mathbb{Z}^2\}$.
Next, since the Fourier transform of Ψ is $\widehat{\Psi}(\omega) = \widehat{\Phi}(\omega)\widehat{\chi}(\mathbf{e}\cdot\omega)$ for all $\omega\in\mathbb{R}^2$, from $\widehat{\Phi}(0)=1$ it trivially follows that $\sum_{\alpha\in\mathbb{Z}^2}\Psi(\cdot-\alpha) = \widehat{\Psi}(0) = \widehat{\Phi}(0) = 1$ which is the partition of unity for the function Ψ. □

A theorem is now dealing with the stability of Ψ. We recall that the function Ψ is L_2-stable if there exist two constants $0 < A \le B < \infty$ such that

$$0 < A||\mathbf{c}||_2 \le ||\sum_{\alpha\in\mathbb{Z}^2} c_\alpha \Psi(\cdot-\alpha)||_2 \le B||\mathbf{c}||_2 \tag{5}$$

for any real sequence $\mathbf{c} = \{c_\alpha,\ \alpha\in\mathbb{Z}^2\}$ in $\ell^2(\mathbb{Z}^2)$.

THEOREM 1. *Let* $\{\Phi(\cdot-\alpha), \alpha\in\mathbb{Z}^2\}$ *be linear independent and such that* $\widehat{\Phi}(2\pi k) = \delta_{0,k}$, *where* $\delta_{0,k}$ *is the Kronecker symbol. Then, the integer translates of* Ψ *are linearly independent. Furthermore,* $\{\Psi(\cdot-\alpha), \alpha\in\mathbb{Z}^2\}$ *is a* L_2*-stable basis.*

Proof. To prove the linear independence, it is sufficient to show that the set of the complex periodic zeros of $\widehat{\Psi}$ is empty, that is

$$Z_{\Psi}^{C} = \{\theta \in \mathbb{C}^2 | \widehat{\Psi}(\theta + 2\pi k) = 0, \forall k \in \mathbb{Z}^2\} = \{\emptyset\}$$

(see [12] for details). Now, since

$$\widehat{\Psi}(\theta + 2\pi k) = \widehat{\Phi}(\theta + 2\pi k)\widehat{\chi}(\mathbf{e} \cdot (\theta + 2\pi k)) ,$$

if θ is not a multiple of 2π, then $\theta + 2\pi k \notin 2\pi\mathbb{Z}^2$ and $\widehat{\Phi}(\theta + 2\pi k) \neq 0$, $\widehat{\chi}(\mathbf{e} \cdot (\theta + 2\pi k)) \neq 0$, so that θ is not a periodic zero. If θ is a multiple of 2π, then $\theta + 2\pi k \in 2\pi\mathbb{Z}^2$ and

$$\widehat{\Phi}(\theta + 2\pi k) = \begin{cases} 0, & \text{if } k \neq K, \\ 1, & \text{if } k = K, \end{cases}$$

where $K := -\frac{\theta}{2\pi}$. Now, for $k = K$ one has $\widehat{\chi}(\mathbf{e} \cdot (\theta + 2\pi K)) = \widehat{\chi}(0) = 1$, so that θ is not a periodic zero. It follows the set Z_{Ψ}^{C} is empty.
We conclude with the observation that, obviously, also the set of the *real periodic zeros* of $\widehat{\Psi}$ is empty, that is

$$Z_{\Psi}^{R} = \{\theta \in \mathbb{R}^2 | \widehat{\Psi}(\theta + 2\pi k) = 0, \forall k \in \mathbb{Z}^2\} = \{\emptyset\} ,$$

which implies the L_2-stable stability of the system of the integer translates of Ψ as shown, again, in [12]. □

As a consequence of Theorem 1, the following corollary holds.

COROLLARY 1. *The refinable function Ψ generates a multi-resolution analysis on $L^2(\mathbb{R}^2)$.*

3. A new class of bivariate refinable functions

Aim of this Section is the construction of a specific class of refinable functions having all the properties of the Ψ function discussed in the previous section. As Φ refinable function we consider a tensor product of particular univariate functions, that is

$$\Phi^{\mathbf{H}_1,\mathbf{H}_2}(x) := \varphi^{\mathbf{H}_1}(x_1)\varphi^{\mathbf{H}_2}(x_2) , \tag{6}$$

where $\mathbf{H}_1 = (n_1, h_1)$, $\mathbf{H}_2 = (n_2, h_2)$, and $x = (x_1, x_2)$, and where $\varphi^{\mathbf{H}_1}$, $\varphi^{\mathbf{H}_2}$ are univariate functions belonging to the class of one parameter refinable functions introduced in [9]. We recall that the refinement mask of a GP function of type $\varphi^{\mathbf{H}}$, $\mathbf{H} = (n, h)$, is supported on $[0, n+1]$ and has positive entries

$$a_\alpha^{\mathbf{H}} = \frac{1}{2^h}\left[\binom{n+1}{\alpha} + 4(2^{h-n} - 1)\binom{n-1}{\alpha-1}\right], \quad \alpha = 0, \ldots, n+1, \tag{7}$$

so that, whenever $n = h$, the function $\varphi^{\mathbf{H}}$ reduces to the B-splines of degree n.

It is worthwhile to note that the real parameter h, $h \geq n \geq 2$, is an additional parameter which turns out to be useful for getting higher flexibility in the applications.

It is easy to see that the symbol associated with the refinement mask in (7) is

$$p^{\mathbf{H}}(z) = \frac{1}{2^h}(1+z)^{n-1}(z^2 + (2^{h-n+2} - 2)z + 1)\,. \tag{8}$$

For any n and h, ($h \geq n > 2$), the function $\varphi^{\mathbf{H}}$ belongs to $C^{n-2}(\mathbb{R})$, is centrally symmetric and the function system $\{\varphi^{\mathbf{H}}(x-\alpha), \alpha \in \mathbb{Z}\}$ is linearly independent, stable and satisfies $\sum_{\alpha\in\mathbb{Z}} \varphi^{\mathbf{H}}(x-\alpha) = 1$ for all $x \in \mathbb{R}$. Moreover, the Fourier transform $\widehat{\varphi}^{\mathbf{H}}(\omega)$ vanishes if and only if $\omega \in 2\pi\mathbb{Z} \setminus \{0\}$.

With the $\varphi^{\mathbf{H}}$ refinable functions at hand we are able to construct a new class of bivariate refinable functions using the direction convolution product with direction $\mathbf{e} = (1, 1)$. We define the function $\Psi^{\mathbf{H}_1,\mathbf{H}_2}$ as

$$\begin{aligned} \Psi^{\mathbf{H}_1,\mathbf{H}_2}(x) \quad &:= (\Phi^{\mathbf{H}_1,\mathbf{H}_2} *_{\mathbf{e}} \chi)(x) \\ &= \int_0^1 \Phi^{\mathbf{H}_1,\mathbf{H}_2}(x - \mathbf{e}t)\,dt = \int_0^1 \varphi^{\mathbf{H}_1}(x_1 - t)\varphi^{\mathbf{H}_2}(x_2 - t)\,dt\,. \end{aligned} \tag{9}$$

Note that the support of $\Psi^{\mathbf{H}_1,\mathbf{H}_2}$ satisfies

$$supp(\Psi^{\mathbf{H}_1,\mathbf{H}_2}) \subset supp(\Phi^{\mathbf{H}_1,\mathbf{H}_2}) + [0,1]^2\,,$$

where $supp(\Phi^{\mathbf{H}_1,\mathbf{H}_2}) = [0, n_1+1] \times [0, n_2+1]$. Moreover, the function $\Psi^{\mathbf{H}_1,\mathbf{H}_2}$ is such that

$$\widehat{\Psi}^{\mathbf{H}_1,\mathbf{H}_2}(\omega_1,\omega_2) = \widehat{\varphi}^{\mathbf{H}_1}(\omega_1)\widehat{\varphi}^{\mathbf{H}_2}(\omega_2)\widehat{\chi}(\omega_1+\omega_2)\,, \tag{10}$$

and its refinement mask and associated symbol are

$$\begin{aligned} \mathbf{b}^{\mathbf{H}_1,\mathbf{H}_2} &= \left\{ \frac{(a_\alpha^{\mathbf{H}_1,\mathbf{H}_2} + a_{\alpha-\mathbf{e}}^{\mathbf{H}_1,\mathbf{H}_2})}{2},\ \alpha \in \mathbb{Z}^2 \right\}, \\ P^{\mathbf{H}_1,\mathbf{H}_2}(z) &= p^{\mathbf{H}_1}(z_1)p^{\mathbf{H}_2}(z_2)\tfrac{1}{2}(1 + z_1 z_2)\,, \end{aligned} \tag{11}$$

where $\mathbf{a}^{\mathbf{H}_1,\mathbf{H}_2} = \{a_{\alpha_1}^{\mathbf{H}_1} a_{\alpha_2}^{\mathbf{H}_2},\ \alpha = (\alpha_1,\alpha_2) \in \mathbb{Z}^2\}$ is the mask of the tensor product.
Last, due to the results in Section 2, $\Psi^{\mathbf{H}_1,\mathbf{H}_2}$ has linearly independent integer translates and it generates a multi-resolution analysis on $L^2(\mathbb{R}^2)$.

4. Examples

In this Section we show the refinement masks and the graphs of some refinable functions constructed using the directional convolution strategy.

We start by setting $\mathbf{H}_1 = \mathbf{H}_2 = (3, h)$, and $h \geq 3$. The refinement mask $\mathbf{a}^{(3,h)}$ of the univariate refinable functions for different values of h are listed below while the graphs of these functions, obtained by performing five steps of the subdivision algorithm, are shown in Figure 1.

$$\mathbf{a}^{(3,3)} = \tfrac{1}{2^3}\{1, 4, 6, 4, 1\}, \quad \mathbf{a}^{(3,4)} = \tfrac{1}{2^4}\{1, 8, 14, 8, 1\},$$

$$\mathbf{a}^{(3,8)} = \tfrac{1}{2^8}\{1, 128, 254, 128, 1\}.$$

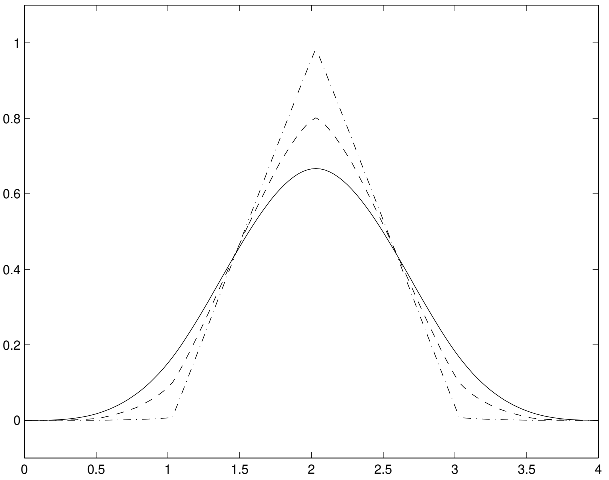

Figure 1: Graphs of the functions $\varphi^{(3,3)}(-)$, $\varphi^{(3,4)}(--)$ and $\varphi^{(3,8)}(.-)$

Note that $\varphi^{(3,3)}$ is just the cubic B-spline with uniform knots.

The bivariate refinement masks corresponding to the tensor product refinable functions $\Phi^{\mathbf{H}_1,\mathbf{H}_2}$ we construct from the previous functions for $\mathbf{H}_1 = \mathbf{H}_2 = (3, 3)$ and $\mathbf{H}_1 = \mathbf{H}_2 = (3, 4)$ are

$$\mathbf{a}^{(3,3),(3,3)} = \frac{1}{2^6}\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix},$$

$$\mathbf{a}^{(3,4),(3,4)} = \frac{1}{2^8}\begin{bmatrix} 1 & 8 & 14 & 8 & 1 \\ 8 & 64 & 112 & 64 & 8 \\ 14 & 112 & 196 & 112 & 14 \\ 8 & 64 & 112 & 64 & 8 \\ 1 & 8 & 14 & 8 & 1 \end{bmatrix}$$

while for $\mathbf{H}_1 = \mathbf{H}_2 = (3, 8)$ the refinement mask is

$$\mathbf{a}^{(3,8),(3,8)} = \frac{1}{2^{16}} \begin{bmatrix} 1 & 128 & 254 & 128 & 1 \\ 128 & 16384 & 32512 & 16384 & 128 \\ 254 & 32512 & 64516 & 32512 & 254 \\ 128 & 16384 & 32512 & 16384 & 128 \\ 1 & 128 & 254 & 128 & 1 \end{bmatrix}.$$

The associated refinable functions obtained by three steps of the corresponding subdivision algorithm are shown in Fig. 2, Fig. 3 and Fig. 4 where, for shortness, the function $\Phi^{\mathbf{H}_1,\mathbf{H}_2}$ with $\mathbf{H}_1 = \mathbf{H}_2$ is denoted just as $\Phi^{\mathbf{H}_1}$.

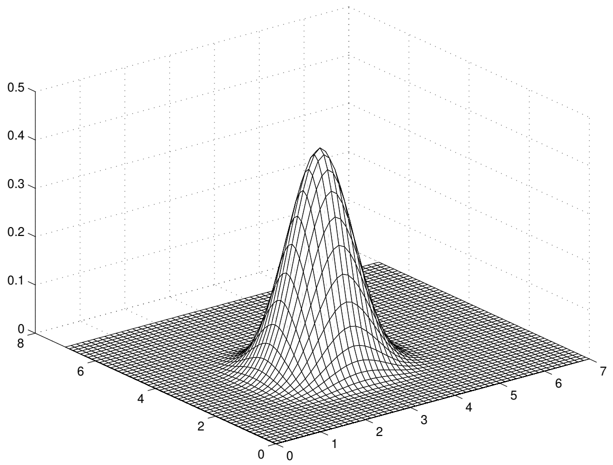

Figure 2: Graph of the function $\Phi^{(3,3)}$

Finally, the refinement mask of the convolved functions for $\mathbf{H}_1 = \mathbf{H}_2 = (3, 3)$ and $\mathbf{H}_1 = \mathbf{H}_2 = (3, 4)$ are

$$\mathbf{b}^{(3,3),(3,3)} = \frac{1}{2^{7}} \begin{bmatrix} 0 & 1 & 4 & 6 & 4 & 1 \\ 1 & 8 & 22 & 28 & 17 & 4 \\ 4 & 22 & 48 & 52 & 28 & 6 \\ 6 & 28 & 52 & 48 & 22 & 4 \\ 4 & 17 & 28 & 22 & 8 & 1 \\ 1 & 4 & 6 & 4 & 1 & 0 \end{bmatrix},$$

$$\mathbf{b}^{(3,4),(3,4)} = \frac{1}{2^{9}} \begin{bmatrix} 0 & 1 & 8 & 14 & 8 & 1 \\ 1 & 16 & 78 & 120 & 65 & 8 \\ 8 & 78 & 224 & 260 & 120 & 14 \\ 14 & 120 & 260 & 224 & 78 & 8 \\ 8 & 65 & 120 & 78 & 16 & 1 \\ 1 & 8 & 14 & 8 & 1 & 0 \end{bmatrix}$$

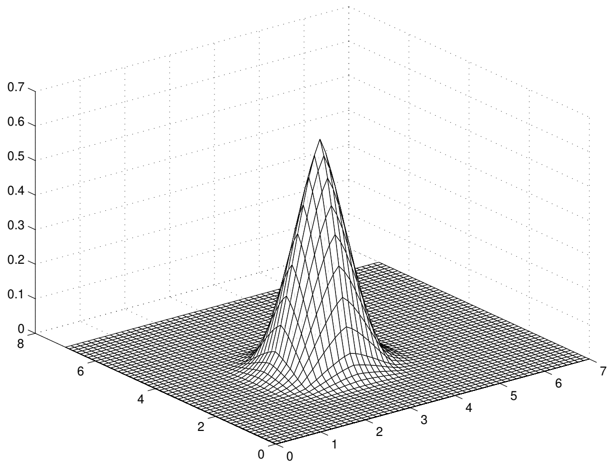

Figure 3: Graph of the function $\Phi^{(3,4)}$

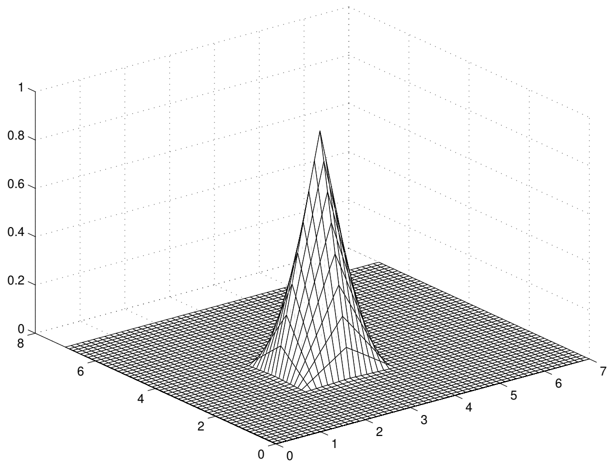

Figure 4: Graph of the function $\Phi^{(3,8)}$

and for $\mathbf{H}_1 = \mathbf{H}_2 = (3, 8)$

$$\mathbf{b}^{(3,8),(3,8)} = \frac{1}{2^{17}} \begin{bmatrix} 0 & 1 & 128 & 254 & 128 & 1 \\ 1 & 256 & 16638 & 32640 & 16385 & 128 \\ 128 & 16638 & 65024 & 80900 & 32640 & 254 \\ 254 & 32640 & 80900 & 65024 & 16638 & 128 \\ 128 & 16385 & 32640 & 16638 & 256 & 1 \\ 1 & 128 & 254 & 128 & 1 & 0 \end{bmatrix},$$

with corresponding graphs in Fig. 5, Fig. 6 and Fig. 7 (obtained, again, by three steps of the corresponding subdivision algorithm).

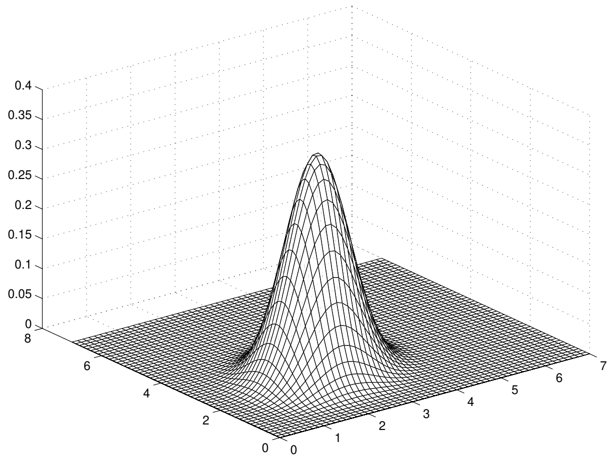

Figure 5: Graph of the function $\Psi^{(3,3)}$

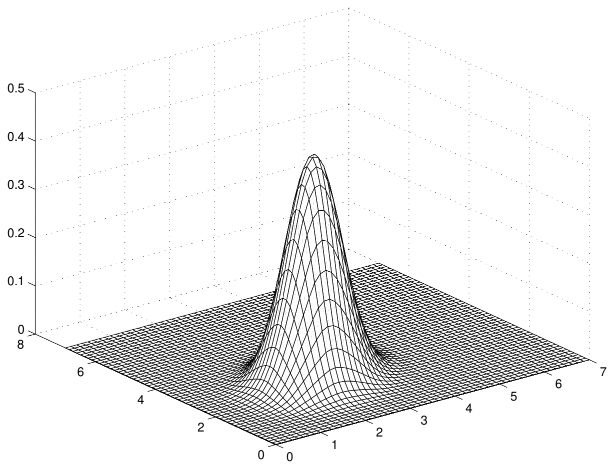

Figure 6: Graph of the function $\Psi^{(3,4)}$

Applications of the new refinable functions of type $\Psi^{\mathbf{H}_1,\mathbf{H}_2}$ are presently under investigation.

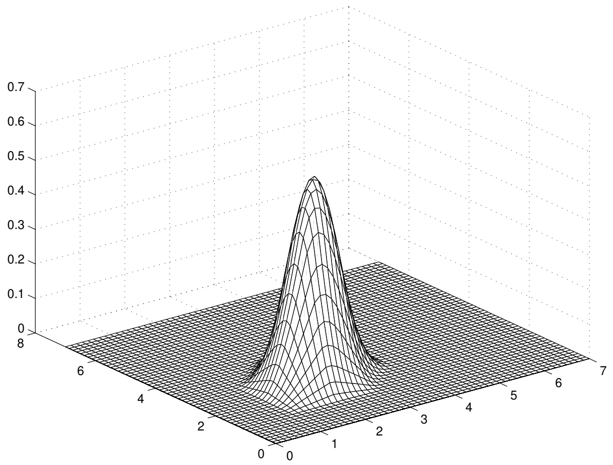

Figure 7: Graph of the function $\Psi^{(3,8)}$

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Costanza CONTI
Dip. di Energetica "Sergio Stecco"
Università di Firenze
Via Lombroso 6/17
50134 Firenze, ITALY
e-mail: `c.conti@ing.unifi.it`

Laura GORI, Francesca PITOLLI
Dip. Me.Mo.Mat.
Università di Roma "La Sapienza"
Via A. Scarpa 10
00161 Roma, ITALY
e-mail: `gori@dmmm.uniroma1.it`
`pitolli@dmmm.uniroma1.it`