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CONTENTS

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Preface

The present issue of the Rendiconti del Seminario Matematico, Università e Politecnico Torino, contains the texts of the courses by T. Gramchev, M. Reissig and M. Yoshino, delivered at the "Bimestre Intensivo Microlocal Analysis and Related Subjects".

The Bimestre was held in the frame of the activities INDAM, Istituto Nazionale di Alta Matematica, at the Departments of Mathematics of the University and Politecnico of Torino, during May and June 2003. More than 100 lecturers were given during the Bimestre, concerning different aspects of the Microlocal Analysis and related topics. We do not intend to present here the full Proceedings, and limit publication to the following 3 articles, representative of the high scientific level of the activities; they are devoted to new aspects of the general theory of the partial differential equations: perturbative methods in scales of Banach spaces, hyperbolic equations with non-Lipschitz coefficients, singular differential equations and Diophantine phenomena.

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PERTURBATIVE METHODS IN SCALES OF BANACH SPACES: APPLICATIONS FOR GEVREY REGULARITY OF SOLUTIONS TO SEMILINEAR PARTIAL DIFFERENTIAL EQUATIONS

Abstract. We outline perturbative methods in scales of Banach spaces of Gevrey functions for dealing with problems of the uniform Gevrey regularity of solutions to partial differential equations and nonlocal equations related to stationary and evolution problems. The key of our approach is to use suitably chosen Gevrey norms expressed as the limit for $N \to \infty$ of partial sums of the type

$$\sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| \leq N} \frac{T^{|\alpha|}}{(\alpha!)^\sigma} \| D_x^\alpha u \|_{H^s(\mathbb{R}^n)}$$

for solutions to semilinear elliptic equations in $\mathbb{R}^n$. We also show (sub)exponential decay in the framework of Gevrey functions from Gelfand-Shilov spaces $S_\nu^\mu(\mathbb{R}^n)$ using sequences of norms depending on two parameters

$$\sum_{\alpha, \beta \in \mathbb{Z}_+^n, |\alpha|+|\beta| \leq N} \frac{\varepsilon^{|\beta|} T^{|\alpha|}}{(\alpha!)^\mu (\beta!)^\nu} \| x^\beta D^\alpha u \|_{H^s(\mathbb{R}^n)}.$$

For solutions $u(t, \cdot)$ of evolution equations we employ norms of the type

$$\sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| \leq N} \sup_{0<t<T} \Big(\frac{t^\theta (\rho(t))^{|\alpha|}}{(\alpha!)^\sigma} \| D_x^\alpha u(t, \cdot) \|_{L^p(\mathbb{R}^n)} \Big)$$

for some $\theta \geq 0$, $1 < p < \infty$, $\rho(t) \searrow 0$ as $t \searrow 0$.

The use of such norms allows us to implement a Picard type scheme for seemingly different problems of uniform Gevrey regularity and to reduce the question to the boundedness of an iteration sequence $z_N(T)$ (which is one of the N-th partial sums above) satisfying inequalities of the type

$$z_{N+1}(T) \leq \delta_0 + C_0 T z_N(T) + g(T; z_N(T))$$

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with T being a small parameter, and g being at least quadratic in u near $u = 0$.

We propose examples showing that the hypotheses involved in our abstract perturbative approach are optimal for the uniform Gevrey regularity.

1. Introduction

The main aim of the present work is to develop a unified approach for investigating problems related to the uniform G^σ Gevrey regularity of solutions to PDE on the whole space $\mathbb{R}^n$ and the uniform Gevrey regularity with respect to the space variables of solutions to the Cauchy problem for semilinear parabolic systems with polynomial nonlinearities and singular initial data. Our approach works also for demonstrating exponential decay of solutions to elliptic equations provided we know a priori that the decay for $|x| \to \infty$ is of the type $o(|x|^{-\tau})$ for some $0 < \tau \ll 1$.

The present article proposes generalizations of the body of iterative techniques for showing Gevrey regularity of solutions to nonlinear PDEs in Mathematical Physics in papers of H.A. Biagioni* and the author.

We start by recalling some basic facts about the Gevrey spaces. We refer to [50] for more details. Let $\sigma \geq 1$, $\Omega \subset \mathbb{R}^n$ be an open domain. We denote by $G^\sigma(\mathbb{R}^n)$ (the Gevrey class of index σ) the set of all $f \in C^\infty(\Omega)$ such that for every compact subset $K \subset\subset \Omega$ there exists $C = C_{f,K} > 0$ such that

$$\sup_{\alpha \in \mathbb{Z}_+^n} \left(\frac{C^{|\alpha|}}{(\alpha!)^\sigma} \sup_{x \in K} |\partial_x^\alpha f(x)| \right) < +\infty,$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

Throughout the present paper we will investigate the regularity of solutions of stationary PDEs in $\mathbb{R}^n$ in the frame of the L^2 based uniformly Gevrey G^σ functions on $\mathbb{R}^n$ for $\sigma \geq 1$. Here $f \in G^\sigma_{un}(\mathbb{R}^n)$ means that for some $T > 0$ and $s \geq 0$

$$\sup_{\alpha \in \mathbb{Z}_+^n} \left(\frac{T^{|\alpha|}}{(\alpha!)^\sigma} \|\partial_x^\alpha f\|_s \right) < +\infty, \tag{1}$$

where $\|f\|_s = \|f\|_{H^s(\mathbb{R}^n)}$ stands for a $H^s(\mathbb{R}^n) = H_2^s(\mathbb{R}^n)$ norm for some $s \geq 0$. In particular, if $\sigma = 1$, we obtain that every $f \in G^1_{un}(\mathbb{R}^n)$ is extended to a holomorphic function in $\{z \in \mathbb{C}^n;\ |Im\, z| < T\}$. Note that given $f \in G^\sigma_{un}(\mathbb{R}^n)$ we can define

$$\rho_\sigma(f) = \sup\{T > 0 : \text{ such that (1) holds}\}. \tag{2}$$

One checks easily by the Sobolev embedding theorem and the Stirling formula that the definition (2) is invariant with respect to the choice of $s \geq 0$. One may call $\rho_\sigma(f)$ the uniform G^σ Gevrey radius of $f \in G^\sigma_{un}(\mathbb{R}^n)$.

*She has passed away on June 18, 2003 after struggling with a grave illness. The present paper is a continuation following the ideas and methods contained in [6], [7] and especially [8] and the author dedicates it to her memory.

We will use scales of Banach spaces of G^σ functions with norms of the following type

$$\sum_{k=0}^{\infty} \frac{T^{|k|}}{(k!)^\sigma} \sum_{j=0}^{n} \|D_j^k u\|_s, \quad D_j = D_{x_j}.$$

For global L^p based Gevrey norms of the type (1) we refer to [8], cf. [27] for local L^p based norms of such type, [26] for $|f|_\infty := \sup_\Omega |f|$ based Gevrey norms for the study of degenerate Kirchhoff type equations, see also [28] for similar scales of Banach spaces of periodic G^σ functions. We stress that the use $\sum_{j=1}^n \|D_j^k u\|_s$ instead of $\sum_{|\alpha|=k} \|D_x^\alpha u\|_s$ allows us to generalize with simpler proofs hard analysis type estimates for $G^\sigma_{un}(\mathbb{R}^n)$ functions in [8].

We point out that exponential G^σ norms of the type

$$\|u\|_{\sigma,T;exp} := \sqrt{\int_{\mathbb{R}^n} e^{2T|\xi|^{1/\sigma}} |\hat{u}(\xi)|^2 \, d\xi}$$

have been widely (and still are) used in the study of initial value problems for weakly hyperbolic systems, local solvability of semilinear PDEs with multiple characteristics, semilinear parabolic equations, (cf. [23], [6], [30] for $\sigma = 1$ and [12], [20], [27], [28] when $\sigma > 1$ for applications to some problems of PDEs and Dynamical Systems).

The abstract perturbative methods which will be exposed in the paper aim at dealing with 3 seemingly different problems. We write down three model cases.

1. First, given an elliptic linear constant coefficients partial differential operator P in $\mathbb{R}^n$ and an entire function f we ask whether one can find $s_{cr} > 0$ such that

$$(P1) \qquad \begin{array}{c} Pu + f(u) = 0, \; u \in H^s(\mathbb{R}^n), \, s > s_{cr} \\ \text{implies} \\ u \in \mathcal{O}\{z \in \mathbb{C}^n : |\Im z| < T\} \text{ for some } T > 0 \end{array}$$

while for (some) $s < s_{cr}$ the implication is false.

Recall the celebrated KdV equation

$$u_t - u_{xxx} - auu_x = 0 \qquad x \in \mathbb{R}, \; t > 0, \; a > 0 \tag{3}$$

or more generally the generalized KdV equation

$$u_t - u_{xxx} + au^p u_x = 0 \qquad x \in \mathbb{R}, \; t > 0 \; a > 0 \tag{4}$$

where p is an odd integer (e.g., see [34] and the references therein). We recall that a solution u in the form $u(x,t) = v(x+ct)$, $v \neq 0$, $c \in \mathbb{R}$, is called solitary (traveling) wave solution. It is well known that v satisfies the second order Newton equation (after plugging $v(x+ct)$ in (4) and integrating)

$$v'' - cv + \frac{a}{p+1} v^{p+1} = 0, \tag{5}$$

and if $c > 0$ we have a family of explicit solutions

$$v_c(x) = \frac{C_{p,a}}{(\cosh((p-1)\sqrt{c}x))^{2/(p+1)}} \qquad x \in \mathbb{R}, \tag{6}$$

for explicit positive constant $C_{p,a}$.

Incidentally, $u_c(t,x) = e^{ict}v_c(x)$, $c > 0$ solves the nonlinear Schrödinger equation

$$iu_t - u_{xx} + a|u|^p u = 0 \qquad x \in \mathbb{R},\ t > 0\ a > 0 \tag{7}$$

and is called also stationary wave solution cf. [11], [34] and the references therein.

Clearly the solitary wave v_c above is uniformly analytic in the strip $|\Im x| \le T$ for all $0 < T < \pi/((p-1)\sqrt{c})$. One can show that the uniform G^1 radius is given by $\rho_1[v_s] = \pi/(((p-1)\sqrt{c})$.

In the recent paper of H. A. Biagioni and the author [8] an abstract approach for attacking the problem of uniform Gevrey regularity of solutions to semilinear PDEs has been proposed. One of the key ingredients was the introduction of L^p based norms of $G^\sigma_{un}(\mathbb{R}^n)$ functions which contain infinite sums of fractional derivatives in the non-analytic case $\sigma > 1$. Here we restrict our attention to simpler L^2 based norms and generalize the results in [8] with simpler proofs. The hard analysis part is focused on fractional calculus (or generalized Leibnitz rule) for nonlinear maps in the framework of $L^2(\mathbb{R}^n)$ based Banach spaces of uniformly Gevrey functions $G^\sigma_{un}(\mathbb{R}^n)$, $\sigma \ge 1$. In particular, we develop functional analytic approaches in suitable scales of Banach spaces of Gevrey functions in order to investigate the $G^\sigma_{un}(\mathbb{R}^n)$ regularity of solutions to semilinear equations with Gevrey nonlinearity on the whole space $\mathbb{R}^n$:

$$Pu + f(u) = w(x), \qquad x \in \mathbb{R}^n \tag{8}$$

where P is a Gevrey G^σ pseudodifferential operator or a Fourier multiplier of order m, and $f \in G^\theta$ with $1 \le \theta \le \sigma$. The crucial hypothesis is some G^σ_{un} estimates of commutators of P with $D^\alpha_j := D^\alpha_{x_j}$

If $n = 1$ we capture large classes of dispersive equations for solitary waves (cf. [4], [21], [34], [42], for more details, see also [1], [2], [10] and the references therein).

Our hypotheses are satisfied for: $P = -\Delta + V(x)$, where the real potential $V(x) \in G^\sigma_{un}(\mathbb{R}^n)$ is real valued, bounded from below and $\lim_{|x|\to\infty} V(x) = +\infty$; P being an arbitrary linear elliptic differential operator with constant coefficients. We allow also the order m of P to be less than one (cf. [9] for the so called fractal Burgers equations, see also [42, Theorem 10, p.51], where G^σ, $\sigma > 1$, classes are used for the Whitham equation with antidissipative terms) and in that case the Gevrey index σ will be given by $\sigma \ge 1/m > 1$. We show $G^\sigma_{un}(\mathbb{R}^n)$ regularity of every solution $u \in H^s(\mathbb{R}^n)$ with $s > s_{cr}$, depending on n, the order of P and the type of nonlinearity. For general analytic nonlinearities, $s_{cr} > n/p$. However, if $f(u)$ is polynomial, s_{cr} might be taken less than $n/2$, and in that case s_{cr} turns out to be related to the critical index of the singularity of the initial data for semilinear parabolic equations, cf. [15], [5] [49] (see also [25] for $H^s(\mathbb{R}^n) := H^s_2(\mathbb{R}^n)$, $0 < s < n/2$ solutions in $\mathbb{R}^n$, $n \ge 3$, to semilinear elliptic equations).

The proof relies on the nonlinear Gevrey calculus and iteration inequalities of the type $z_{N+1}(T) \le z_0(T) + g(T, z_N(T))$, $N \in \mathbb{Z}_+$, $T > 0$ where $g(T, 0) = 0$ and

$$z_N(T) = \sum_{k=0}^{N} \frac{T^k}{(k!)^\sigma} \sum_{j=1}^{n} \|D_j^k u\|_s. \tag{9}$$

Evidently the boundedness of $\{z_N(T)\}_{N=1}^{\infty}$ for some $T > 0$ implies that $z_{+\infty}(T) = \|u\|_{\sigma,T;s} < +\infty$, i.e., $u \in G^\sigma_{un}(R^n)$. We recover the results of uniform analytic regularity of dispersive solitary waves (cf. J. Bona and Y. Li, [11], [40]), and we obtain $G^\sigma_{un}(\mathbb{R}^n)$ regularity for $u \in H^s(\mathbb{R}^n)$, $s > n/2$ being a solution of equations of the type $-\Delta u + V(x)u = f(u)$, where $f(u)$ is polynomial, $\nabla V(x)$ satisfies (1) and for some $\mu \in \mathbb{C}$ the operator $(-\Delta + V(x) - \mu)^{-1}$ acts continuously from $L^2(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$. An example of such $V(x)$ is given by $V(x) = V_\sigma(x) =< x >^\rho \exp(-\frac{1}{|x|^{1/(\sigma-1)}})$ for $\sigma > 1$, and $V(x) =< x >^\rho$ if $\sigma = 1$, for $0 < \rho \le 1$, where $< x >= \sqrt{1 + x^2}$. In fact, we can capture also cases where $\rho > 1$ (like the harmonic oscillator), for more details we send to Section 3.

We point out that our results imply also uniform analytic regularity $G^1_{un}(\mathbb{R})$ of the $H^2(\mathbb{R})$ solitary wave solutions $r(x - ct)$ to the fifth order evolution PDE studied by M. Groves [29] (see Remark 2 for more details).

Next, modifying the iterative approach we obtain also new results for the analytic regularity of stationary type solutions which are bounded but not in $H^s(\mathbb{R}^n)$. As an example we consider Burgers' equation (cf. [32])

$$u_t - \nu u_{xx} + u u_x = 0, \qquad x \in \mathbb{R},\ t > 0 \tag{10}$$

which admits the solitary wave solution $\varphi_c(x + ct)$ given by

$$\varphi_c(x) = \frac{2c}{ae^{-cx} + 1}, \qquad x \in \mathbb{R}. \tag{11}$$

for $a \ge 0$, $c \in \mathbb{R} \setminus 0$. Clearly φ_c extends to a holomorphic function in the strip $|\Im x| < \pi/|c|$ while $\lim_{x \to \mathrm{sign}(c)\infty} \varphi_c(x) = 2c$ and therefore $\varphi_c \notin L^2(\mathbb{R})$. On the other hand

$$\varphi_c'(x) = \frac{2cae^{-cx}}{(ae^{-cx} + 1)^2}, \qquad x \in \mathbb{R}. \tag{12}$$

One can show that $\varphi_c' \in G^1_{un}(\mathbb{R}^n)$. It was shown in [8], Section 5, that if a bounded traveling wave satisfies in addition $v' \in H^1(\mathbb{R})$ then $v' \in G^1_{un}(\mathbb{R}^n)$. We propose generalizations of this result. We emphasize that we capture as particular cases the bore-like solutions to dissipative evolution PDEs (Burgers' equation, the Fisher-Kolmogorov equation and its generalizations cf. [32], [37], [31], see also the survey [55] and the references therein).

We exhibit an explicit recipe for constructing strongly singular solutions to higher order semilinear elliptic equations with polynomial nonlinear terms, provided they have suitable homogeneity properties involving the nonlinear terms (see Section 6). In such a way we generalize the results in [8], Section 7, where strongly singular solutions of $-\Delta u + cu^d = 0$ have been constructed. We give other examples of weak nonsmooth solutions to semilinear elliptic equations with polynomial nonlinearity which are in $H^s(\mathbb{R}^n)$, $0 < s < n/2$ but with $s \leq s_{cr}$ cf. [25] for the particular case of $-\Delta u + cu^{2k+1} = 0$ in $\mathbb{R}^n$, $n \geq 3$. The existence of such classes of singular solutions are examples which suggest that our requirements for initial regularity of the solution are essential in order to deduce uniform Gevrey regularity. This leads to, roughly speaking, a kind of dichotomy for classes of elliptic semilinear PDE's in $\mathbb{R}^n$ with polynomial nonlinear term, namely, that any solution is either extendible to a holomorphic function in a strip $\{z \in \mathbb{C}^n : |Imz| \leq T\}$, for some $T > 0$, or for some specific nonlinear terms the equation admits solutions with singularities (at least locally) in $H^s_p(\mathbb{R}^n)$, $s < s_{cr}$.

2. The second aim is motivated by the problem of the type of decay - polynomial or exponential - of solitary (traveling) waves (e.g., cf. [40] and the references therein), which satisfy frequently nonlocal equations. We mention also the recent work by P. Rabier and C. Stuart [48], where a detailed study of the pointwise decay of solutions to second order quasilinear elliptic equations is carried out (cf also [47]).

The example of the solitary wave (6) shows that we have both uniform analyticity and exponential decay. In fact, by the results in [8], Section 6, one readily obtains that v_c defined in (6) belongs to the Gelfand–Shilov class $S^1(\mathbb{R}^n) = S^1_1(\mathbb{R}^n)$. We recall that given $\mu > 0, \nu > 0$ the Gelfand-Shilov class $S^\nu_\mu(\mathbb{R}^n)$ is defined as the set of all $f \in G^\mu(\mathbb{R}^n)$ such that there exist positive constants C_1 and C_2 satisfying

$$|\partial^\alpha_x f(x)| \leq C_1^{|\alpha|+1}(\alpha!)^\nu e^{-C_2|x|^{1/\mu}}, \qquad x \in \mathbb{R}^n, \alpha \in \mathbb{Z}^n_+. \tag{13}$$

We will use a characterization of $S^\nu_\mu(\mathbb{R}^n)$ by scales of Banach spaces with norms

$$|||f|||_{\mu,\nu;\varepsilon,T} \quad = \quad \sum_{j,k\in\mathbb{Z}^n_+} \frac{\varepsilon^{|j|}T^{|k|}}{(j!)^\nu(k!)^\mu}\|x^j D^k_x u\|_s.$$

In particular, $S^\nu_\mu(\mathbb{R}^n)$ contains nonzero functions iff $\mu+\nu \geq 1$ (for more details on these spaces we refer to [24], [46], see also [17], [18] for study of linear PDE in $S_\theta(\mathbb{R}^n) := S^\theta_\theta(\mathbb{R}^n)$).

We require three essential conditions guaranteeing that every solution $u \in H^s(\mathbb{R}^n)$, $s > s_{cr}$ of (8) for which it is known that it decays polynomially for $|x| \to \infty$ necessarily belongs to $S^\mu_\nu(\mathbb{R}^n)$ (i.e., it satisfies (13) or equivalently $|||u|||_{\mu,\nu;\varepsilon,T} < +\infty$ for some $\varepsilon > 0$, $T > 0$). Namely: the operator P is supposed to be invertible; f has no linear term, i.e., f is at least quadratic near the origin; and finally, we require that the $H^s(\mathbb{R}^n)$ based norms of commutators of P^{-1} with operators of the type $x^\beta D^\alpha_x$ satisfy certain analytic–Gevrey estimates for all $\alpha, \beta \in \mathbb{Z}^n_+$. The key is again an iterative approach, but this time one has to derive more subtle estimates involving partial sums for the Gevrey norms $|||f|||_{\mu,\nu;\varepsilon,T}$ of the type

$$z_N(\mu,\nu;\varepsilon,T) \quad = \quad \sum_{|j+k|\leq N} \frac{\varepsilon^{|j|}T^{|k|}}{(j!)^\nu(k!)^\mu}\|x^j D^k_x u\|_s.$$

The (at least) quadratic behaviour is crucial for the aforementioned gain of the rate of decay for $|x| \to 0$ and the technical arguments resemble some ideas involved in the Newton iterative method. If $\mu = \nu = 1$ we get the decay estimates in [8], and as particular cases of our general results we recover the well known facts about polynomial and exponential decay of solitary waves, and obtain estimates for new classes of stationary solutions of semilinear PDEs. We point out that different type of G^1_{un} Gevrey estimates have been used for getting better large time decay estimates of solutions to Navier–Stokes equations in $\mathbb{R}^n$ under the assumption of initial algebraic decay (cf. M. Oliver and E. Titi [44]).

As it concerns the sharpness of the three hypotheses, examples of traveling waves for some nonlocal equations in Physics having polynomial (but not exponential) decay for $|x| \to 0$ produce counterexamples when (at least some of the conditions) fail.

3. The third aim is to outline iterative methods for the study of the Gevrey smoothing effect of semilinear parabolic systems for positive time with singular initial data. More precisely, we consider the Cauchy problem of the type

$$\partial_t u + (-\Delta)^m u + f(u) = 0, \quad u|_{t=0} = u^0, \qquad t > 0, \; x \in \Omega, \tag{14}$$

where $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{T}^n$. We investigate the influence of the elliptic dissipative terms of evolution equations in $\mathbb{R}^n$ and $\mathbb{T}^n$ on the critical L^p, $1 \le p \le \infty$, index of the singularity of the initial data u^0, the analytic regularity with respect to $x \in \Omega$ for positive time and the existence of self-similar solutions. The approach is based again on the choice of suitable L^p based Banach spaces with timedepending Gevrey norms with respect to the space variables x and then fixed point type iteration scheme.

The paper is organized as follows. Section 2 contains several nonlinear calculus estimates for Gevrey norms. Section 3 presents an abstract approach and it is dedicated to the proof of uniform Gevrey regularity of a priori $H^s(\mathbb{R}^n)$ solutions u to semilinear PDEs, while Section 4 deals with solutions u which are bounded on $\mathbb{R}^n$ such that $\nabla u \in H^s(\mathbb{R}^n)$. We prove Gevrey type exponential decay results in the frame of the Gelfand-Shilov spaces $S^\mu_\nu(\mathbb{R}^n)$ in Section 5. Strongly singular solutions to semilinear elliptic equations are constructed in Section 6. The last two sections deal with the analytic-Gevrey regularizing effect in the space variables for solutions to Cauchy problems for semilinear parabolic systems with polynomial nonlinearities and singular initial data.

2. Nonlinear Estimates in Gevrey Spaces

Given $s > n$, $T > 0$ we define

$$G^\sigma(T; H^s) = \{v : \|v\|_{\sigma,T;s} := \sum_{k=0}^{\infty} \sum_{j=1}^{n} \frac{T^k}{(k!)^\sigma} \|D^k_{x_j} v\|_s < +\infty\}, \tag{15}$$

and

$$(16)\quad G^{\sigma}_{\infty}(T;H^s)=\{v:|||v|||_{\sigma,T;s}=\|v\|_{L^\infty}+\sum_{k=0}^{+\infty}\sum_{j=0}^{n}\frac{T^k}{(k!)^\sigma}\|D_j^k\nabla v\|_s<+\infty\}.$$

We have

LEMMA 1. *Let $s>n/2$. Then the spaces $G^\sigma(T;H^s)$ and $G^\sigma_\infty(T;H^s)$ are Banach algebras.*

We omit the proof since the statement for $G^\sigma(T;H^s)$ is a particular case of more general nonlinear Gevrey estimates in [27]) while the proof for $G^\sigma_\infty(T;H^s)$ is essentially the same.

We need also a technical assertion which will play a crucial role in deriving some nonlinear Gevrey estimates in the next section.

LEMMA 2. *Given $\rho\in(0,1)$, we have*

$$(17)\qquad \|<D>^{-\rho}D_j^k u\|_s\le\varepsilon\|D_j^k u\|_s+(1-\rho)\left(\frac{\rho}{\varepsilon}\right)^{1/(1-\rho)}\|D_j^{k-1}u\|_s$$

for all $k\in\mathbb{N}$, $s\ge 0$, $u\in H^{s+k}(\mathbb{R}^n)$, $j=1,\dots,n$, $\varepsilon>0$. Here $<D>$ stands for the constant p.d.o. with symbol $<\xi>=(1+|\xi|^2)^{1/2}$.

Proof. We observe that $<\xi>^{-\rho}|\xi_j|^k\le|\xi_j|^{k-\rho}$ for $j=1,\dots,n$, $\xi\in\mathbb{R}^n$. Set $g_\varepsilon(t)=\varepsilon t-t^\rho$, $t\ge 0$. Straightforward calculations imply

$$\min_{g\in\mathbb{R}}g(t)=g((\frac{\rho}{\varepsilon})^{1/1-\rho})=-(1-\rho)\left(\frac{\rho}{\varepsilon}\right)^{1/(1-\rho)}$$

which concludes the proof. □

We show some combinatorial inequalities which turn out to be useful in for deriving nonlinear Gevrey estimates (cf [8]).

LEMMA 3. *Let $\sigma\ge 1$. Then there exists $C>0$ such that*

$$(18)\qquad \frac{\ell!(\sigma\ell_\mu+r)!\prod_{\nu\ne\mu}(\sigma\ell_\nu)!}{\ell_1!\cdots\ell_j!(\sigma\ell+r)!}\le C^j,$$

for all $j\in\mathbb{N}$, $\ell=\ell_1+\cdots+\ell_j$, $\ell_i\in\mathbb{N}$, $\mu\in\{1,\dots,j\}$ and $0\le r<\sigma$, with $k!:=\Gamma(k+1)$, $\Gamma(z)$ being the Gamma function.

Proof. By the Stirling formula, we can find two constants $C_2>C_1>0$ such that

$$C_1\frac{k^{k+\frac12}}{e^k}\le k!\le C_2\frac{k^{k+\frac12}}{e^k}$$

for all $k \in \mathbb{N}$. Then the left–hand side in (18) can be estimated by:

$$
\begin{aligned}
&\frac{C_2^{j+1}\ell^{\ell+\frac{1}{2}}(\sigma\ell_\mu+r)^{\sigma\ell_\mu+r+\frac{1}{2}}\prod_{\nu\neq\mu}(\sigma\ell_\nu)^{\sigma\ell_\nu+\frac{1}{2}}}{C_1^{j+1}\ell_1^{\ell_1+\frac{1}{2}}\cdots\ell_j^{\ell_j+\frac{1}{2}}(\sigma\ell+r)^{\sigma\ell+r+\frac{1}{2}}}\\
=\ &\left(\frac{C_2}{C_1}\right)^{j+1}\frac{\ell^\ell(\sigma\ell_\mu+r)^{\sigma\ell_\mu+r}\prod_{\nu\neq\mu}(\sigma\ell_\nu)^{\sigma\ell_\nu}}{\prod_{\nu=1}^j\ell_\nu^{\ell_\nu}(\sigma\ell+r)^{\sigma\ell+r}}\left[\frac{\ell(\sigma\ell_\mu+r)}{\ell_\mu(\sigma\ell+r)}\right]^{\frac{1}{2}}\sigma^{\frac{j-1}{2}}\\
\leq\ &C_3^j\frac{\ell^\ell(\sigma\ell_\mu+r)^{\sigma\ell_\mu+r}\sigma^{\sigma(\ell-\ell_\mu)}[\prod_{\nu\neq\mu}\ell_\nu^{\ell_\nu}]^\sigma}{\prod_{\nu=1}^j\ell_\nu^{\ell_\nu}(\sigma\ell+r)^{\sigma\ell+r}}\\
\leq\ &C_3^j\frac{\ell^\ell(\sigma\ell_\mu+r)^{\sigma\ell_\mu}\sigma^{\sigma(\ell-\ell_\mu)}[\prod_{\nu\neq\mu}\ell_\nu^{\ell_\nu}]^{\sigma-1}}{\ell_\mu^{\ell_\mu}(\sigma\ell+r)^{\sigma\ell}}\\
=\ &C_3^j\frac{\ell^\ell(\ell_\mu+\frac{r}{\sigma})^{\sigma\ell_\mu}\left[\prod_{\nu\neq\mu}\ell_\nu^{\ell_\nu}\right]^{\sigma-1}}{\ell_\mu^{\ell_\mu}(\ell+\frac{r}{\sigma})^{\sigma\ell}}\\
=\ &C_3^j\frac{\ell^\ell(\ell_\mu+\frac{r}{\sigma})^{(\sigma-1)\ell_\mu}(\ell_\mu+\frac{r}{\sigma})^{\ell_\mu}\left[\prod_{\nu\neq\mu}\ell_\nu^{\ell_\nu}\right]^{\sigma-1}}{(\ell+\frac{r}{\sigma})^\ell(\ell+\frac{r}{\sigma})^{(\sigma-1)\ell}\ell_\mu^{\ell_\mu}}\\
\leq\ &C_3^je^{\frac{r}{\sigma}}\left[\frac{(\ell_\mu+\frac{r}{\sigma})^{\ell_\mu}\prod_{\nu\neq\mu}\ell_\nu^{\ell_\nu}}{(\ell+\frac{r}{\sigma})^{\ell_1+\cdots+\ell_j}}\right]^{\sigma-1}\leq C_3^je^{\frac{r}{\sigma}},\qquad N\in\mathbb{N}
\end{aligned}
$$

which implies (18) since $0 < r \leq \sigma$.

□

Given $s > n/2$ we associate two N-th partial sums for the norm in (15)

$$S_N^\sigma[v;T,s] \quad = \quad \sum_{k=0}^N\frac{T^k}{(k!)^\sigma}\sum_{j=1}^n\|D_{x_j}^kv\|_s, \tag{19}$$

$$\widetilde{S}_N^\sigma[v;T,s] \quad = \quad \sum_{k=1}^N\frac{T^k}{(k!)^\sigma}\sum_{j=1}^n\|D_{x_j}^kv\|_s. \tag{20}$$

Clearly (19) and (20) yield

$$S_N^\sigma[v;T,s] \quad = \quad \|v\|_s+\widetilde{S}_N^\sigma[v;T,s]. \tag{21}$$

LEMMA 4. *Let $f \in G^\theta(Q)$ for some $\theta \geq 1$, where $Q \subset \mathbb{R}^p$ is an open neighbourhood of the origin in $\mathbb{R}^p$, $p \in \mathbb{N}$ satisfying $f(0) = 0$, $\nabla f(0) = 0$. Then for $v \in H^\infty(\mathbb{R}^n : \mathbb{R}^p)$ there exists a positive constant A_0 depending on $\|v\|_s$, $\rho_\theta(f|_{B_{|v|_\infty}})$, where B_R stands for the ball with radius R, such that*

$$\widetilde{S}_N^\sigma[f(v);T,s] \leq |\nabla f(v)|_\infty\widetilde{S}_N^\sigma[v;T,s]+\sum_{j\in\mathbb{Z}_+^p,2\leq|j|\leq N}\frac{A_0^j}{(j!)^{\sigma-\theta}}(\widetilde{S}_{N-1}^\sigma[v;T,s])^j, \tag{22}$$

for $T > 0$, $N \in \mathbb{N}$, $N \geq 2$.

Proof. Without loss of generality, in view of the choice of the H^s norm, we will carry out the proof for $p = n = 1$. First, we recall that

$$
\begin{aligned}
D^k(f(v(x))) &= \sum_{j=1}^{k} \frac{(D^j f)(v(x))}{j!} \sum_{\substack{k_1+\cdots+k_j=k \\ k_1\geq 1,\cdots,k_j\geq 1}} \prod_{\mu=1}^{j} \frac{D^{k_\mu} v(x)}{k_\mu!} \\
&= f'(v(x)) D^k v(x) \\
&+ \sum_{j=2}^{k} \frac{(D^j f)(v(x))}{j!} \sum_{\substack{k_1+\cdots+k_j=k \\ k_1\geq 1,\cdots,k_j\geq 1}} \prod_{\mu=1}^{j} \frac{D^{k_\mu} v(x)}{k_\mu!}.
\end{aligned} \tag{23}
$$

Thus

$$
\begin{aligned}
\widetilde{S}_N^\sigma[f(v);T,s] &\leq \omega_s \|f'(v)\|_s \widetilde{S}_N^\sigma[v;T,s] + \sum_{k=1}^{N} \sum_{j=1}^{k} \frac{\|(D^j f)(v)\|_s)}{(j!)^\theta} \frac{\omega_s^j}{(j)!^{\sigma-\theta}} \\
&\times \sum_{\substack{k_1+\cdots+k_j=k \\ k_1\geq 1,\cdots,k_j\geq 1}} M^{\sigma,j}_{k_1,\dots,k_j} \prod_{\mu=1}^{j} \frac{T^{k_\mu} \|D^{k_\mu} v\|_s}{(k\mu!)^\sigma}
\end{aligned} \tag{24}
$$

where ω_s is the best constant in the Schauder Lemma for $H^s(\mathbb{R}^n)$, $s > n/2$, and

$$
M^{\sigma,j}_{k_1,\dots,k_j} = \left(\frac{k_1!\cdots k_j! j!}{(k_1+\cdots+k_j)!} \right)^{\sigma-1}, \qquad j, k_\mu \in \mathbb{N}, k_\mu \geq 1, \mu = 1,\dots,j. \tag{25}
$$

We get, thanks to the fact that $k_\mu \geq 1$ for every $\mu = 1,\dots,j$, that

$$
M^{\sigma,j}_{k_1,\dots,k_j} \leq 1, \qquad k_\mu \in \mathbb{N}, k_\mu \geq 1, \mu = 1,\dots,j \tag{26}
$$

(see [27]). Combining (26) with nonlinear superposition Gevrey estimates in [27] we obtain that there exists $A_0 = A_0(f, \|v\|_s) > 0$ such that

$$
\omega_s^j \frac{\|(D^j f)(v)\|_s}{(j!)^\theta} \leq A_0^j, \qquad j \in \mathbb{N}. \tag{27}
$$

We estimate (24) by

$$\begin{aligned}
\widetilde{S}_N^\sigma[f(v);T,s] &\le \omega_s\|f'(v)\|_s\widetilde{S}_N^\sigma[v;T,s]\\
&+ \sum_{k=2}^{N}\sum_{j=2}^{k}\frac{\|(D^jf)(v)\|_s)}{(j!)^\theta}\frac{\omega_s^j}{(j)!^{\sigma-\theta}}\\
&\times \sum_{\substack{k_1+\cdots+k_j=k\\ k_1\ge 1,\cdots,k_j\ge 1}}\prod_{\mu=1}^{j}\frac{T^{k_\mu}\|D^{k_\mu}v\|_s}{(k\mu!)^\sigma}\\
&\le \omega_s\|f'(v)\|_s\widetilde{S}_N^\sigma[v;T,s]\\
&+ \sum_{j=2}^{N}\frac{A_0^j}{(j)!^{\sigma-\theta}}(\widetilde{S}_{N-1}^\sigma[v;T,s])^j. \qquad (28)
\end{aligned}$$

The proof is complete.

□

We also propose an abstract lemma which will be useful for estimating Gevrey norms by means of classical iterative Picard type arguments.

LEMMA 5. *Let $a(T)$, $b(T)$, $c(T)$ be continuous nonnegative functions on* $[0,+\infty[$ *satisfying $a(0)=0$, $b(0)<1$, and let $g(z)$ be a nonzero real–valued nonnegative $C^1[0,+\infty)$ function, such that $g'(z)$ is nonnegative increasing function on* $(0,+\infty)$ *and*

$$g(0)=g'(0)=0.$$

Then there exists $T_0>0$ such that

a) for every $T\in]0,T_0]$ the set $F_T=\{z>0; z=a(T)+b(T)z+c(T)g(z)\}$ is not empty.

b) Let $\{z_k(T)\}_1^{+\infty}$ be a sequence of continuous functions on $[0,+\infty[$ *satisfying*

$$z_{k+1}(T)\le a(T)+b(T)z_k(T)+g(z_k(T)),\ z_0(T)\le a(T), \qquad (29)$$

for all $k\in\mathbb{Z}_+$. Then necessarily $z_k(T)$ is bounded sequence for all $T\in]0,T_0]$.

The proof is standard and we omit it (see [8], Section 3 for a similar abstract lemma).

3. Uniform Gevrey regularity of $H^s(\mathbb{R}^n)$ solutions

We shall study semilinear equations of the following type

$$Pv(x)=f[v](x)+w(x), \qquad x\in\mathbb{R}^n \qquad (30)$$

where $w \in G^\sigma(T; H^s)$ for some fixed $\sigma \geq 1$, $T_0 > 0$, $s > 0$ to be fixed later, P is a linear operator on $\mathbb{R}^n$ of order $\tilde{m} > 0$, i.e. acting continuously from $H^{s+\tilde{m}}(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$ for every $s \in \mathbb{R}$, and $f[v] = f(v, \ldots, D^\gamma v, \ldots)_{|\gamma| \leq m_0}$, $m_0 \in \mathbb{Z}_+$, with $0 \leq m_0 < \tilde{m}$ and

$$f \in G^\theta(\mathbb{C}^L), \qquad f(0) = 0 \tag{31}$$

where $L = \sum_{\gamma \in \mathbb{Z}_+^n} 1$.

We suppose that there exists $m \in]m_0, \tilde{m}]$ such that P admits a left inverse P^{-1} acting continuously

$$P^{-1} : H^s(\mathbb{R}^n) \to H^{s+m}(\mathbb{R}^n), \qquad s \in \mathbb{R}. \tag{32}$$

We note that since $f[v]$ may contain linear terms we have the freedom to replace P by $P + \lambda$, $\lambda \in \mathbb{C}$. By (32) the operator P becomes hypoelliptic (resp., elliptic if $\tilde{m} = m$) globally in $\mathbb{R}^n$ with $\tilde{m} - m$ being called the loss of regularity (derivatives) of P. We define the critical Gevrey index, associated to (30) and (32) as follows

$$\sigma_{crit} = \max\{1, (m - m_0)^{-1}, \theta\}.$$

Our second condition requires Gevrey estimates on the commutators of P with D_j^k, namely, there exist $s > n/2 + m_0$, $C > 0$ such that

$$\|P^{-1}[P, D_p^k]v\|_s \leq (k!)^\sigma \sum_{0 \leq \ell \leq k-1} \frac{C^{k-\ell+1}}{(\ell!)^\sigma} \sum_{j=1}^n \|D_j^\ell v\|_s \tag{33}$$

for all $k \in \mathbb{N}$, $p = 1, \ldots, n$, $v \in H^{k-1}(\mathbb{R}^n)$.

We note that all constant p.d.o. and multipliers satisfy (33). Moreover, if P is analytic p.d.o. (e.g., cf. [13], [50]), then (33) holds as well for the L^2 based Sobolev spaces $H^s(\mathbb{R}^n)$.

If $v \in H^s(\mathbb{R}^n)$, $s > m_0 + \frac{n}{2}$, solves (30), standard regularity results imply that $v \in H^\infty(\mathbb{R}^n) = \bigcap_{r>0} H^r(\mathbb{R}^n)$.

We can start by $v \in H^{s_0}(\mathbb{R}^n)$ with $s_0 \leq m_0 + \frac{n}{2}$ provided f is polynomial. More precisely, we have

LEMMA 6. *Let $f[u]$ satisfy the following condition: there exist $0 < s_0 < m_0 + \frac{n}{2}$ and a continuous nonincreasing function*

$$\kappa(s),\ s \in [s_0, \frac{n}{2} + m_0[, \quad \kappa(s_0) < m - m_0, \quad \lim_{s \to \frac{n}{p} + m_0} \kappa(s) = 0$$

such that

$$f \in C(H^s(\mathbb{R}^n) : H^{s-m_0-\kappa(s)}(\mathbb{R}^n)), \qquad s \in [s_0, \frac{n}{2} + m_0[. \tag{34}$$

Then every $v \in H^{s_0}(\mathbb{R}^n)$ solution of (30) belongs to $H^\infty(\mathbb{R}^n)$.

Proof. Applying P^{-1} to (30) we get $v = P^{-1}(f[v] + w)$. Therefore, (34) and (32) lead to $v \in H^{s_1}$ with $s_1 = s_0 - m_0 - \kappa(s_0) + m > s_0$. Since the gain of regularity $m - m_0 - \kappa(s) > 0$ increases with s, after a finite number of steps we surpass $\frac{n}{2}$ and then we get $v \in H^\infty(\mathbb{R}^n)$.

□

REMARK 1. Let $f[u] = (D_x^{m_0} u)^d,\ d \in \mathbb{N},\ d \geq 2$. In this case $\kappa(s) = (d - 1)(\frac{n}{2} - (s - m_0))$, for $s \in [s_0, \frac{n}{2} + m_0[$, with $\kappa(s_0) < m - m_0$ being equivalent to $s_0 > m_0 + \frac{n}{2} - \frac{m-m_0}{d-1}$. This is a consequence of the multiplication rule in $H^s(\mathbb{R}^n)$, $0 < s < \frac{n}{p}$, namely: if $u_j \in H^{s_j}(\mathbb{R}^n),\ s_j \geq 0,\ \frac{n}{p} > s_1 \geq \cdots \geq s_d$, then

$$\prod_{j=1}^{d} u_j \in H^{s_1 + \cdots + s_d - (d-1)\frac{n}{2}}(\mathbb{R}^n),$$

provided

$$s_1 + \cdots + s_d - (d-1)\frac{n}{2} > 0.$$

Suppose now that $f[u] = u^{d-1} D_x^{m_0} u$ (linear in $D_x^{m_0} u$), $m_0 \in \mathbb{N}$. In this case, by the rules of multiplication, we choose $\kappa(s)$ as follows: $s_0 > n/2$ (resp., $s_0 > m_0/2$), $\kappa(s) \equiv 0$ for $s \in]s_0, n/2 + m_0[$ provided $n \geq m_0$ (resp., $n < m_0$); $s_0 \in]n/2 - (m - m_0)/(d-1), n/2[$, $\kappa(s) = (d-1)(n/2 - s)$ for $s \in [s_0, n/2[$, $\kappa(s) = 0$ if $s \in [n/2, n/2 + m_0[$ provided $\frac{n}{p} - \frac{m-m_0}{d-1} > 0$ and $ds_0 - (d-2)n/2 - m_0 > 0$.

We state the main result on the uniform G^σ regularity of solutions to (30).

THEOREM 1. *Let $w \in G^\sigma(T_0; H^s)$, $s > n/2 + m_0$, $T_0 > 0$, $\sigma \geq \sigma_{crit}$. Suppose that $v \in H^\infty(\mathbb{R}^n)$ is a solution of (30). Then there exists $T_0' \in]0, T_0]$ such that*

$$v \in G^\sigma(T_0'; H^s), \qquad T \in]0, T_0']. \tag{35}$$

In particular, if $m - m_0 \geq 1$, which is equivalent to $\sigma_{crit} = 1$, and $\sigma = 1$, v can be extended to a holomorphic function in the strip $\{z \in \mathbb{C}^n : |Im\ z| < T_0'\}$. If $m < 1$ or $\theta > 1$, then $\sigma_{crit} > 1$ and v belongs to $G^\sigma_{un}(\mathbb{R}^n)$.

Proof. First, by standard arguments we reduce to $(m_0 + 1) \times (m_0 + 1)$ system by introducing $v_j =< D >^j v$, $j = 0, \ldots, m_0$ (e.g., see [33], [50]) with the order of the inverse of the transformed matrix valued–operator P^{-1} becoming $m_0 - m$, while σ_{crit} remains invariant. So we deal with a semilinear system of $m_0 + 1$ equations

$$Pv(x) = f(\kappa_0(D)v_0, \ldots, \kappa_{m_0}(D)v_{m_0}) + w(x), \qquad x \in \mathbb{R}^n$$

where κ_j's are zero order constant p.d.o., $f(z)$ being a G^θ function in $\mathbb{C}^{m_0+1} \mapsto \mathbb{C}^{m_0+1}$, $f(0) = 0$. Since $\kappa_j(D)$, $j = 0, \ldots, m_0$, are continuous in $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, and the nonlinear estimates for $f(\kappa_0(D)v_0, \ldots, \kappa_{m_0}(D)v_{m_0})$ are the same as for

$f(v_0, \dots, v_{m_0})$ (only the constants change), we consider $\kappa_j(D) \equiv 1$. Hence, without loss of generality we may assume that we are reduced to

$$Pv(x) = f(v) + w(x), \qquad x \in \mathbb{R}^n \tag{36}$$

Let $v \in H^\infty(\mathbb{R}^n)$ be a solution to (36). Equation (36) is equivalent to

$$P(D_j^k v) = -[P, D_j^k]v + D_j^k(f(v)) + D_j^k w. \tag{37}$$

which yields

$$D_j^k v = -P^{-1}[P, D_j^k]v + P^{-1}D_j^k(f(v)) + P^{-1}D_j^k w. \tag{38}$$

In view of (33), we readily obtain the following estimates with some constant $C_0 > 0$

$$\frac{T^k}{(k!)^\sigma}\|P^{-1}[P, D_j^k]v\|_s \quad \leq \quad C_0 T \sum_{\ell=0}^{k-1}(C_0 T)^{k-\ell-1}\frac{T^\ell}{(\ell!)^\sigma}\sum_{q=1}^{n}\|D_q^\ell v\|_s \tag{39}$$

for all $k \in \mathbb{N}$, $j = 1, \dots, n$. Therefore

$$\begin{aligned}
S_N^{comm}[v; T] \quad &:= \quad \sum_{k=1}^{N}\sum_{j=1}^{n}\frac{T^k}{(k!)^\sigma}\|P^{-1}[P, D_j^k]v\|_s \\
&\leq \quad \sum_{k=1}^{N}\sum_{\ell=0}^{k-1}(C_0 T)^{k-\ell-1}\frac{T^\ell}{(\ell!)^\sigma}\sum_{q=1}^{n}\|D_q^\ell v\|_s \\
&= \quad nC_0 T \sum_{\ell=1}^{N-1}\frac{T^\ell}{(\ell!)^\sigma}\sum_{q=1}^{n}\|D_q^\ell v\|_s \sum_{k=\ell+1}^{N}(C_0 T)^{k-\ell-1} \\
&\leq \quad \frac{nC_0 T}{1 - C_0 T} S_{N-1}^\sigma[v; T, H^s] \\
&\leq \quad \frac{nC_0 T}{1 - C_0 T}\|v\|_s + \frac{nC_0 T}{1 - C_0 T}\widetilde{S}_{N-1}^\sigma[v; T, H^s] \qquad (40)
\end{aligned}$$

for all $N \in \mathbb{N}$ provided $0 < T < C_0^{-1}$.

Now, since the case $\theta = 1$ is easier to deal with, we shall treat the case $\theta > 1$, hence $\sigma_{cr} > 1$.

Next, by Lemma 2, one gets that for $N_s := \|P^{-1}\|_{H^{s-1/\sigma_{crit}} \to H^s}$

$$\begin{aligned}
\|P^{-1}D_j^k(f(v))\|_s \quad &\leq \quad N_s\||D_j|^{k-1/\sigma_{cr}}(f(v))\|_s \\
&\leq \quad \varepsilon\|D_j^k(f(v))\|_s + C(\varepsilon)\|D_j^{k-1}(f(v))\|_s, \qquad \varepsilon > 0 \qquad (41)
\end{aligned}$$

where

$$C(\varepsilon) = (1-\rho)(\frac{N_s\rho}{\varepsilon})^{1/1-\rho}.$$

Set

$$L_0 = |f'(v)|_\infty.$$

Therefore, if $N \geq 3$, in view of (22) we can write

$$\widetilde{S}^\sigma_N[v; T, s] \leq \widetilde{S}^\sigma_N[w; T, s] + \frac{nC_0T}{1 - C_0T}\|v\|_s + \frac{nC_0T}{1 - C_0T}\widetilde{S}^\sigma_{N-1}[v; T, H^s]$$

$$(42) \quad +\varepsilon L_0\widetilde{S}^\sigma_N[v; T, s] + \varepsilon \sum_{j=2}^{N} \frac{A_0^j}{(j!)^{\sigma-\theta}}(\widetilde{S}^\sigma_{N-1}[v; T, s])^j$$

$$+C(\varepsilon)T\left(\|f(v)\|_s + \varepsilon L_0\widetilde{S}^\sigma_{N-1}[v; T, s] + \varepsilon \sum_{j=2}^{N-1} \frac{A_0^j}{(j!)^{\sigma-\theta}}(\widetilde{S}^\sigma_{N-2}[v; T, s])^j\right)$$

for $0 < T < \min\{C_0^{-1}, T_0\}$. Now we fix $\varepsilon > 0$ to satisfy

$$(43) \qquad \varepsilon L_0 < 1$$

Then by (42) we obtain that

$$(44) \qquad \widetilde{S}^\sigma_N[v; T, s] \leq a(T) + b(T)\widetilde{S}^\sigma_{N-1}[v; T, s] + g(\widetilde{S}^\sigma_{N-1}[v; T, s], T)$$

where

$$(45) \qquad a(T) = \frac{\|w\|_{\sigma,T,s} - \|w\|_s + nC_0T(1 - C_0T)^{-1}\|v\|_s + C_\varepsilon T\|f(v)\|_s}{1 - \varepsilon L_0}$$

$$(46) \qquad b(T) = T\frac{nC_0 + (1 - C_0T)\varepsilon C_\varepsilon L_0}{(1 - C_0T)(1 - \varepsilon T)}$$

$$(47) c(T)g(z) = \frac{\varepsilon(1 + \varepsilon C(\varepsilon)L_0T)}{1 - \varepsilon L_0}\sum_{j=2}^{\infty} \frac{A_0^j}{(j!)^{\sigma-\theta}}z^j$$

for $0 < T < \min\{C_0^{-1}, T_0\}$. Now we are able to apply Lemma 3 for $0 < T < T_0'$, by choosing T_0' small enough , $T_0' < \min\{T_0, C_0^{-1}\}$ so that the sequence $\widetilde{S}^\sigma_N[v; T_0', s]$ is bounded . This implies the convergence since $\widetilde{S}^\sigma_N[v; T_0', s]$ is nondecreasing for $N \to \infty$.

□

REMARK 2. The operator P appearing in the ODEs giving rise to traveling wave solutions for dispersive equations is usually a constant p.d.o. or a Fourier multiplier (cf. [11], [40], [29]), and in that case the commutators in the LHS of (33) are zero. Let now $V(x) \in G^\sigma(\mathbb{R}^n : \mathbb{R})$, $\inf_{x\in\mathbb{R}^n} V(x) > 0$. Then it is well known (e.g., cf. [52]) that the operator $P = -\Delta + V(x)$ admits an inverse satisfying $P^{-1} : H^s(\mathbb{R}^n) \to H^{s+1}(\mathbb{R}^n)$. One checks via straightforward calculations that the Gevrey commutator hypothesis is satisfied if there exists $C > 0$ such that

$$(48) \qquad \|P^{-1}(D_x^\beta V D_x^\gamma u)\|_s \leq C^{|\beta|+1}(\beta!)^\sigma\|D_x^\gamma u\|_s,$$

for all $\beta, \gamma \in \mathbb{Z}_+^n$, $\beta \neq 0$.

We point out that, as a corollary of our theorem, we obtain for $\sigma = 1$ and $f[v] \equiv 0$ a seemingly new result, namely that every eigenfunction $\phi_j(x)$ of $-\Delta + V(x)$ is extended to a holomorphic function in $\{z \in \mathbb{C}^n : |\Im z| < T_0\}$ for some $T_0 > 0$. Next, we have a corollary from our abstract result on uniform G^1 regularity for the $H^2(\mathbb{R})$ solitary wave solutions $r(x + ct)$, $c > 0$ in [29] satisfying

$$Pu = r'''' + \mu r'' + cr = f(r, r', r'') = f_0(r, r') + f_1(r, r')r'', \ \mu \in \mathbb{R} \tag{49}$$

with f_j being homogeneous polynomial of degree $d - j$, $j = 0, 1$, $d \geq 3$ and $|\mu| < 2\sqrt{-c}$. Actually, by Lemma 6, we find that every solution r to (49) belonging to $H^s(\mathbb{R})$, $s > 3/2$, is extended to a holomorphic function in $\{z \in \mathbb{C} : |Im\, z| < T\}$ for some $T > 0$.

4. Uniform Gevrey regularity of L^∞ stationary solutions

It is well known that the traveling waves to dissipative equations like Burgers, Fisher–Kolmogorov, Kuramoto–Sivashinsky equations have typically two different nonzero limits for $x \to \pm\infty$ (see the example (6)). Now we investigate the $G^\sigma_{un}(\mathbb{R}^n)$ regularity of such type of solutions for semilinear elliptic equations.

We shall generalize Theorem 4.1 in [8] for G^θ nonlinear terms f. We restrict our attention to (30) for $n = 1$, $m_0 = 0$, $P = P(D)$ being a constant coefficients elliptic p.d.o. or Fourier multiplier of order m.

THEOREM 2. *Let $\theta \geq 1$, $\sigma \geq \theta$, $m - m_0 \geq 1$, $f \in G^\theta(\mathbb{C}^L)$, $f(0) = 0$, $w \in G^\sigma_\infty(T_0; H^s)$ for some $T_0 > 0$. Suppose that $v \in L^\infty(\mathbb{R})$ is a weak solution of (30) satisfying $\nabla v \in H^s(\mathbb{R})$. Then there exists T_0', depending on T_0, $P(D)$, f, $\|v\|_\infty$ and $\|\nabla v\|_s$ such that $v \in G^\sigma_\infty(H^s(\mathbb{R}); T_0')$. In particular, if $\sigma = \theta = 1$ then v can be extended to a holomorphic function in $\{z \in \mathbb{C} : |Im\, z| < T_0'\}$.*

Without loss of generality we suppose that $n = 1$, $m_0 = 0$. It is enough to show that $v' = D_x v \in G^\sigma(H^s, T)$ for some $T > 0$.

We need an important auxiliary assertion, whose proof is essentially contained in [27].

LEMMA 7. *Let $g \in G^\theta(\mathbb{R}^p : \mathbb{R})$, $1 \leq \theta \leq \sigma$, $g(0) = 0$. Then there exists a positive continuous nondecreasing function $G(t)$, $t \geq 0$ such that*

$$\|(D^\alpha g)(v)w\|_s \leq |(D^\alpha g)(v)|_\infty \|w\|_s + G(|v|_\infty)^\alpha (\alpha!)^\theta (\|w\|_{s-1} + \|\nabla v\|^s_{s-1}) \tag{50}$$

for all $v \in (L^\infty(\mathbb{R}^n : \mathbb{R}))^p$, $v' \in (H^s(\mathbb{R}^n : \mathbb{R}))^p$, $w \in (H^s(\mathbb{R}))^p$, $\alpha \in \mathbb{Z}^p_+$, provided $s > n/2 + 1$.

Proof of Theorem 2. Write $u = v'$. We observe that $(f(v))' = f'(v)u$ and the hypotheses imply that $g(v) := f'(v) \in L^\infty(\mathbb{R})$ and $u \in H^s(\mathbb{R})$. Thus differentiating k

times we obtain that u satisfies

$$PD^k u \quad = \quad D^k(g(v)u) + D^k w'$$

which leads to

$$D^k u \quad = \quad P^{-1}D^k(g(v)u) + P^{-1}D^k w'.$$

Hence, since $m \geq 1$ and $P^{-1}D$ is bounded in $H^s(\mathbb{R})$ we get the following estimates

$$\begin{aligned}
\|D^k u\|_s &\leq C\|D^{k-1}(g(v)u)\|_s + \|D^{k-1}w'\|_s \\
&\leq \sum_{j=0}^{k-1} \binom{k-1}{j} \|D^j(g(v))D^{k-1-j}u\|_s + \|D^{k-1}w'\|_s \\
&\leq \sum_{j=0}^{k-1} \binom{k-1}{j} \sum_{\ell=0}^{j} \frac{C^{\ell-1}}{\ell!} \\
&\times \sum_{\substack{p_1+\cdots+p_\ell=j \\ p_1\geq 1,\cdots,p_\ell\geq 1}} \prod_{\mu=1}^{\ell} \frac{\|D^{p_\mu-1}u\|_s}{p_\mu!} \|(D^\ell g)(v))D^{k-1-j}u\|_s + \|D^{k-1}w'\|_s. \qquad (51)
\end{aligned}$$

Now, by Lemma 7 and (51) we get, with another positive constant C,

$$\begin{aligned}
\frac{T^k}{(k!)^\sigma}\|D^k u\|_s &\leq CTk^{-\sigma}\|v\|_s \sum_{j=0}^{k-1} \binom{k-1}{j}^{-\sigma+1} \sum_{\ell=0}^{j} \frac{C^{\ell-1}}{(\ell!)^{\sigma-\theta}} \\
&\times \sum_{\substack{p_1+\cdots+p_\ell=j \\ p_1\geq 1,\cdots,p_\ell\geq 1}} \prod_{\mu=1}^{\ell} \frac{T^{p_\mu}|D^{p_\mu-1}u|_s}{(p_\mu!)^\sigma} (G(|v|_\infty))^\ell \frac{T^{k-1-j}|D^{k-1-j}u|_s}{((k-1-j)!)^\sigma} \\
&+ \|w'\|_{\sigma,T;s}. \qquad (52)
\end{aligned}$$

Next, we conclude as in [8].

□

REMARK 3. As a corollary from our abstract theorem we obtain apparently new results on the analytic $G^1_{un}(\mathbb{R})$ regularity of traveling waves of the Kuramoto–Sivashinsky equation cf. [41], and the Fisher–Kolmogorov equation and its generalizations (cf. [37], [31]).

5. Decay estimates in Gelfand–Shilov spaces

In the paper [8] new functional spaces of Gevrey functions were introduced which turned out to be suitable for characterizing both the uniform analyticity and the exponential decay for $|x| \to \infty$. Here we will show regularity results in the framework of the Gelfand–Shilov spaces $S^\nu_\mu(\mathbb{R}^n)$ with

$$\mu \geq 1, \nu \geq 1. \qquad (53)$$

Let us fix $s > n/2$, $\mu, \nu \geq 1$. Then for every $\varepsilon > 0$, $T > 0$ we set

$$D^{\nu}_{\mu}(\varepsilon, T) = \{v \in S(\mathbb{R}^n) : |\!|\!| v |\!|\!|_{\varepsilon,T} < +\infty\}$$

where

$$|\!|\!| v |\!|\!|_{\varepsilon,T} = \sum_{j,k\in\mathbb{Z}^n_+}^{\infty} \frac{\varepsilon^{|j|}T^{|k|}}{(j!)^{\nu}(k!)^{\mu}} \|x^j D^k v\|_s.$$

We stress that (53) implies that $S^{\nu}_{\mu}(\mathbb{R}^n)$ becomes a ring with respect to the pointwise multiplication and the spaces $D^{\nu}_{\mu}(\varepsilon, T)$ become Banach algebras.

Using the embedding of $H^s(\mathbb{R}^n)$ in $L^{\infty}(\mathbb{R}^n)$ and standard combinatorial arguments, we get that one can find $c > 0$ such that

$$|D^k v(x)| \leq cT^{-|k|}(k!)^{\mu} e^{-\varepsilon|x|^{1/\nu}} |\!|\!| v |\!|\!|_{\varepsilon,T},\ x \in \mathbb{R}^n, k \in \mathbb{Z}^n_+,\ v \in D(\varepsilon, T). \tag{54}$$

Clearly $S^{\nu}_{\mu}(\mathbb{R}^n)$ is inductive limit of $D^{\nu}_{\mu}(\varepsilon, T)$ for $T \searrow 0$, $\varepsilon \searrow 0$.

We set

$$E^{\nu}_{\mu;N}[v; \varepsilon, T] = \sum_{\substack{j,k\in\mathbb{Z}^n_+ \\ |j|+|k|\leq N}} \frac{\varepsilon^{|j|}T^{|k|}}{(j!)^{\nu}(k!)^{\mu}} \|x^j D^k v\|_s.$$

We will study the semilinear equation (30), with $w \in D^{\nu}_{\mu}(\varepsilon_0, T_0)$. The linear operator P is supposed to be of order $\tilde{m} = m$, to be elliptic and invertible, i.e. (32) holds. The crucial hypothesis on the nonlinearity $f(u)$ in order to get decay estimates is the lack of linear part in the nonlinear term. For the sake of simplicity we assume that f is entire function and quadratic near 0, i.e.,

$$f(z) = \sum_{j\in\mathbb{Z}^L_+, |j|\geq 2} f_j z^j \tag{55}$$

and for every $\delta > 0$ there exists $C_\delta > 0$ such that

$$|f_j| \leq C_\delta \delta^{|j|}, \qquad j \in \mathbb{Z}^L_+.$$

Next, we introduce the hypotheses on commutators of P^{-1}.

We suppose that there exist $\mathcal{A}_0 > 0$, and $\mathcal{B}_0 > 0$ such that

$$\|P^{-1}, [P, x^{\beta} D^{\alpha}_x]v\|_s \leq (\alpha!)^{\mu}(\beta!)^{\nu} \sum_{\substack{\rho\leq\alpha,\theta\leq\beta \\ \rho+\theta\neq\alpha+\beta}} \frac{\mathcal{A}_0^{|\alpha-\rho|}\mathcal{B}_0^{|\beta-\theta|}}{(\rho!)^{\mu}(\theta!)^{\nu}} \|x^{\theta} D^{\rho} v\|_s \tag{56}$$

for all $\alpha, \beta \in \mathbb{Z}^n_+$.

The next lemma, combined with well known L^p estimates for Fourier multipliers and L^2 estimates for pseudodifferential operators, indicates that our hypotheses on the commutators are true for a large class of pseudodifferential operators.

LEMMA 8. *Let P be defined by an oscillatory integral*

$$
\begin{aligned}
Pv(x) &= \int e^{ix\xi} P(x,\xi)\hat{v}(\xi)\bar{d}\xi \\
&= \iint e^{i(x-y)\xi} P(x,\xi)v(y)dy\bar{d}\xi, \qquad (57)
\end{aligned}
$$

where $P(x,\xi)$ is global analytic symbol of order m, i.e. for some $C>0$

$$
\sup_{\alpha,\beta\in\mathbb{Z}^n_+}\left(\sup_{(x,\xi)\in\mathbb{R}^{2n}}\left(\frac{<\xi>^{-m+|\beta|}\, C^{|\alpha|+|\beta|}}{\alpha!\beta!}\,|D^\alpha_x D^\beta_\xi P(x,\xi)|\right)\right) < +\infty. \qquad (58)
$$

Then the following relations hold

$$
[P, x^\beta D^\alpha_x]v(x) = \alpha!\beta! \sum_{\substack{\rho\le\alpha,\theta\le\beta\\ |\rho+\theta|<|\alpha+\beta|}} \frac{(-1)^{|\beta-\theta|}(-i)^{|\alpha-\rho|}}{(\alpha-\rho)!(\beta-\theta)!} P^{(\beta-\theta)}_{(\alpha-\rho)}(x,D)(x^\theta D^\rho_x v) \qquad (59)
$$

for all $\alpha,\beta\in\mathbb{Z}^n_+$, where $P^{(\beta)}_{(\alpha)}(x,\xi) := D^\beta_\xi \partial^\alpha_x P(x,\xi)$.

Proof. We need to estimate the commutator $[P, x^\beta D^\alpha_x]v = P(x^\beta D^\alpha_x v) - x^\beta D^\alpha_x P(v)$. We have

$$
P(x^\beta D^\alpha_x v) = \iint e^{i(x-y)\xi} P(x,\xi) y^\beta D^\alpha_y v(y)dy\bar{d}\xi;
$$

$$
\begin{aligned}
x^\beta D^\alpha_x P(v) &= \iint x^\beta D^\alpha_x (e^{i(x-y)\xi} P(x,\xi))v(y)dy\bar{d}\xi \\
&= \iint \sum_{\rho\le\alpha} \binom{\alpha}{\rho} x^\beta e^{i(x-y)\xi} \xi^\rho D^{\alpha-\rho}_x P(x,\xi)v(y)dy\bar{d}\xi \\
&= \iint \sum_{\rho\le\alpha} \binom{\alpha}{\rho} x^\beta (-D_y)^\rho (e^{i(x-y)\xi}) D^{\alpha-\rho}_x P(x,\xi)v(y)dy\bar{d}\xi \\
&= \iint \sum_{\rho\le\alpha} \binom{\alpha}{\rho} x^\beta e^{i(x-y)\xi} D^{\alpha-\rho}_x P(x,\xi) D^\rho_y v(y)dy\bar{d}\xi \\
&= \iint \sum_{\rho\le\alpha} \binom{\alpha}{\rho} D^\beta_\xi (e^{ix\xi})(e^{-iy\xi} D^{\alpha-\rho}_x P(x,\xi)) D^\rho_y v(y)dy\bar{d}\xi \\
&= \iint \sum_{\rho\le\alpha} \binom{\alpha}{\rho} e^{ix\xi}(-D_\xi)^\beta (e^{-iy\xi}) D^{\alpha-\beta}_x P(x,\xi)) D^\rho_y v(y)dy\bar{d}\xi \\
&= \iint \sum_{\substack{\rho\le\alpha\\ \theta\le\beta}} \binom{\alpha}{\rho}\binom{\beta}{\theta} e^{i(x-y)\xi} y^\theta (-1)^{|\beta-\theta|} D^{\beta-\theta}_\xi D^{\alpha-\rho}_x P(x,\xi) D^\rho_y v(y)dy\bar{d}\xi \\
&= \sum_{\substack{\rho\le\alpha\\ \theta\le\beta}} \binom{\alpha}{\rho}\binom{\beta}{\theta} \iint e^{i(x-y)\xi} (-1)^{|\beta-\theta|} D^{\beta-\theta}_\xi D^{\alpha-\rho}_x P(x,\xi) y^\theta D^\rho_y v(y)dy\bar{d}\xi
\end{aligned}
$$

which concludes the proof of the lemma.

□

REMARK 4. We point out that if $P(D)$ has nonzero symbol, under the additional assumption of analyticity, namely $P(\xi) \neq 0$, $\xi \in \mathbb{R}^n$ and there exists $C > 0$ such that

$$\frac{|D_\xi^\alpha(P(\xi))|}{|P(\xi)|} \leq C^{|\alpha|}\alpha!(1+|\xi|)^{-|\alpha|}, \quad \alpha \in \mathbb{Z}_+^n,\ \xi \in \mathbb{R}^n, \tag{60}$$

the condition (56) holds. More generally, (56) holds if $P(x,\xi)$ satisfies the following global Gevrey $S_\nu^\mu(\mathbb{R}^n)$ type estimates: there exists $C > 0$ such that

$$\sup_{(x,\xi)\in R^{2n}} (<\xi>^{-m+|\alpha|} (\alpha!)^\mu (\beta!)^{-\nu} |\partial_x^\alpha \partial_\xi^\beta P(x,\xi)|) \leq C^{|\alpha|+|\beta|+1}, \qquad \alpha,\beta \in \mathbb{Z}_+^n.$$

The latter assertion is a consequence from the results on $L^2(\mathbb{R}^n)$ estimates for p.d.o.-s (e.g., cf. [19]).

Let now $P(D)$ be a Fourier multiplier with the symbol $P(\xi) = 1 + i\ \mathrm{sign}(\xi)\xi^2$ (such symbol appears in the Benjamin-Ono equation). Then (56) fails.

Since our aim is to show (sub)exponential type decay in the framework of the Gelfand-Shilov spaces, in view of the preliminary results polynomial decay in [8], we will assume that

$$<x>^N v \in H^\infty(\mathbb{R}^n), \qquad N \in \mathbb{Z}_+. \tag{61}$$

Now we extend the main result on exponential decay in [8].

THEOREM 3. *Let f satisfy (55) and $w \in S_\mu^\nu(\mathbb{R}^n)$ with μ, ν satisfying (53), i.e., $w \in D_\mu^\nu(\varepsilon_0, T_0)$ for some $\varepsilon_0 > 0$, $T_0 > 0$. Suppose that the hypothesis (56) is true. Let now $v \in H^\infty(\mathbb{R}^n)$ satisfy (61) and solve (30) with the RHS w as above. Fix $\varepsilon \in]0, \min\{\varepsilon_0, \mathcal{B}_0^{-1}\}[$. Then we can find $T_0'(\varepsilon) \in]0, \min\{\tilde{T}_0, \mathcal{A}_0^{-1}\}[$ such that $v \in D_\mu^\nu(\varepsilon, T)$ for $T \in]0, T_0'(\varepsilon)[$. In particular, $v \in S_\mu^\nu(\mathbb{R}^n)$.*

Proof. For the sake of simplicity we will carry out the argument in the one dimensional case. We write for $\beta \geq 1$

$$\begin{aligned} P(x^\beta D^\alpha v) &= x^\beta D_x^\alpha w - [P, x^\beta D_x^\alpha]v + x^\beta D_x^\alpha(f(v)) \\ &= x^\beta D_x^\alpha w - [P, x^\beta D_x^\alpha]v \\ &+ D_x(x^\beta D_x^{\alpha-1}(f(v))) - \beta x^{\beta-1} D_x^{\alpha-1}(f(v)). \end{aligned} \tag{62}$$

Thus

$$\begin{aligned} x^\beta D^\alpha v &= P^{-1}x^\beta D_x^\alpha w - P^{-1}[P, x^\beta D_x^\alpha]v \\ &+ P^{-1}D_x(x^\beta D_x^{\alpha-1}(f(v))) - \beta P^{-1}(x^{\beta-1}D_x^{\alpha-1}(f(v))) \end{aligned} \tag{63}$$

which implies, for some constant depending only on the norms of P^{-1} and $P^{-1}D$ in H^s, the following estimates

$$
\begin{aligned}
\frac{\varepsilon^\beta T^\alpha}{(\alpha!)^\mu(\beta!)^\nu}\|x^\beta D^\alpha v\|_s \quad &\le \quad C\frac{\varepsilon^\beta T^\alpha}{(\alpha!)^\mu(\beta!)^\nu}\|x^\beta D^\alpha w\|_s \\
&+ \sum_{\substack{\rho\le\alpha,\theta\le\beta\\ \rho+\theta\neq\alpha+\beta}} \frac{\mathcal{A}_0^{\alpha-\rho}\mathcal{B}_0^{\beta-\theta}}{(\rho!)^\mu(\theta!)^\nu}\|x^\theta D^\rho v\|_s \\
&+ C\frac{T}{\alpha^\mu}\frac{\varepsilon^\beta T^{\alpha-1}}{((\alpha-1)!)^\mu(\beta!)^\nu}\|x^\beta D^{\alpha-1}(f(v))\|_s \\
&+ \frac{\varepsilon T}{\beta^{\nu-1}}\frac{\varepsilon^{\beta-1}T^{\alpha-1}}{((\alpha-1)!)^\mu((\beta-1)!)^\nu}\|x^{\beta-1}D^{\alpha-1}(f(v))\|_s
\end{aligned} \tag{64}
$$

for all $\alpha, \beta \in \mathbb{Z}_+$, $\beta \ge 1$, $\alpha \ge 1$.

On the other hand, if $\alpha = 0$ we have

$$
\begin{aligned}
\frac{\varepsilon^\beta}{(\beta!)^\nu}\|x^\beta v\|_s \quad &\le \quad C\frac{\varepsilon^\beta}{(\beta!)^\nu}\|x^\beta w\|_s \\
&+ \sum_{0\le\theta<\beta}\frac{\mathcal{B}_0^{\beta-\theta}}{(\theta!)^\nu}\|x^\theta v\|_s + C\frac{\varepsilon^\beta}{(\beta!)^\nu}\|x^\beta(f(v))\|_s
\end{aligned} \tag{65}
$$

for all $\beta \in \mathbb{Z}_+$, $\beta \ge 1$.

Now use the (at least) quadratic order of $f(u)$ at $u = 0$, namely, there exist $C_1 > 0$ depending on s and f, and a positive nondecreasing function $G(t)$, $t \ge 0$, such that

$$
\frac{\varepsilon^\beta}{(\beta!)^\nu}\|x^\beta f(v)\|_s \quad \le \quad C_1\varepsilon\|xv\|_s G(\|v\|_s)\left(\frac{\varepsilon^{\beta-1}}{((\beta-1)!)^\nu}\|x^{\beta-1}v)\|_s\right) \tag{66}
$$

This, combined with (64), allows us to gain an extra $v\epsilon > 0$ and after summation with respect to $\alpha, \beta, \alpha+\beta \le N+1$, to obtain the following iteration inequalities for some $C_0 > 0$

$$
E^\mu_{\nu;N+1}[v;\varepsilon,T] \quad \le \quad \|v\|_s + C_0\varepsilon\|xv\|_s E^\mu_{\nu;N}[v;\varepsilon,T] + TG(E^\mu_{\nu;N}[v;\varepsilon,T]) \tag{67}
$$

for all $N \in \mathbb{Z}_+$. We can apply the iteration lemma taking $0 < T \le T_0'(\varepsilon)$ with $0 < T_0'(\varepsilon) \ll 1$.

□

REMARK 5. We recall that the traveling waves for the Benjamin-Ono equation decay as $O(x^{-2})$ for $|x| \to +\infty$, where $P(\xi) = c + i\ \mathrm{sign}(\xi)\xi^2$ for some $c > 0$. Clearly (H_3) holds with $\mu = 2$ but it fails for $\mu \ge 3$. Next, if a traveling wave solution $\varphi(x) \in H^2(\mathbb{R})$ in [29] decays like $|x|^{-\varepsilon}$ as $x \to \infty$ for some $0 < \varepsilon \ll 1$, then by Theorem 2 it should decay exponentially and will belong to the Gelfand–Shilov class $S_1^1(\mathbb{R})$. Finally, we recall that $u_\nu(x) = -4\nu x(x^2+\nu^2)^{-1}$, $x \in \mathbb{R}$, solves the stationary

Sivashinsky equation $|D_x|u + \nu\partial_x^2 u = u\partial_x u$, $\nu > 0$ (cf. [51]). Clearly $u_\nu(x)$ extends to a holomorphic function in the strip $|Im\, z| < \nu$ and decays (exactly) like $O(|x|^{-1})$ for $x \to \infty$. Although the full symbol $-|\xi| + \nu\xi^2$ is not invertible in $L^2(\mathbb{R})$, we can invert P in suitable subspaces of odd functions and check that (56) holds iff $\mu = 1$.

6. Strongly singular solutions

First we consider a class of semilinear ODE with polynomial nonlinear terms on the real line

$$P[y](x) := D^m y(x) + \sum_{\ell=2}^{d} \sum_{j\in\{0,1,\dots,m-1\}^\ell} p_j^\ell D^{j_1} y(x) D^{j_2} y(x) \cdots D^{j_m} y(x) = 0,$$

$$p_j^\ell \in \mathbb{C} \tag{68}$$

where $m, d \in \mathbb{N}$, $m \geq 2$, $d \geq 2$, $Dy(x) = (1/i)y'(x)$. We require the following homogeneity type condition: there exists $\tau > 0$ such that

$$-\tau - m = -\ell\tau - j_1 - \cdots - j_\ell \qquad \text{if } p_j^d \neq 0. \tag{69}$$

Thus, by the homogeneity we obtain after substitution in (68) and straightforward calculations that $y_\pm(x) = c_\pm(\pm x)^{-\tau}$ solves (68) for $\pm x > 0$ provided $c \neq 0$ is zero of the polynomial $P_m^\pm(\lambda)$, where

$$P_{m,\tau}^\pm(\lambda) = \lambda\tau(\tau-1)\cdots(\tau-m+1)(-1)^m + \sum_{\ell=2}^{d} \sum_{j\in\{0,1,\dots,m-1\}^\ell} \lambda^\ell \tilde{p}_j^\ell(\tau) \tag{70}$$

where

$$\tilde{p}_j^\ell(\tau) = p_j^\ell(-1)^{|j|}\tau(\tau-j_1)\cdots(\tau-j_\ell). \tag{71}$$

If $\tau < 1$ the singularity of the type $|x|^{-\tau}$ near $x = 0$ is in $L^p_{loc}(\mathbb{R})$, $p \geq 1$, provided $p\tau < 1$. In this case we deduce that one can glue together y_+ and y_- into one $y \in L^p_{loc}(\mathbb{R})$ function. However, the products in (68) are in general not in $L^1_{loc}(\mathbb{R})$ near the origin, so we have no real counterexample of singular solutions to (68) on $\mathbb{R}$. We shall construct such solutions following the approach in [8], namely, using homogeneous distributions on the line (for more details on homogeneous distributions see L. Hörmander [33], vol. I). We recall that if $u \in \mathcal{S}'(\mathbb{R})$ is homogeneous distribution of order r, then $u(x) = u_\pm|x|^r$ for $\pm x > 0$, $u_\pm \in \mathbb{C}$, and $\hat{u}(\xi)$ is a homogeneous distribution of order $-1-r$.

Given $\mu > -1$ we set

$$h^\pm_{-1+\mu}(x) := \mathcal{F}^{-1}_{\xi\to x}(H^\mu_\pm(\xi)), \tag{72}$$

where

$$H^\mu_\pm(\xi) = H(\pm\xi)|\xi|^\mu, \tag{73}$$

with $H(t)$ standing for the Heaviside function. Since $\tau > 0$ if $\mu = -1 + \tau > -1$ we get that $|\xi|^{-1+\mu}$ is L^1_{loc} near $\xi = 0$, therefore $H(\pm\xi)|\xi|^{-1+\mu}$ belongs to $\mathcal{S}'(\mathbb{R})$ and $h^{\pm}_{-\tau}$ are homogeneous of degree $-\tau$. Moreover, since $supp(\widehat{h^{+}_{-\tau}}) = [0, +\infty[$ (resp. $supp(\widehat{h^{-}_{-\tau}})(\xi) =]-\infty, 0[$), $h^{+}_{-\tau}$ (resp. $h^{-}_{-\tau}$) satisfies a well known condition, guaranteeing that the product $(h^{+}_{-\tau})^m$ (resp. $(h^{-}_{-\tau})^m$), or equivalently the convolutions

$$\widehat{D^{j_1}h^{+}_{-\tau}} * \cdots * \widehat{D^{j_\ell}h^{+}_{-\tau}}$$

(resp.

$$\widehat{D^{j_1}h^{-}_{-\tau}} * \cdots * \widehat{D^{j_1}h^{-}_{-\tau}})$$

are well defined in $\mathcal{S}'(\mathbb{R})$ for any $m \in \mathbb{N}$ (cf. [43], see also [33]). In view of the equivalence between (68) and (74), the order of homogeneity $-\tau$ of $y_\pm$, and (72), we will look for solutions to

$$\widehat{P[y]}(\xi) = \xi^m \hat{y}(\xi) + \sum_{\ell=2}^{d} \sum_{j\in\{0,1,\ldots,m-1\}^\ell} p^\ell_j \xi^{j_1}\hat{y} * \cdots * \xi^{j_2}\hat{y} \cdots \xi^{j_m}\hat{y} = 0 \tag{74}$$

proportional to $H(\pm\xi)|\xi|^{-1+\tau}$ homogeneous of order $-1+\tau$ with support in $\pm\xi \geq 0$. Following [8], we set

$$h^{\pm,a}_{-\tau}(\xi) = a_\pm H(\pm\xi)|\xi|^{-1+\tau}, \tag{75}$$

with $a_\pm \in \mathbb{C}$ to be determined later on.

Using the homogeneity of $h^{\pm,a}_{-\tau}$, the definition of τ in (69) and the convolutions identities derived in [8], Section 7, we readily obtain that

$$a_\pm \mathcal{F}^{-1}_{\xi\to x}(h^{\pm}_{-\tau}) = c_\pm(-\pm x)^\tau \tag{76}$$

where $c_\pm$ is a complex constant and by substituting $\hat{y}(\xi) = h^{\pm,a}_{-\tau}(\xi)$ in (74)

$$\widehat{P[y]}(\xi) = a_\pm H(\pm\xi)|\xi|^{m+\tau}\tilde{P}^{\pm}_{m,\tau}(a_\pm) = 0 \tag{77}$$

where $\tilde{P}^{\pm}_{m,\tau}(\lambda)$ is a polynomial such that $a_\pm$ is zero of $\tilde{P}^{\pm}_{m,\tau}(\lambda)$ iff $c_\pm$ in (76) is zero of $P^{\pm}_{m,\tau}(\lambda)$ (defined in (70)).

Therefore we have constructed explicit homogeneous solutions to (68)

$$u_{a_\pm}(x) = a_\pm \mathcal{F}_{\xi\to x}(H^{-1+\tau}_{\pm}(\xi)) \tag{78}$$

Consider now a semilinear PDE with polynomial nonlinearities

$$Pu = P_m(D)u + F(u, \ldots, D^\alpha_x u, \ldots)|_{|\alpha|\leq m-1} \tag{79}$$

where $P_m(D)$ is constant linear partial differential operator homogeneous of order m and where F is polynomial of degree $d \geq 2$.

Given $\theta \in S^{n-1}$ we shall define the ODE $P_\theta(D)$ in the following way: let Q be an orthogonal matrix such that $Q^*\theta = (1, 0, \dots, 0)$. Then for $x = y_1$ we have

$$P_\theta(D)u(y_1) = (P_m(D_y)U_\theta + F(U_\theta, \dots, D_y^\alpha U_\theta, \dots)|_{|\alpha| \le m-1}) \tag{80}$$

with $U_\theta(y) = u(< Q, (y_1, 0, \dots, 0) >)$.

Let now $\theta \in S^{n-1}$, $n \ge 2$ and let $L = L_\theta$ be the hyperplane orthogonal to θ. We define, as in [8], $U_\pm^\theta = a_\pm \delta_P \otimes h^\pm_{-\tau,\theta} \in \mathcal{S}'(\mathbb{R})$, $j = 1, \dots, d-1$, by the action on $\phi(x) \in \mathcal{S}(\mathbb{R}^n)$

$$(U^\theta_{\pm;j}, \phi) := \int_{\mathbb{R}^{n-1}_{y'}} \int_{\mathbb{R}_{y_n}} \phi({}^t Q y) dy') u_{-\tau,\pm}(y_n) dy_n dy' \tag{81}$$

where Q is an orthogonal matrix transforming θ into $(0, \dots, 0, 1)$. We have proved

PROPOSITION 1. *Suppose that P_θ satisfies the homogeneity property (69) for some $\theta \in S^{n-1}$ and denote by L the hyperplane orthogonal to θ. Then every homogeneous distribution defined by (81) solves (79).*

We propose examples of semilinear elliptic PDEs with singular solutions as above:

1) $Pu = \Delta u + u^d = 0$, $d \in \mathbb{N}$, $d \ge 2$, $\tau = -2/(d-1)$, $\vartheta \in S^{n-1}$;

2) $P = (-\Delta)^m u + D^p_{x_1} u D^q_{x_1} u + u^d = 0$, with $m, d, p, q \in \mathbb{N}$, $d \ge 2$ satisfying $2m = (2m - p - q)(d-1)$. In that case

$$\tau = -\frac{2m}{d-1} = -2m + p + q, \qquad \theta = (1, 0, \dots, 0).$$

7. Analytic Regularization for Semilinear Parabolic Systems

We consider the initial value problem for systems of parabolic equations

$$\begin{gathered} \partial_t u_j + P_j(D) u_j + \sum_{\ell=1}^{L} \kappa_{j,\ell}(D)(F_{j,\ell}(\vec{u})) = 0, \\ \vec{u}|_{t=0} = \vec{u^0}, \ t > 0, \ x \in \Omega, \ j = 1, \dots, N, \end{gathered} \tag{82}$$

where $\vec{u} = (u_1, \dots, u_N)$; $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$. $P_j(D)$ is differential operator of order $m \in 2\mathbb{N}$, $Re(P_j(D))$ is positive elliptic of order m for all $j = 1, \dots, m$.

The nonlinear terms $F_{j,\ell} \in C^1(\mathbb{C}^N : \mathbb{C})$, $j = 1, \dots, N$, are homogeneous of order $s_\ell > 1$. We write $ord_z F(z) = s$ for F (positively) homogeneous of order s.

In the case we study the analytic regularity of the solutions for positive time we will assume that $F_{j,\ell}$, $j = 1, \dots, N$, are homogeneous polynomials of degree $s_\ell \ge 2$, namely

$$F_{j,\ell}(z) = \sum_{\beta \in \mathbb{Z}^N_+, |\beta| = s_\ell} F^\beta_{j,\ell} z^\beta, \qquad F^\beta_{j,\ell} \in \mathbb{C}, \ z \in \mathbb{C}^N, \tag{83}$$

for $j = 1, \dots, N$, $\ell = 1, \dots, L$.

The operators $\kappa_{j,\ell}$ satisfy

$$\kappa_{j,\ell}(D) \in \Psi_h^{d_\ell}(\Omega),\ 0 \le d_\ell < m \tag{84}$$

for $j = 1, \dots, N$, $\ell = 1, \dots, L$. Here $\Psi_h^\nu(\mathbb{R}^n)$) (resp. $\Psi_h^\nu(\mathbb{T}^n)$) stands for the space of all smooth homogeneous p.d.o. on $\mathbb{R}^n$ (resp. restricted on $\mathbb{T}^n$) of order $\nu \ge 0$. We suppose that

$$\text{either } d_\ell > 0 \text{ or } \kappa_{j,\ell}(\xi) \equiv const,\ \ell = 1, \dots L,\ j = 1, \dots, N \quad \text{if } \Omega = \mathbb{T}^n. \tag{85}$$

The initial data $\vec{u^0} \in \mathcal{S}'(\Omega)$ will be prescribed later on. Such systems contain as particular cases semilinear parabolic equations, the Navier-Stokes equations for an incompressible fluid, Burgers type equations, the Cahn-Hilliard equation, the Kuramoto-Sivashinsky type equations and so on.

For given $q \in [1, +\infty]$, $\gamma \ge 0$, $\theta \in \mathbb{R}$, $\mu \ge 1$ and $T \in]0, +\infty]$ we define the analytic-Gevrey type Banach space $A^\gamma_{\theta,q}(T;\mu)$ as the set of all $\vec{u} \in C(]0, T[: (L^q(\Omega))^N)$ such that the norm

$$\|\vec{u}\|_{A^\gamma_{\theta,q}(T;\mu)} = \sum_{\alpha \in \mathbb{Z}^n_+} \frac{\gamma^{|\alpha|}}{\alpha!} \sup_{0<t\le T} (t^{\frac{|\alpha|}{\mu}+\theta} \|\partial^\alpha \vec{u}(t)\|_{L^q}) \tag{86}$$

is finite. The Sobolev embedding theorems and the Cauchy formula for the radius of convergence of power series imply, for $\gamma > 0$, that, if $\vec{u} \in A^\gamma_{\theta,q}(T;\mu)$ then $\vec{u}(t,\cdot) \in \mathcal{O}(\Gamma_{\gamma t^{\frac{1}{\mu}}})$, $t \in]0, T]$ where $\Gamma_\rho := \{x \in \mathbb{C}^n : |Im(x)| < \rho\}$, $\rho > 0$ and $\mathcal{O}(\Gamma)$ stands for the space of all holomorphic functions in Γ, Γ being an open set in $\mathbb{C}^n$, while for $\gamma = 0$, with the convention $0^0 = 1$, we obtain that $A^0_{\theta,q}(T;\mu)$ coincides with the usual Kato-Fujita weighted type space $C_\theta(L^q;T)$ and μ is irrelevant. Given $u \in C(]0, T[: L^1_{loc}(\Omega))$ and $t \in]0, T[$, we define

$$\rho_{[u]}(t) = \sup\{\rho > 0 : u(t,\cdot) \in \mathcal{O}(\Gamma_\rho)\}$$

with $\rho_{[u]}(t) := 0$ if it cannot be extended to a function in $\mathcal{O}(\Gamma_\rho)$ for any $\rho > 0$. Clearly for each $u \in A^\gamma_{\theta,q}(T;\mu)$ we have $\rho_{[u]}(t) \ge \gamma t^{\frac{1}{\mu}}$, $t \in]0, T]$. We define $A^{\gamma,\mathcal{S}'}_{\theta,q}(T;\mu) := C([0, T[: (\mathcal{S}'(\Omega))^N) \bigcap A^\gamma_{\theta,q}(T;\mu)$, $C^{\mathcal{S}'}_\theta(L^q;T) := A^{0,\mathcal{S}'}_{\theta,q}(T;\mu)$. One motivation for the introduction of $A^{\gamma,\mathcal{S}'}_{\theta,q}(T;\mu)$ is that $(L^p(\Omega))^N \ni f \to (-\Delta)^{\frac{k}{2}} E^\Omega_P[f] \in A^{\gamma,\mathcal{S}'}_{\frac{1}{m}(k+\frac{1}{p}-\frac{1}{q}),q}(T;m)$, for all $1 \le p \le q \le +\infty$, $\gamma \ge 0$, $k \ge 0$ where

$$E^\Omega_P[f](t) := e^{-tP(D)} f = \mathcal{F}^{-1}_{\xi \to x}(e^{-tP(\xi)} \hat{f}(\xi)),$$

$f \in (\mathcal{S}'(\Omega))^N$.

We denote by $B^{\rho,\infty}_q(\mathbb{R}^n)$ (resp. $\dot{B}^{\rho,\infty}_q(\mathbb{R}^n)$) the Besov (resp. homogeneous Besov) spaces, cf. [54]. Typically for perturbative methods dealing with (82), given $\gamma \ge 0$,

$\theta \geq 0$, $q \geq 1$, we want to find the space of all $f \in (\mathcal{S}'(\Omega))^N$ such that

$$E_P[f] \in A^{\gamma}_{\frac{\theta}{m},q}(T;m) \tag{87}$$

for some (all) $T \in]0,+\infty[$ (respectively

$$E_P[f] \in A^{\gamma}_{\frac{\theta}{m},q}(+\infty;m) \tag{88}$$

if P is homogeneous). These spaces depend on Ω, θ, q and m but not on $\gamma \geq 0$ and P and we denote them by $\mathcal{B}^{-\theta,\infty}_q(\Omega) = \mathcal{B}^{-\theta,\infty}_q(\Omega;m)$ (resp. $\dot{\mathcal{B}}^{-\theta,\infty}_q(\Omega) = \dot{\mathcal{B}}^{-\theta,\infty}_q(\Omega;m)$). It is well known, for instance, that $\dot{\mathcal{B}}^{-\theta,\infty}_q(\mathbb{R}^n) = \dot{B}^{-\theta,\infty}_q(\mathbb{R}^n)$ if $P = -\Delta,\ N = 1,\ \theta > 0$, cf. [54]. However $\dot{\mathcal{B}}^{0,\infty}_q(\mathbb{R}^n) \neq \dot{B}^{0,\infty}_q(\mathbb{R}^n)$. One shows that $\mathcal{B}^{-\theta_1,\infty}_{q_1}(\Omega) \hookrightarrow \mathcal{B}^{-\theta_2,\infty}_{q_2}(\Omega)$, (resp. $\dot{\mathcal{B}}^{-\theta_1,\infty}_{q_1}(\Omega) \hookrightarrow \dot{\mathcal{B}}^{-\theta_2,\infty}_{q_2}(\Omega)$) if $\theta_2 \geq$ (resp. $\theta_2 =$) $\theta_1 + \frac{n}{q_1} - \frac{n}{q_2}$, $1 \leq q_1 \leq q_2$; $\dot{\mathcal{B}}^{-\theta,\infty}_q(\mathbb{T}^n) \subset \mathcal{S}'_0(\mathbb{T}^n) := \{f \in \mathcal{S}'(\mathbb{T}^n) \text{ s.t. } \int_{\mathbb{T}^n} f = 0\}$ for all $\theta \geq 0$, $1 \leq q \leq \infty$ and if $b(D) \in \Psi^r_h(\mathbb{T}^n)$, $r \geq 0$, then

$$b(D) : L^p(\mathbb{T}^n) \longrightarrow \dot{\mathcal{B}}_q^{-\frac{n}{p}+\frac{n}{q}-r,\infty}(\mathbb{T}^n), \qquad q \geq p > 1$$

and

$$b(D) : \mathcal{M}(\mathbb{T}^n) \longrightarrow \dot{\mathcal{B}}_q^{-n+\frac{n}{q}-r,\infty}(\mathbb{T}^n), \qquad q \geq 1,\ q > 1 \text{ if } r = 0.$$

Set $H^{\rho}_{\mathcal{S}'}(\mathbb{R}^n)$, $\rho \in \mathbb{R}$, to be the space of all Schwartz distributions homogeneous of order ρ.

We put

$$s = \max\{s_1,\ldots,s_\ell\}, \qquad p_{cr} = n \max_{\ell=1,\ldots,L} \frac{s_\ell - 1}{m - d_\ell}$$

and define $\mathcal{C}^{m,s}_{p_{cr}}(n)$ as the set of all (q,θ) s.t. $q \geq \max\{1, p_{cr}\}$, $\theta \geq 0$, $s\theta < m$, $\theta + \frac{n}{q} \leq \frac{n}{p_{cr}}$ with $q > p_{cr}$ if $\theta = 0$; $\partial\mathcal{C}^{m,s}_{p_{cr}}(n) := \{(q,\theta(q)) \in \mathcal{C}^{m,s}_{p_{cr}}(n)\}$, $\theta(q) := \frac{n}{p_{cr}} - \frac{n}{q}\}$; $\dot{\mathcal{C}}^{m,s}_{p_{cr}}(n) = \mathcal{C}^{m,s}_{p_{cr}}(n) \backslash \partial\mathcal{C}^{m,s}_{p_{cr}}(n)$; $q_{max} = \sup\{\tau > q : s(\theta(q) + \frac{n}{q} - \frac{n}{\tau}) < m\}$. Throughout the section we will tacitly assume that $F_{j,\ell}$'s are polynomials as in (83) when we state analytic regularity results for (82) in the framework of the Gevrey spaces $A^{\gamma}_{\frac{\theta}{m},q}(T;m)$, $\gamma > 0$.

THEOREM 4. *There exists an absolute constant $a > 0$ such that:*

i) if $(q,\theta) \in \dot{\mathcal{C}}^{m,s}_{p_{cr}}(n)$ and $\vec{u^0} \in \mathcal{B}^{-\theta,\infty}_q(\Omega)$ then $\exists T^* > 0$ s.t. (82) admits a solution

$$\vec{u} \in \bigcap_{\gamma \geq 0} A^{\gamma,\mathcal{S}'}_{\frac{\theta}{m},q}(T^* \exp(-a\gamma^{\frac{m-1}{m}});m). \tag{89}$$

The solution is unique in $C^{\mathcal{S}'}_{\frac{\theta}{m}}(L^q;T^*)$ provided $q \geq s$;

ii) if $(q,\theta(q)) \in \partial\mathcal{C}^{m,s}_{p_{cr}}(n)$ *then* $\exists C' > 0$ *s.t. if* $\vec{u^0} \in \mathcal{B}^{-\theta(q),\infty}_q(\Omega)$ *satisfies*

$$\lim_{T\searrow 0} \|E^{\Omega}_P[\vec{u^0}]\|_{C_{\frac{\theta(q)}{m}}(L^q;T)} = c'_0 \le C' \tag{90}$$

then $\exists T' > 0$ *s.t. (82) admits a solution* $\vec{u} \in C^{\mathcal{S}'}_{\frac{\theta(q)}{m}}(L^q;T')$ *satisfying* $\vec{u} \in A^{\gamma}_{\frac{\theta(q)}{m},q}(T'_\gamma;m)$ *for some* $T'_\gamma \in]0,T']$, $\gamma \in [0,(\frac{1}{a}\ln\frac{C'}{c_0})^{\frac{m}{m-1}}]$. *The solution is unique in* $C^{\mathcal{S}'}_{\frac{\theta(q)}{m}}(L^q;T')$ *provided* $q \ge s$.

For the next two theorems we require that the operator $P(D)$ is homogeneous and

$$\frac{m-d_1}{s_1-1} = \ldots = \frac{m-d_\ell}{s_\ell-1} = \frac{n}{p_{cr}}.$$

THEOREM 5. *Let* $(q,\theta(q)) \in \partial\mathcal{C}^{m,s}_{p_{cr}}(n)$. *We claim that there exists* $C'' > 0$ *s.t. if* $\vec{u^0} \in \dot{\mathcal{B}}^{-\theta(q),\infty}_q(\Omega)$ *satisfies*

$$\|E^{\Omega}_P[\vec{u^0}]\|_{C_{\frac{\theta(q)}{m}}(L^q;+\infty)} = c''_0 \le C''_q \tag{91}$$

then (82) admits a global solution

$$\vec{u} \in \bigcap_{0\le\gamma\le\overline{\gamma}} A^{\gamma}_{\frac{\theta(q)}{m},q}(+\infty;m), \qquad \overline{\gamma} := (\frac{1}{a}\ln\frac{C''}{c''_0})^{\frac{m}{m-1}}.$$

Furthermore, the solution is unique if $q \ge s$.

THEOREM 6. *Let* $\vec{u}^0 \in (H^{-\frac{n}{p_{cr}}}_{\mathcal{S}'}(\mathbb{R}^n))^N \bigcap \dot{\mathcal{B}}^{-\theta(q),\infty}_q(\mathbb{R}^n)$ *for some* $q \in]\max\{p_{cr},s\},q_{max}[$, *and let* $\|E^{\mathbb{R}^n}_P[\vec{u^0}]\|_{C_{\frac{\theta(q)}{m}}(L^q;+\infty)} = c''_0 \le C''$. *Then the unique solution in the previous theorem satisfies*

$$\vec{u}(t,x) = t^{-\frac{n}{mp_{cr}}}\vec{g}(\frac{x}{\sqrt[m]{t}}),\ t>0 \qquad \vec{g}(z) \in L^q(\mathbb{R}^n)\bigcap(L^\infty(\mathbb{R}^n)\bigcap\mathcal{O}(\Gamma_\gamma))^N, \tag{92}$$

$$\vec{w} := E^{\mathbb{R}^n}_P[\vec{u^0}](1) - \vec{g} \in (L^{\overline{q}}(\mathbb{R}^n)\bigcap\mathcal{O}(\Gamma_\gamma))^N, \qquad \overline{q} = \max\{p_{cr},\frac{q^*}{s}\} \tag{93}$$

for all $\gamma \in [0,(\frac{1}{a}\ln\frac{C''}{c''_0})^{\frac{m}{m-1}}]$. *Assume now that* $F_{j,\ell}$*'s are polynomials and*

$$s_\ell \le p_{cr} \le 2s_\ell,\ \ s < p_{cr},\ \ 2d_\ell \ge m, \qquad \ell = 1,\ldots,L. \tag{94}$$

Then there exists $\varepsilon > 0$ *s.t. for all*

$$\vec{u^0} \in (H^{-\frac{n}{p_{cr}}}_{\mathcal{S}'}(\mathbb{R}^n)\bigcap \dot{B}^{0,\infty}_{p_{cr}}(\mathbb{R}^n))^N$$

with

$$\|\vec{u^0}\|_{\dot{B}^{0,\infty}_{p_{cr}}} = \varepsilon_0 < \varepsilon$$

the IVP (82) has a unique solution

$$\vec{u} \in BC_w([0, +\infty[: (\dot{B}^{0,\infty}_{p_{cr}}(\mathbb{R}^n))^N)$$

satisfying (92) and $\vec{w} \in L^{p_{cr}}(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$. *Furthermore,* $\vec{g} \in (\mathcal{O}(\Gamma_\gamma))^N$ *for* $\gamma \in [0, \overline{\gamma}]$. *Here the subscript w in* BC_w *means that we have continuity in the weak topology* $\sigma(\dot{B}^{0,\infty}_{p_{cr}}(\mathbb{R}^n), (\dot{B}^{0,\infty}_{p'_{cr}}(\mathbb{R}^n)))$ *and* $p'_{cr} = \frac{p_{cr}}{p_{cr}-1}$.

8. Sketch of the proofs of the Gevrey regularity for parabolic systems

The main idea is to reduce (82) to the system of integral equations

$$u_j(t) = E^\Omega_{P_j}[u^0_j](t) + \sum_{\ell=1}^{L} K^\Omega_{j,\ell}[\vec{u}](t), \qquad j = 1, \dots, N \tag{95}$$

where

$$K^\Omega_{j,\ell}[\vec{u}](t) = \int_0^t E^\Omega_{j,\ell}(t-\tau) * F_{j,\ell}(\vec{u}(\tau))d\tau,$$

$E^\Omega_{j,\ell}(t) = \kappa_{j,\ell}(D)E^\Omega_{P_j}$. We assume that P_j is homogeneous.

We write a Picard type iterative scheme

$$u^{k+1}_j(t) = E^\Omega_{P_j}[u^0_j](t) + \sum_{\ell=1}^{L} K^\Omega_{j,\ell}[\vec{u^k}](t), \qquad j = 1, \dots, N \tag{96}$$

for $k = 0, 1, \dots$ with $\vec{u^0} := 0$.

We need two crucial estimates, namely for some absolute constant $a > 0$

$$\max_{j,\ell} \|E^\Omega_{j,\ell}\|_{A^\gamma_{\frac{d_\ell}{m}+\frac{n}{m}(1-\frac{1}{r}),r}(+\infty;m)} \le C_1 \exp(a\gamma^{\frac{m-1}{m}}), \qquad \forall\gamma \ge 0 \tag{97}$$

$$\|K^\Omega_{j,\ell}[\vec{u}]\|_{A^\gamma_{\theta,q}(T;m)} \le C_2 \|E^\Omega_{j,\ell}\|_{A^\gamma_{\frac{d_\ell}{m}+\frac{n(s_\ell-1)}{mq},\frac{q}{q-s_\ell+1}}(T;m)} (\|\vec{u}\|_{A^\gamma_{\theta,q}(T;m)})^{s_\ell} T^{\rho_\ell} \tag{98}$$

where $r \in [1, +\infty]$ (resp. $r \in]1, +\infty]$) if $d_\ell > 0$ or $d_\ell = 0$ and $\kappa_{j,\ell}(\xi) \equiv const$ (resp. $d_\ell = 0$, $\kappa_{j,\ell}(\xi) \not\equiv const$, $\Omega = \mathbb{R}^n$),

$$(q, \theta) \in \partial\mathcal{C}^{m,s}_{p_{cr}}(n), \qquad \rho_\ell = \frac{m - d_\ell - (\theta + \frac{n}{q})(s_\ell - 1)}{m},$$

$C_1 = C_1(r) > 0$, $C_2 = C_2(\{F_{j,\ell}\}, r) > 0$. We note that in the case $\Omega = \mathbb{R}^n$ we have

$$\partial^\alpha_x E^{\mathbb{R}^n}_{j,\ell}(t,x) = \int e^{ix\xi - tP_j(\xi)}\kappa_{j,\ell}(\xi)\xi^\alpha \, \bar{d}\xi = t^{-\frac{d_\ell+|\alpha|+n}{m}}\varphi_{j,\ell}(\frac{x}{t^{\frac{1}{m}}}), \ \bar{d}\xi = (2\pi)^{-n}d\xi, \tag{99}$$

with $\mathcal{F}\varphi_{j,\ell}(\xi) = e^{-P_j(\xi)}\kappa_{j,\ell}(\xi)\xi^\alpha$. If $r \ge 2$ we estimate $\|\varphi_{j,\ell}\|_{L^r}$ by means of the Fourier transformation, the Young theorem and the Stirling formula. For the case $1 \le$

$r < 2$ we deduce the same result using integration by parts, the properties of the Fourier transform of homogeneous functions and the Stirling formula again. The case $\Omega = \mathbb{T}^n$ and $r \in [2, +\infty]$ is evident while (85), (97) for $\Omega = \mathbb{R}^n$, $r = 1$ and the representation

$$\partial_x^\alpha E_{j,\ell}^{\mathbb{T}^n}(t,x) = \sum_{\xi \in \mathbb{Z}^n} \partial_x^\alpha E_{j,\ell}^{\mathbb{R}^n}(t, x + 2\pi\xi), \qquad x \in \mathbb{T}^n \sim [-\pi, \pi]^n \tag{100}$$

yield the $L^1(\mathbb{T}^n)$ estimate (97) (see [39] for similar arguments). The Riesz-Thorin theorem concludes the proof of (97) for $\Omega = \mathbb{T}^n$. The key argument in showing (98) is a series of nonlinear superposition estimates in the framework of $A^\gamma_{\theta,q}(T; m)$. We note that $m \geq 1$ is essential for the validity of such estimates. Next, for given $R > 0$ we define

$$B_q^\gamma(R : T) = \{\vec{u} \in A^\gamma_{\frac{\theta}{m},q}(T; m) : \ \|\vec{u}\|_{A^\gamma_{\frac{\theta}{m},q}(T;m)} \leq R\}.$$

At the end we are reduced to find $R > 0$ and $T > 0$ such that

$$\|E_P^\Omega[\vec{u^0}]\|_{A^\gamma_{\frac{\theta}{m},q}(T;m)} + C_1 \exp(a\gamma^{\frac{m-1}{m}}) \sum_{\ell=1}^{L} T^{\rho_\ell} R^{s_\ell} \quad \leq \quad R, \tag{101}$$

$$C_2 \exp(a\gamma^{\frac{m-1}{m}}) \sum_{\ell=1}^{L} T^{\rho_\ell} R^{s_\ell - 1} \quad < \quad 1. \tag{102}$$

The estimates (102) allows us to show the convergence of the scheme above which leads to the existence–uniqueness statements for local and global solutions.

The self-similar solutions in the first part of Theorem 6 are obtained by the uniqueness and the homogeneity, while (92) and (93) are deduced by a suitable generalization of arguments used in [49] and [5].

Concerning the last part of Theorem 6, we follow the idea in [16], namely setting $g = v + w$, $v = E_P[u^0](1)$ (we consider the scalar case $g = \vec{g}$, $L = 1$) we obtain for w, an equation modeled by

$$w = \mathcal{H}_P^\kappa[(v + w)^s], \quad \mathcal{H}_P^\kappa[f] = \int_0^1 \int_{\mathbb{R}^n} \kappa(D) E_P^{\mathbb{R}^n}(1 - \tau, y) \tau^{-\frac{ns}{mp_{cr}}} f\left(\frac{y}{\sqrt[m]{t}}\right) dy\, d\tau \tag{103}$$

where $\kappa(D) \in \Psi_h^d(\mathbb{R}^n)$. The condition (94) allows us to generalize Lemma 6, p. 187 in [16], namely we show that $\mathcal{H}$ acts continuously from $L^{\frac{p_{cr}}{s}}(\mathbb{R}^n)$ to $L^{p_{cr}}(\mathbb{R}^n)$ using the Littlewood-Paley analysis and the characterization of the L^p spaces.

We point out, that if $\vec{u^0} \in (H_p^r(\Omega))^N$ and $p > 1$ we show that $\lim_{T \to 0} \|E_P[\vec{u^0}]\|_{A^\gamma_{\frac{\theta}{m},q}(T;m)} = 0$ for all $\theta = r^- + \frac{n}{p} - \frac{n}{q}$, $q \geq \max\{p, \frac{pn}{n-rp}\}$, $\gamma \geq 0$. Thus we recover and/or generalize the known local and global results for the semilinear heat equations when $r \geq r_{cr}(p)$ (see [38], [3], [5], [21] and [49] and the references therein). In particular, we extend the result of THEOREM 2.1 in [39] on the complex Ginzburg-Landau equation in $\mathbb{T}^n$, since THEOREM 2 $ii)$ allows initial data $u^0 \in H_2^{r_{cr}(2)}(\mathbb{T}^n) = H_2^{\frac{n}{2}-\frac{1}{\sigma}}(\Omega)$, provided $\sigma > \max\{\frac{1}{n}, \frac{4}{n+\sqrt{n^2+16n}}\}$. Furthermore, our local results on the analytic regularity yield $\rho_{[u]}(t) = O(t^{\frac{1}{2}})$, $t \searrow 0$ which improves the corresponding results for the Navier-Stokes equation for an incompressible fluid in $\Omega = \mathbb{T}^n$, $n = 2, 3$ while for the Ginzburg-Landau equation we get $\rho_{[u]}(t) = O(t^{\frac{1}{2}})$, $t \searrow 0$, the same rate as in [53], where the initial data are $L^\infty(\mathbb{R}^n)$. If $m = 4$ Theorem 4 and Theorem 5 yield new results for the Cahn-Hilliard equation $\partial_t u + \Delta^2 u + \Delta(u^s) = 0$. Here $p_{cr} = \frac{n(s-1)}{2}$, and $r_{cr}(p_{cr}) \in I_p$ iff $s > \frac{4+n}{n+2}$ which is always fulfilled since $s \geq 2$. Hence if $u^0 = \beta|D|^{r_{cr}(p)}\omega$, $\omega \in L^p(\mathbb{R}^n)$ if $p > 1$, $\omega \in \mathcal{M}(\mathbb{R}^n)$, $p \in [\max\{1, p_{cr}\}, p_{max}[$, $\beta \in \mathbb{R}$, (82) admits unique global solution $u(t, x)$ which belongs to $\mathcal{O}(\Gamma_{\gamma t^{\frac{1}{4}}})$ for all $t > 0$ provided $\|\omega\|_{L^p} < c\exp(-a\gamma^{\frac{1}{4}})$. We could consider fractional derivatives of measures as initial data iff $p_{cr} \leq 1$ which is equivalent to $s \in]\frac{4+n}{2+n}, \frac{n+2}{n}]$.

Our estimates on the analytic regularity globally in $t > 0$ seem to be completely new. We have examples for $\Omega = \mathbb{R}^n$ showing that our estimates on $\rho_{[u]}(t)$ are sharp at least within certain classes of solutions. If $\Omega = \mathbb{T}^n$ we could give in some cases better estimates of $\rho_{[u]}(t)$ as $t \to +\infty$.

Comparing Theorem 6 with the results in [16] for self-similar solutions, we point out that we allow initial data

$$\vec{u^0} \in (H_{\mathcal{S}'}^{-1}(\mathbb{R}^n))^N$$

such that $\vec{u^0}|_{S^{n-1}} \notin (L^\infty(S^{n-1}))^N$. We construct also self-similar solutions for the Cahn-Hilliard equation of the form $u(t, x) = t^{-\frac{1}{2(s-1)}} g(\frac{x}{\sqrt[4]{t}})$. As it concerns the last part of Theorem 6, it is an extension of Theorem 2, p. 181 in [16].

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HYPERBOLIC EQUATIONS WITH NON-LIPSCHITZ COEFFICIENTS

Abstract. The goal of this article is to present new trends in the theory of solutions valued in Sobolev spaces for strictly hyperbolic Cauchy problems of second order with non-Lipschitz coefficients. A very precise relation between oscillating behaviour of coefficients and loss of derivatives of solution is given. Several methods as energy method together with sharp Gårding's inequality and construction of parametrix are used to get optimal results. Counter-examples complete the article.

1. Introduction

In this course we are interested in the Cauchy problem

$$(1)\qquad \begin{aligned} &u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t,x) u_{x_k x_l} = 0 \qquad \text{on } (0,T)\times\mathbb{R}^n\ , \\ &u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x) \qquad \text{for } x \in \mathbb{R}^n\ . \end{aligned}$$

Setting $a(t,x,\xi) := \sum\limits_{k,l=1}^{n} a_{kl}(t,x)\xi_k\xi_l$ we suppose with a positive constant C the strict hyperbolicity assumption

$$(2)\qquad a(t,x,\xi) \geq C\ |\xi|^2$$

with $a_{kl} = a_{lk}\ ,\ k,l = 1,\cdots,n$.

DEFINITION 1. *The Cauchy problem (1) is well-posed if we can fix function spaces A_1, A_2 for the data φ, ψ in such a way that there exists a uniquely determined solution $u \in C([0,T], B_1) \cap C^1([0,T], B_2)$ possessing the domain of dependence property.*

The question we will discuss in this course is how the regularity of the coefficients $a_{kl} = a_{kl}(t,x)$ is related to the well-posedness of the Cauchy problem (1).

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2. Low regularity of coefficients

2.1. L_1-property with respect to t

In [10] the authors studied the Cauchy problem

$$u_{tt} - \sum_{k,l=1}^{n} \partial_{x_k}(a_{kl}(t,x)\partial_{x_l}u) = 0 \quad \text{on } (0,T)\times\Omega\,,$$
$$u(0,x) = \varphi(x)\,,\ u_t(0,x) = \psi(x) \quad \text{on } \Omega\,,$$

where Ω is an arbitrary open set of $\mathbb{R}^n$ and $T > 0$. The coefficients of the elliptic operator in self-adjoint form satisfy the next analyticity assumption:
For any compact set K of Ω and for any multi-index β there exist a constant A_K and a function $\Lambda_K = \Lambda_K(t)$ belonging to $L^1(0,T)$ such that

$$|\sum_{k,l=1}^{n} \partial_x^\beta a_{kl}(t,x)| \le \Lambda_K(t) A_K^{|\beta|}|\beta|!\,.$$

Moreover, the strict hyperbolicity condition

$$\lambda_0|\xi|^2 \le \sum_{k,l=1}^{n} a_{kl}(t,x)\xi_k\xi_l \le \Lambda(t)|\xi|^2$$

is satisfied with $\lambda_0 > 0$ and $\sqrt{\Lambda(t)} \in L^1(0,T)$.

THEOREM 1. *Let us suppose these assumptions. If the data φ and ψ are real analytic on Ω, then there exists a unique solution $u = u(t,x)$ on the conoid $\Gamma_\Omega^T \subset \mathbb{R}^{n+1}$. The conoid is defined by*

$$\Gamma_\Omega^T = \Big\{(t,x) :\ \text{dist}\,(x, \mathbb{R}^n\setminus\Omega) > \int_0^T \sqrt{\Lambda(s)}ds\ ,\ t\in[0,T]\Big\}\,.$$

The solution is C^1 in t and real analytic in x.

QUESTIONS. The Cauchy problem can be studied for elliptic equations in the case of analytic data. Why do we need the hyperbolicity assumption? What is the difference between the hyperbolic and the elliptic case?

We know from the results of [8] for the Cauchy problem

$$u_{tt} - a(t)u_{xx} = 0\,,\quad u(0,x) = \varphi(x)\,,\quad u_t(0,x) = \psi(x)\,,$$

that assumptions like $a \in L^p(0,T)$, $p > 1$, or even $a \in C[0,T]$, don't allow to weaken the analyticity assumption for data φ, ψ to get well-posedness results.

THEOREM 2. *For any class $\mathcal{E}\{M_h\}$ of infinitely differentiable functions which strictly contains the space $\mathcal{A}$ of real analytic functions on $\mathbb{R}$ there exists a coefficient $a = a(t) \in C[0,T]$, $a(t) \geq \lambda > 0$, such that the above Cauchy problem is not well-posed in $\mathcal{E}\{M_h\}$.*

2.2. C^κ-property with respect to t

Let us start with the Cauchy problem

$$u_{tt} - a(t)u_{xx} = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x),$$

where $a \in C^\kappa[0,T]$, $\kappa \in (0,1)$. From [5] we have the following result:

THEOREM 3. *If $a \in C^\kappa[0,T]$, then this Cauchy problem is well-posed in Gevrey classes G^s for $s < \frac{1}{1-\kappa}$. To φ, $\psi \in G^s$ we have a uniquely determined solution $u \in C^2([0,T], G^s)$.*

We can use different definitions for G^s (by the behaviour of derivatives on compact subsets, by the behaviour of Fourier transform). If

- $s = \frac{1}{1-\kappa}$, then we should be able to prove local existence in t;
- $s > \frac{1}{1-\kappa}$, then there is no well-posedness in G^s.

The paper [22] is concerned with the strictly hyperbolic Cauchy problem

$$\begin{aligned} &u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t,x)u_{x_k x_l} + \text{ lower order terms } = f(t,x) \\ &u(0,x) = \varphi(x), \; u_t(0,x) = \psi(x), \end{aligned}$$

with coefficients depending Hölderian on t and Gevrey on x. It was proved well-posedness in Gevrey spaces G^s. Here G^s stays for a scale of Banach spaces.

One should understand

- how to define the Gevrey space with respect to x, maybe some suitable dependence on t is reasonable, thus scales of Gevrey spaces appear;
- the difference between $s = \frac{1}{1-\kappa}$ and $s < \frac{1}{1-\kappa}$, in the first case the solution should exist locally, in the second case globally in t if we constructed the right scale of Gevrey spaces.

REMARK 1. In the proof of Theorem 3 we use instead of $a \in C^\kappa[0,T]$ the condition $\int_0^{T-\tau} |a(t+\tau) - a(t)|dt \leq A\,\tau^\kappa$ for $\tau \in [0, T/2]$. But then the solution belongs only to $H^{2,1}([0,T], G^s)$.

3. High regularity of coefficients

3.1. Lip-property with respect to t

Let us suppose $a \in C^1[0,T],\ a(t) \geq C > 0$, in the strictly hyperbolic Cauchy problem

$$u_{tt} - a(t)u_{xx} = 0,$$
$$u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x).$$

Using the energy method and Gronwall's Lemma one can prove immediately the well-posedness in Sobolev spaces H^s, that is, if $\varphi \in H^{s+1}(\mathbb{R}^n),\ \psi \in H^s(\mathbb{R}^n)$, then there exists a uniquely determined solution $u \in C([0,T], H^{s+1}) \cap C^1([0,T], H^s)\ \ (s \in N_0)$. A more precise result is given in [20].

THEOREM 4. *If the coefficients* $a_{kl} \in C([0,T], B^s) \cap C^1([0,T], B^0)$ *and* $\varphi \in H^{s+1},\ \psi \in H^s$*, then there exists a uniquely determined solution* $u \in C([0,T], H^{s+1}) \cap C^1([0,T], H^s)$*. Moreover, the energy inequality* $E_k(u)(t) \leq C_k E_k(u)(0)$ *holds for* $0 \leq k \leq s$*, where* $E_k(u)$ *denotes the energy of k'th order of the solution* u*.*

By B^∞ we denote the space of infinitely differentiable functions having bounded derivatives on $\mathbb{R}^n$. Its topology is generated by the family of norms of spaces $B^s,\ s \in \mathbb{N}$, consisting of functions with bounded derivatives up to order s.

REMARK 2. For our starting problem we can suppose instead of $a \in C^1[0,T]$ the condition $\int_0^{T-\tau} |a(t+\tau) - a(t)|dt \leq A\ \tau$ for $\tau \in [0, T/2]$. Then we have the same statement as in Theorem 4. The only difference is that the solution belongs to $C([0,T], H^{s+1}) \cap H^{1,2}([0,T], H^s) \cap H^{2,1}([0,T], H^{s-1})$.

PROBLEM 1. Use the literature to get information about whether one can weaken the assumptions for a_{kl} from Theorem 4 to show the energy estimates $E_k(u)(t) \leq C_k E_k(u)(0)$ for $0 \leq k \leq s$.

All results from this section imply that no loss of derivatives appears, that is, the energy $E_k(u)(t)$ of k-th order can be estimated by the energy $E_k(u)(0)$ of k-th order.

Let us recall some standard arguments:

- If the coefficients have more regularity $C^1([0,T], B^\infty)$, and the data φ and ψ are from H^∞, then the Cauchy problem is H^∞ well-posed, that is, there exists a uniquely determined solution from $C^2([0,T], H^\infty)$.
  This result follows from the energy inequality.
- Together with the domain of dependence property from H^∞ well-posedness we conclude C^∞ well-posedness, that is, to arbitrary data φ and ψ from C^∞ there exists a uniquely determined solution from $C^2([0,T], C^\infty)$.

This result follows from the energy inequality and the domain of dependence property.

Results for domain of dependence property:

THEOREM 5 ([5]). *Let us consider the strictly hyperbolic Cauchy problem*

$$u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t) u_{x_k x_l} = f(t,x),\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x)\ .$$

The coefficients $a_{kl} = a_{lk}$ are real and belong to $L^1(0,T)$. Moreover, $\sum\limits_{k,l=1}^{n} a_{kl}(t)\xi_k\xi_l \geq \lambda_0|\xi|^2$ with $\lambda_0 > 0$. If $u \in H^{2,1}([0,T],\mathcal{A}')$ is a solution for given $\varphi, \psi \in \mathcal{A}'$ and $f \in L^1([0,T],\mathcal{A}')$, then from $\varphi \equiv \psi \equiv f \equiv 0$ for $|x-x_0| < \rho$ it follows that $u \equiv 0$ on the set

$$\{(t,x) \in [0,T] \times \mathbb{R}^n : |x-x_0| < \rho - \int_0^t \sqrt{|a(s)|}ds\}.$$

Here $|a(t)|$ denotes the Euclidean matrix norm, $\mathcal{A}'$ denotes the space of analytic functionals.

THEOREM 6 ([20]). *Let us consider the strictly hyperbolic Cauchy problem*

$$u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t,x) u_{x_k x_l} = f(t,x),\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x).$$

The real coefficients $a_{kl} = a_{lk}$ satisfy $a_{kl} \in C^{1+\sigma}([0,T]\times\mathbb{R}^n) \cap C([0,T],B^0)$. Let us define $\lambda_{\max}^2 := \sup\limits_{|\xi|=1,[0,T]\times\mathbb{R}^n} a_{kl}(t,x)\xi_k\xi_l$. Then $\varphi = \psi \equiv 0$ on $D \cap \{t=0\}$ and $f \equiv 0$ on D implies $u \equiv 0$ on D, where D denotes the interior for $t \geq 0$ of the backward cone $\{(x,t) : |x-x_0| = \lambda_{\max}(t_0-t),\ (x_0,t_0) \in (0,T] \times \mathbb{R}^n\}$.

3.2. Finite loss of derivatives

In this section we are interested in weakening the Lip-property for the coefficients $a_{kl} = a_{kl}(t)$ in such a way, that we can prove energy inequalities of the form $E_{s-s_0}(u)(t) \leq E_s(u)(t)$, where $s_0 > 0$. The value s_0 describes the so-called *loss of derivatives*.

Global condition

The next idea goes back to [5]. The authors supposed the so-called LogLip-property, that is, the coefficients a_{kl} satisfy

$$|a_{kl}(t_1) - a_{kl}(t_2)| \leq C|t_1 - t_2|\,|\ln|t_1-t_2|\,| \quad \text{for all} \quad t_1, t_2 \in [0,T], t_1 \neq t_2\ .$$

More precisely, the authors used the condition

$$\int_0^{T-\tau} |a_{kl}(t+\tau) - a_{kl}(t)|dt \le C\ \tau(|\ln\ \tau| + 1) \quad \text{for} \quad \tau \in (0, T/2]\ .$$

Under this condition well-posedness in C^∞ was proved.

As far as the author knows there is no classification of LogLip-behaviour with respect to the related loss of derivatives. He expects the following classification for solutions of the Cauchy problem $u_{tt} - a(t)u_{xx} = 0,\ u(0, x) = \varphi(x),\ u_t(0, x) = \psi(x)$:

Let us suppose $|a(t_1) - a(t_2)| \le C|t_1 - t_2|\ |\ln|t_1 - t_2||^\gamma$ *for all* $t_1, t_2 \in [0, T]$, $t_1 \ne t_2$. *Then the energy estimates* $E_{s-s_0}(u)(t) \le C\ E_s(u)(0)$ *should hold, where*

- $s_0 = 0$ *if* $\gamma = 0$,
- s_0 *is arbitrary small and positive if* $\gamma \in (0, 1)$,
- s_0 *is positive if* $\gamma = 1$,
- *there is no positive constant* s_0 *if* $\gamma > 1$ *(infinite loss of derivatives).*

The statement for $\gamma = 0$ *can be found in [5]. The counter-example from [9] implies the statement for* $\gamma > 1$.

OPEN PROBLEM 1. Prove the above statement for $\gamma \in (0, 1)$!

OPEN PROBLEM 2. The results of [9] show that $\gamma = 1$ gives a finite loss of derivatives. Do we have a concrete example which shows that the solution has really a finite loss of derivatives?

We already mentioned the paper [9]. In this paper the authors studied strictly hyperbolic Cauchy problems with coefficients of the principal part depending LogLip on spatial and time variables.

- If the principal part is as in (1.1) but with an elliptic operator in divergence form, then the authors derive energy estimates depending on a suitable low energy of the data and of the right-hand side.
- If the principal part is as in (1.1) but with coefficients which are B^∞ in x and LogLip in t, then the energy estimates depending on arbitrary high energy of the data and of the right-hand side.
- In all these energy estimates which exist for $t \in [0, T^*]$, where T^* is a suitable positive constant independent of the regularity of the data and right-hand side, the loss of derivatives depends on t.

It is clear, that these energy estimates are an important tool to prove (locally in t) well-posedness results.

Local condition

A second possibility to weaken the Lip-property with respect to t goes back to [6]. Under the assumptions

$$a \in C[0,T] \cap C^1(0,T],\ |ta'(t)| \leq C \quad \text{for} \quad t \in (0,T], \tag{3}$$

the authors proved a C^∞ well-posedness result for $u_{tt} - a(t)u_{xx} = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x)$ (even for more general Cauchy problems). They observed the effect of a finite loss of derivatives.

REMARK 3. Let us compare the local condition with the global one from the previous section. If $a = a(t) \in$ LogLip$[0,T]$, then the coefficient may have an irregular behaviour (in comparison with the Lip-property) on the whole interval $[0,T]$. In (3) the coefficient has an irregular behaviour only at $t = 0$. Away from $t = 0$ it belongs to C^1. Coefficients satisfying (3) don't fulfil the non-local condition

$$\int_0^{T-\tau} |a(t+\tau) - a(t)|dt \leq C\tau(|\ln\tau| + 1) \quad \text{for} \quad \tau \in (0, T/2]\,.$$

We will prove the next theorem by using the energy method and the following generalization of Gronwall's inequality to differential inequalities with singular coefficients. The method of proof differs from that of [6].

LEMMA 1 (LEMMA OF NERSESJAN [21]). *Let us consider the differential inequality*

$$y'(t) \leq K(t)y(t) + f(t)$$

for $t \in (0,T)$, where the functions $K = K(t)$ and $f = f(t)$ belong to $C(0,T]$, $T > 0$. Under the assumptions

- $\int_0^\delta K(\tau)d\tau = \infty\,,\ \int_\delta^T K(\tau)d\tau < \infty\,,$
- $\lim_{\delta\to+0} \int_\delta^t \exp\left(\int_s^t K(\tau)d\tau\right) f(s)ds$ *exists,*
- $\lim_{\delta\to+0} y(\delta)\exp\left(\int_\delta^t K(\tau)d\tau\right) = 0$

for all $\delta \in (0,t)$ and $t \in (0,T]$, every solution belonging to $C[0,T] \cap C^1(0,T]$ satisfies

$$y(t) \leq \int_0^t \exp\Big(\int_s^t K(\tau)d\tau\Big) f(s)ds\,.$$

THEOREM 7. *Let us consider the strictly hyperbolic Cauchy problem*

$$u_{tt} - a(t)u_{xx} = 0,\ u(0, x) = \varphi(x), u_t(0, x) = \psi(x)\ ,$$

where $a = a(t)$ satisfies with $\gamma \geq 0$ the conditions

$$a \in C[0, T] \cap C^1(0, T],\ |t^\gamma a'(t)| \leq C \quad \text{for} \quad t \in (0, T]\ . \tag{4}$$

Then this Cauchy problem is C^∞ well-posed iff $\gamma \in [0, 1]$. If

- $\gamma \in [0, 1)$*, then we have no loss of derivatives, that is, the energy inequalities $E_s(u)(t) \leq C_s\ E_s(u)(0)$ hold for $s \geq 0$;*
- $\gamma = 1$*, then we have a finite loss of derivatives, that is, the energy inequalities $E_{s-s_0}(u)(t) \leq C_s\ E_s(u)(0)$ hold for large s with a positive constant s_0.*

Proof. The proof will be divided into several steps.

Step 1. Cone of dependence

Let $u \in C^2([0, T], C^\infty(\mathbb{R}))$ be a solution of the Cauchy problem. If $\chi = \chi(x) \in C_0^\infty(\mathbb{R})$ and $\chi \equiv 1$ on $[x_0 - \rho, x_0 + \rho]$, then $v = \chi\, u \in C^2([0, T], \mathcal{A}')$ is a solution of

$$v_{tt} - a(t)v_{xx} = f(t, x)\ ,\quad v(0, x) = \tilde{\varphi}(x),\ v_t(0, x) = \tilde{\psi}(x)\ ,$$

where $\tilde{\varphi}, \tilde{\psi} \in C_0^\infty(\mathbb{R})$ and $f \in C([0, T], \mathcal{A}')$. Due to Theorem 5 we know that $v = v(t, x)$ is uniquely determined in $\{(t, x) \in [0, T] \times \mathbb{R}^n\colon |x - x_0| < \rho - \int_0^t \sqrt{|a(s)|}ds\}$. Hence, u is uniquely determined in this set, too. This implies $\varphi \equiv \psi \equiv 0$ on $[x_0 - \rho, x_0 + \rho]$ gives $u \equiv 0$ in this set. It remains to derive an energy inequality (see Section 3.1).

Step 2. The statement for $\gamma \in [0, 1)$

If $\gamma \in [0, 1)$, then

$$\int_0^{T-\tau} \left|\frac{a(t + \tau) - a(t)}{\tau}\right| dt \leq \int_0^{T-\tau} |a'(\theta(t, \tau))| dt \leq \int_0^{T-\tau} \frac{C}{t^\gamma}\, dt \leq C\ .$$

Thus the results from [5] are applicable.

Step 3. The statement for $\gamma > 1$

From the results of [6], we understand, that there is no C^∞ well-posedness for $\gamma > 1$. One can only prove well-posedness in suitable Gevrey spaces. Now let us consider the remaining case $\gamma = 1$.

Step 4. A family of auxiliary problems

We solve the next family of auxiliary problems:

$$\begin{aligned}
u_{tt}^{(0)} &= 0\ ,\quad u^{(0)}(0, x) = \varphi(x)\ ,\quad u_t^{(0)}(0, x) = \psi(x)\ ,\\
u_{tt}^{(1)} &= a(t)u_{xx}^{(0)}\ ,\quad u^{(1)}(0, x) = u_t^{(1)}(0, x) = 0\ ,\\
u_{tt}^{(2)} &= a(t)u_{xx}^{(1)}\ ,\quad u^{(2)}(0, x) = u_t^{(2)}(0, x) = 0\ , \cdots ,\\
u_{tt}^{(r)} &= a(t)u_{xx}^{(r-1)}\ ,\quad u^{(r)}(0, x) = u_t^{(r)}(0, x) = 0\ .
\end{aligned}$$

For the solution of our starting problem we choose the representation $u = \sum_{k=0}^{r} u^{(k)} + v$. Then v solves the Cauchy problem $v_{tt} - a(t)v_{xx} = a(t)u^{(r)}_{xx}$, $v(0,x) = v_t(0,x) = 0$. Now let us determine the asymptotic behaviour of $u^{(r)}$ near $t = 0$. We have

$$|u^{(0)}(t,x)| \le |\varphi(x)| + t|\psi(x)| \ , \ |u^{(1)}(t,x)| \le C\ t^2(|\varphi_{xx}(x)| + t|\psi_{xx}(x)|),$$
$$|u^{(2)}(t,x)| \le C\ t^4(|\partial_x^4\varphi| + t|\partial_x^4\psi|)$$

and so on.

LEMMA 2. *If* $\varphi \in H^{s+1}$, $\psi \in H^s$, *then* $u^{(k)} \in C^2([0,T], H^{s-2k})$ *and* $\|u^{(k)}\|_{C([0,t],H^{s-2k})} \le C_k t^{2k}$ *for* $k = 0, \cdots, r$ *and* $s \ge 2r+2$.

Step 5. Application of Nersesjan's lemma
Now we are interested in deriving an energy inequality for a given solution $v = v(t,x)$ to the Cauchy problem

$$v_{tt} - a(t)v_{xx} = a(t)u^{(r)}_{xx} \ , \quad v(0,x) = v_t(0,x) = 0 \ .$$

Defining the usual energy we obtain

$$\begin{aligned} E'(v)(t) &\le C_a|a'(t)|\ E(v)(t) + E(v)(t) + C\|u^{(r)}_{xx}(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ &\le \frac{C_a}{t}\ E(v)(t) + E(v)(t) + C_a\|u^{(r)}_{xx}(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ &\le \frac{C_a}{t}\ E(v)(t) + E(v)(t) + C_{a,r}t^{4r} \quad . \end{aligned}$$

If $4r > C_a$ (C_a depends on $a = a(t)$ only), then Lemma 1 is applicable with $y(t) = E(v)(t)$, $K(t) = \frac{C_a}{t}$ and $f(t) = C_{a,r}t^{4r}$. It follows that $E(v)(t) \le C_{a,r}t^{4r}$.

LEMMA 3. *If v is a solution of the above Cauchy problem which has an energy, then this energy fulfils* $E(v)(t) \le C_{a,r}t^{4r}$.

Step 6. Existence of a solution
To prove the existence we consider for $\varepsilon > 0$ the auxiliary Cauchy problems

$$v_{tt} - a(t+\varepsilon)v_{xx} = a(t)u^{(r)}_{xx} \in C([0,T], L^2(\mathbb{R})) \ ,$$

with homogeneous data. Then $a_\varepsilon = a_\varepsilon(t) = a(t+\varepsilon) \in C^1[0,T]$. For solutions $v_\varepsilon \in C^1([0,T], L^2(\mathbb{R}))$ which exist from strictly hyperbolic theory, the same energy inequality from the previous step holds. Usual convergence theorems prove the existence of a solution $v = v(t,x)$. The loss of derivatives is $s_0 = 2r+2$. All statements of our theorem are proved.

□

A refined classification of oscillating behaviour

Let us suppose more regularity for a, let us say, $a \in L^\infty[0, T] \cap C^2(0, T]$. The higher regularity allows us to introduce a refined classification of oscillations.

DEFINITION 2. *Let us assume additionally the condition*

$$|a^{(k)}(t)| \leq C_k \left(\frac{1}{t} \left(\ln \frac{1}{t} \right)^\gamma \right)^k \ , \quad \text{for } \ k = 1, 2 \ . \tag{5}$$

We say, that the oscillating behaviour of a is

- *very slow if $\gamma = 0$,*
- *slow if $\gamma \in (0, 1)$,*
- *fast if $\gamma = 1$,*
- *very fast if condition (3.3) is not satisfied for $\gamma = 1$.*

EXAMPLE 1. If $a = a(t) = 2 + \sin \left(\ln \frac{1}{t} \right)^\alpha$, then the oscillations produced by the sin term are very slow (slow, fast, very fast) if $\alpha \leq 1$ ($\alpha \in (1, 2), \alpha = 2, \alpha > 2$).

Now we are going to prove the next result yielding a connection between the type of oscillations and the loss of derivatives which appears. The proof uses ideas from the papers [7] and [14]. The main goal is the construction of WKB-solutions. We will sketch our approach, which is a universal one in the sense, that it can be used to study more general models from non-Lipschitz theory, weakly hyperbolic theory and the theory of $L_p - L_q$ decay estimates.

THEOREM 8. *Let us consider*

$$u_{tt} - a(t)u_{xx} = 0 \ , \quad u(0, x) = \varphi(x) \ , \quad u_t(0, x) = \psi(x) \ ,$$

where $a = a(t)$ satisfies the condition (5), and the data φ , ψ belong to H^{s+1}, H^s respectively. Then the following energy inequality holds:

$$E(u)(t) \mid_{H^{s-s_0}} \leq C(T) E(u)(0) \mid_{H^s} \quad \textit{for all } \ t \in (0, T] \ , \tag{6}$$

where

- *$s_0 = 0$ if $\gamma = 0$,*
- *s_0 is an arbitrary small positive constant if $\gamma \in (0, 1)$,*
- *s_0 is a positive constant if $\gamma = 1$,*
- *there does not exist a positive constant s_0 satisfying (6) if $\gamma > 1$, that is, we have an infinite loss of derivatives.*

Proof. The proof will be divided into several steps. Without loss of generality we can suppose that T is small. After partial Fourier transformation we obtain

$$v_{tt} + a(t)\xi^2 v = 0 \ , \quad v(0,\xi) = \hat{\varphi}(\xi) \ , \quad v_t(0,\xi) = \hat{\psi}(\xi) \ . \tag{7}$$

Step 1. Zones
We divide the phase space $\{(t,\xi) \in [0,T] \times \mathbb{R} : |\xi| \geq M\}$ into two zones by using the function $t = t_\xi$ which solves $t_\xi \langle\xi\rangle = N(\ln\langle\xi\rangle)^\gamma$. The constant N is determined later. Then the pseudo-differential zone $Z_{pd}(N)$, hyperbolic zone $Z_{hyp}(N)$, respectively, is defined by

$$Z_{pd}(N) = \{(t,\xi) : t \leq t_\xi\}, \quad Z_{hyp}(N) = \{(t,\xi) : t \geq t_\xi\} \ .$$

Step 2. Symbols
To given real numbers $m_1, m_2 \geq 0,\ r \leq 2$, we define

$$\begin{aligned} &S_r\{m_1,m_2\} = \{d = d(t,\xi) \in L^\infty([0,T] \times \mathbb{R}) : \\ &|D_t^k D_\xi^\alpha d(t,\xi)| \leq C_{k,\alpha} \langle\xi\rangle^{m_1 - |\alpha|} \left(\frac{1}{t}\left(\ln\frac{1}{t}\right)^\gamma\right)^{m_2+k}, \ k \leq r, \ (t,\xi) \in Z_{hyp}(N)\}. \end{aligned}$$

These classes of symbols are only defined in $Z_{hyp}(N)$.
Properties:

- $S_{r+1}\{m_1,m_2\} \subset S_r\{m_1,m_2\}$;
- $S_r\{m_1 - p, m_2\} \subset S_r\{m_1,m_2\}$ for all $p \geq 0$;
- $S_r\{m_1 - p, m_2 + p\} \subset S_r\{m_1,m_2\}$ for all $p \geq 0$, this follows from the definition of $Z_{hyp}(N)$;
- if $a \in S_r\{m_1,m_2\}$ and $b \in S_r\{k_1,k_2\}$, then $a\,b \in S_r\{m_1+k_1, m_2+k_2\}$;
- if $a \in S_r\{m_1,m_2\}$, then $D_t a \in S_{r-1}\{m_1, m_2+1\}$, and $D_\xi^\alpha a \in S_r\{m_1 - |\alpha|, m_2\}$.

Step 3. Considerations in $Z_{pd}(N)$
Setting $V = (\xi v, D_t v)^T$ the equation from (7) can be transformed to the system of first order

$$D_t V = \begin{pmatrix} 0 & \xi \\ a(t)\xi & 0 \end{pmatrix} V =: A(t,\xi)V \ . \tag{8}$$

We are interested in the fundamental solution $X = X(t,r,\xi)$ to (8) with $X(r,r,\xi) = I$ (identity matrix). Using the matrizant we can write X in an explicit way by

$$X(t,r,\xi) = I + \sum_{k=1}^{\infty} i^k \int_r^t A(t_1,\xi) \int_r^{t_1} A(t_2,\xi) \cdots \int_r^{t_{k-1}} A(t_k,\xi)\,dt_k \cdots dt_1 \ .$$

The norm $\|A(t,\xi)\|$ can be estimated by $C\langle\xi\rangle$. Consequently

$$\int_0^{t_\xi} \|A(s,\xi)\|ds \le Ct_\xi\langle\xi\rangle = C_N(\ln\langle\xi\rangle)^\gamma .$$

The solution of the Cauchy problem to (8) with $V(0,\xi) = V_0(\xi)$ can be represented in the form $V(t,\xi) = X(t,0,\xi)V_0(\xi)$. Using

$$\|X(t,0,\xi)\| \le \exp(\int_0^t \|A(s,\xi)\|ds) \le \exp(C_N(\ln\langle\xi\rangle)^\gamma)$$

the next result follows.

LEMMA 4. *The solution to (8) with Cauchy condition $V(0,\xi) = V_0(\xi)$ satisfies in $Z_{pd}(N)$ the energy estimate*

$$|V(t,\xi)| \le \exp(C_N(\ln\langle\xi\rangle)^\gamma)|V_0(\xi)| .$$

REMARK 4. In $Z_{pd}(N)$ we are near to the line $t = 0$, where the derivative of the coefficient $a = a(t)$ has an irregular behaviour. It is not a good idea to use the hyperbolic energy $(\sqrt{a(t)}\xi\, v, D_t v)$ there because of the "bad" behaviour of $a' = a'(t)$. To avoid this fact we introduce the energy $(\xi\, v, D_t v)$.

Step 4. Two steps of diagonalization procedure
Substituting $V := (\sqrt{a(t)}\xi\, v, D_t v)^T$ (hyperbolic energy) brings the system of first order

$$D_t V - \begin{pmatrix} 0 & \sqrt{a(t)}\xi \\ \sqrt{a(t)}\xi & 0 \end{pmatrix} V - \frac{D_t a}{2a}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V = 0 . \tag{9}$$

The first matrix belongs to the symbol class $S_2\{1,0\}$, the second one belongs to $S_1\{0,1\}$. Setting $V_0 := MV$, $M = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, this system can be transformed to the first order system

$$D_t V_0 - M\begin{pmatrix} 0 & \sqrt{a(t)}\xi \\ \sqrt{a(t)}\xi & 0 \end{pmatrix} M^{-1} V_0 - M\frac{D_t a}{2a}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M^{-1} V_0 = 0 ,$$

$$D_t V_0 - \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} V_0 - \frac{D_t a}{4a}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V_0 = 0 ,$$

where $\tau_{1/2} := \mp\sqrt{a(t)}\xi + \frac{1}{4}\frac{D_t a}{a}$. Thus we can write this system in the form $D_t V_0 - \mathcal{D}V_0 - R_0 V_0 = 0$, where

$$\mathcal{D} := \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \in S_1\{1,0\} ; \; R_0 = \frac{1}{4}\frac{D_t a}{a}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in S_1\{0,1\} .$$

This step of diagonalization is the *diagonalization of our starting system (9) modulo* $R_0 \in S_1\{0,1\}$.

Let us set

$$\mathcal{N}^{(1)} := -\frac{1}{4}\,\frac{D_t a}{a}\begin{pmatrix} 0 & \frac{1}{\tau_1-\tau_2} \\ \frac{1}{\tau_2-\tau_1} & 0 \end{pmatrix} = \frac{D_t a}{8a^{3/2}\xi}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the matrix $N_1 := I + \mathcal{N}^{(1)}$ is invertible in $Z_{hyp}(N)$ for sufficiently large N. This follows from the definition of $Z_{hyp}(N)$, from

$$\|N_1 - I\| = \|\mathcal{N}^{(1)}\| \le C_a \frac{1}{t|\xi|}\left(\ln\frac{1}{t}\right)^{\gamma} \le \frac{C_a}{N}\left(\frac{\ln\frac{1}{t}}{\ln\langle\xi\rangle}\right)^{\gamma} \le \frac{C_a}{N} \le \frac{1}{2},$$

if N is large, and from

$$\ln\langle\xi\rangle - \ln\frac{1}{t} \ge \ln N + \ln(\ln\langle\xi\rangle)^{\gamma}.$$

We observe that on the one hand $\mathcal{D}N_1 - N_1\mathcal{D} = R_0$ and on the other hand $(D_t - \mathcal{D} - R_0)N_1 = N_1(D_t - \mathcal{D} - R_1)$, where $R_1 := -N_1^{-1}(D_t\mathcal{N}^{(1)} - R_0\mathcal{N}^{(1)})$. Taking account of $\mathcal{N}^{(1)} \in S_1\{-1,1\}$, $N_1 \in S_1\{0,0\}$ and $R_1 \in S_0\{-1,2\}$ the transformation $V_0 =: N_1 V_1$ gives the following first order system:

$$D_t V_1 - \mathcal{D}V_1 - R_1 V_1 = 0\ ,\ \ \mathcal{D} \in S_1\{1,0\}\ ,\ \ R_1 \in S_0\{-1,2\}\ .$$

The second step of diagonalization is the *diagonalization of our starting system (9) modulo* $R_1 \in S_0\{-1,2\}$.

Step 5. Representation of solution of the Cauchy problem
Now let us devote to the Cauchy problem

$$\begin{aligned} &D_t V_1 - \mathcal{D}V_1 - R_1 V_1 = 0\ , \\ &V_1(t_\xi,\xi) = V_{1,0}(\xi) := N_1^{-1}(t_\xi,\xi) M\, V(t_\xi,\xi). \end{aligned} \tag{10}$$

If we have a solution $V_1 = V_1(t,\xi)$ in $Z_{hyp}(N)$, then $V = V(t,\xi) = M^{-1}N_1(t,\xi)V_1(t,\xi)$ solves (9) with given $V(t_\xi,\xi)$ on $t = t_\xi$.
The matrix-valued function

$$E_2(t,r,\xi) := \begin{pmatrix} \exp\left(i\int_r^t(-\sqrt{a(s)}\xi + \frac{D_s a(s)}{4a(s)})ds\right) & 0 \\ 0 & \exp\left(i\int_r^t(\sqrt{a(s)}\xi + \frac{D_s a(s)}{4a(s)})ds\right) \end{pmatrix}$$

solves the Cauchy problem $(D_t - \mathcal{D})E(t,r,\xi) = 0,\ E(r,r,\xi) = I$. We define the matrix-valued function $H = H(t,r,\xi),\ t,r \ge t_\xi$, by

$$H(t,r,\xi) := E_2(r,t,\xi)R_1(t,\xi)E_2(t,r,\xi)\ .$$

Using the fact that $\int\limits_r^t \frac{\partial_s a(s)}{4a(s)}\, ds = \ln\ a(s)^{1/4}\ |_r^t$ (this integral depends only on a, but is independent of the influence of a') the function H satisfies in $Z_{hyp}(N)$ the estimate

$$\|H(t,r,\xi)\| \ \leq\ \frac{C}{\langle\xi\rangle}\ \left(\frac{1}{t}\left(\ln\ \frac{1}{t}\right)^{\gamma}\right)^2\ . \tag{11}$$

Finally, we define the matrix-valued function $Q = Q(t,r,\xi)$ is defined by

$$Q(t,r,\xi) := \sum_{k=1}^{\infty} i^k \int\limits_r^t H(t_1,r,\xi)dt_1 \int\limits_r^{t_1} H(t_2,r,\xi)dt_2 \cdots \int\limits_r^{t_{k-1}} H(t_k,r,\xi)dt_k\ .$$

The reason for introducing the function Q is that

$$V_1 = V_1(t,\xi) := E_2(t,t_\xi,\xi)(I + Q(t,t_\xi,\xi))V_{1,0}(\xi)$$

represents a solution to (10).

Step 6. Basic estimate in $Z_{hyp}(N)$

Using (11) and the estimate $\int\limits_{t_\xi}^t \|H(s,t_\xi,\xi)\|ds\ \leq\ C_N(\ln\langle\xi\rangle)^{\gamma}$ we get from the representation for Q immediately

$$\|Q(t,t_\xi,\xi)\| \leq \exp\Big(\int\limits_{t_\xi}^t \|H(s,t_\xi,\xi)\|ds\Big) \leq \exp(C_N(\ln\langle\xi\rangle)^{\gamma})\ . \tag{12}$$

Summarizing the statements from the previous steps gives together with (12) the next result.

LEMMA 5. *The solution to (9) with Cauchy condition on $t = t_\xi$ satisfies in $Z_{hyp}(N)$ the energy estimate*

$$|V(t,\xi)| \leq C\ \exp(C_N(\ln\langle\xi\rangle)^{\gamma})|V(t_\xi,\xi)|\ .$$

Step 7. Conclusions

From Lemmas 4 and 5 we conclude

LEMMA 6. *The solution $v = v(t,\xi)$ to*

$$v_{tt} + a(t)\xi^2 v = 0\ , \quad v(0,\xi) = \hat{\varphi}(\xi)\ , \quad v_t(0,\xi) = \hat{\psi}(\xi)$$

satisfies the a-priori estimate

$$\left|\begin{pmatrix} \xi\ v(t,\xi) \\ v_t(t,\xi) \end{pmatrix}\right| \leq C\ \exp(C_N(\ln\langle\xi\rangle)^{\gamma})\left|\begin{pmatrix} \xi\ \hat{\varphi}(\xi) \\ \hat{\psi}(\xi) \end{pmatrix}\right|$$

for all $(t,\xi) \in [0,T] \times \mathbb{R}$.

The statement of Lemma 6 proves the statements of Theorem 8 for $\gamma \in [0,1]$. The statement for $\gamma > 1$ follows from Theorem 9 (see next chapter) if we choose in this theorem $\omega(t) = \ln^q \frac{C(q)}{t}$ with $q \geq 2$.

□

REMARKS
1) From Theorem 5 and 8 we conclude the C^∞ well-posedness of the Cauchy problem

$$u_{tt} - a(t)u_{xx} = 0\ , \quad u(0,x) = \varphi(x)\ , \quad u_t(0,x) = \psi(x),$$

under the assumptions $a \in L^\infty[0,T] \cap C^2(0,T]$ and (5) for $\gamma \in [0,1]$.
2) Without any new problems all the results can be generalized to

$$u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t) u_{x_k x_l} = 0\ , \quad u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),$$

with corresponding assumptions for $a_{kl} = a_{kl}(t)$.
3) If we stop the diagonalization procedure after the first step, then we have to assume in Theorem 8 the condition (4). Consequently, we proposed another way to prove the results of Theorem 7. This approach was used in [6].

OPEN PROBLEM 3. In this section we have given a very effective classification of oscillations under the assumption $a \in L^\infty[0,T] \cap C^2(0,T]$. At the moment it does not seem to be clear what kind of oscillations we have if $a \in L^\infty[0,T] \cap C^1(0,T]$ satisfies $|a'(t)| \leq C\,\frac{1}{t}\left(\ln\,\frac{1}{t}\right)^\gamma$, $\gamma > 0$. If $\gamma = 0$, we have a finite loss of derivatives. What happens if $\gamma > 0$? To study this problem we have to use in a correct way the low regularity $C^1(0,T]$ (see next chapters).

OPEN PROBLEM 4. Let us consider the strictly hyperbolic Cauchy problem

$$u_{tt} + b(t)u_{xt} - a(t)u_{xx} = 0\ , \quad u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x)\ .$$

Does the existence of a mixed derivative of second order change the classification of oscillations from Definition 3.1? From the results of [1] we know that $a, b \in$ LogLip $[0,T]$ implies C^∞ well-posedness of the above Cauchy problem.

REMARK 5. *Mixing of different non-regular effects*
The survey article [11] gives results if we mix the different non-regular effects of Hölder regularity of $a = a(t)$ on $[0,T]$ and L_p integrability of a weighted derivative on $[0,T]$. Among all these results we mention only that one which guarantees C^∞ well-posedness of

$$u_{tt} - a(t)u_{xx} = 0\ , \quad u(0,x) = \varphi(x)\ , \quad u_t(0,x) = \psi(x),$$

namely, $a = a(t)$ satisfies $t^q \partial_t a \in L^p(0,T)$ for $q + 1/p = 1$.

4. Hirosawa's counter-example

To end the proof of Theorem 8 we cite a result from [7] which explains that very fast oscillations have a deteriorating influence on C^∞ well-posedness.

THEOREM 9. *[see [7]] Let* $\omega : (0, 1/2] \to (0, \infty)$ *be a continuous, decreasing function satisfying* $\lim \omega(s) = \infty$ *for* $s \to +0$ *and* $\omega(s/2) \le c\,\omega(s)$ *for all* $s \in (0, 1/2]$. *Then there exists a function* $a \in C^\infty(\mathbb{R} \setminus \{0\}) \cap C^0(\mathbb{R})$ *with the following properties:*

- $1/2 \le a(t) \le 3/2$ *for all* $t \in \mathbb{R}$;
- *there exists a suitable positive* T_0 *and to each* p *a positive constant* C_p *such that*
$$|a^{(p)}(t)| \le C_p \omega(t)\Big(\frac{1}{t}\ \ln\ \frac{1}{t}\Big)^p \quad \textit{for all}\ \ t \in (0, T_0)\ ;$$
- *there exist two functions* φ *and* ψ *from* $C^\infty(\mathbb{R})$ *such that the Cauchy problem* $u_{tt} - a(t)u_{xx} = 0$, $u(0, x) = \varphi(x)$, $u_t(0, x) = \psi(x)$, *has no solution in* $C^0([0, r),\ \mathcal{D}'(\mathbb{R}))$ *for all* $r > 0$.

The coefficient $a = a(t)$ possesses the regularity $a \in C^\infty(\mathbb{R} \setminus \{0\})$. To attack the open problem 3 it is valuable to have a counter-example from [14] with lower regularity $a \in C^2(\mathbb{R} \setminus \{0\})$. To understand this counter-example let us devote to the Cauchy problem

$$\begin{aligned} &u_{ss} - b\Big((\ln \frac{1}{s})^q\Big)^2 \triangle u = 0\ , \quad (s, x) \in (0, 1] \times \mathbb{R}^n\ , \\ &u(1, x) = \varphi(x),\ u_s(1, x) = \psi(x), \qquad x \in \mathbb{R}^n\ . \end{aligned} \tag{13}$$

Then the results of [14] imply the next statement.

THEOREM 10. *Let us suppose that* $b = b(s)$ *is a positive, 1-periodic, non-constant function belonging to* C^2. *If* $q > 2$, *then there exist data* $\varphi, \psi \in C^\infty(\mathbb{R}^n)$ *such that (13) has no solution in* $C^2([0, 1], \mathcal{D}'(\mathbb{R}^n))$.

Proof. We divide the proof into several steps.
Due to the cone of dependence property it is sufficient to prove H^∞ well-posedness. We will show that there exist positive real numbers $s_\xi = s(|\xi|)$ tending to 0 as $|\xi|$ tends to infinity and data $\varphi, \psi \in H^\infty(\mathbb{R}^n)$ such that with suitable positive constants C_1, C_2, and C_3,

$$|\xi|\ |\hat{u}(s_\xi, \xi)| + |\hat{u}_s(s_\xi, \xi)| \ge C_1 |\xi|^{\frac{1}{2}} \exp(C_2 (\ln C_3 |\xi|)^\gamma)\ .$$

Here $1 < \gamma < q - 1$. This estimate violates H^∞ well-posedness of the Cauchy problem (13). The assumption $b \in C^2$ guarantees that a unique solution $u \in C^2((0, T], H^\infty(\mathbb{R}^n))$ exists.

Step 1. Derivation of an auxiliary Cauchy problem
After partial Fourier transformation we get from (13)

$$v_{ss} + b\Big((\ln\frac{1}{s})^q\Big)^2 |\xi|^2 v = 0\ , \quad (s,\xi) \in (0,1] \times \mathbb{R}^n\ ,$$
$$v(1,\xi) = \hat{\varphi}(\xi)\ ,\ v_s(1,\xi) = \hat{\psi}(\xi)\ , \quad \xi \in \mathbb{R}^n\ ,$$

where $v(s,\xi) = \hat{u}(s,\xi)$. Let us define $w = w(t,\xi) := \tau(t)^{\frac{1}{2}} v(s(t),\xi)$, where $t = t(s) := (\ln\frac{1}{s})^q$, $\tau = \tau(t) := -\frac{dt}{ds}(s(t))$ and $s = s(t)$ denotes the inverse function to $t = t(s)$. Then w is a solution to the Cauchy problem

$$w_{tt} + b(t)^2 \lambda(t,\xi) w = 0\ , \quad (t,\xi) \in [t(1),\infty) \times \mathbb{R}^n\ ,$$
$$w(t(1),\xi) = \tau(t(1))^{\frac{1}{2}} \hat{\varphi}(\xi)\ , \quad w_t(t(1),\xi) = \tau(t(1))^{-\frac{1}{2}}(\frac{1}{2}\tau_t(t(1))\hat{\varphi}(\xi) - \hat{\psi}(\xi)),$$

where $\lambda = \lambda(t,\xi) = \lambda_1(t,\xi) + \lambda_2(t)$, and

$$\lambda_1(t,\xi) = \frac{|\xi|^2}{\tau(t)^2}\ , \quad \lambda_2(t) = \frac{\theta(t)}{b(t)^2 \tau(t)^2}\ , \quad \theta = \tau'^2 - 2\tau''\tau\ .$$

Simple calculations show that $\tau(t) = q\ t^{\frac{q-1}{q}} \exp(t^{\frac{1}{q}})$ and $\theta(t) \approx -\exp(2t^{\frac{1}{q}})$. Hence, $\lim_{t\to\infty} \lambda_2(t) = 0$. Let λ_0 be a positive real number, and let us define $t_\xi = t_\xi(\lambda_0)$ by the definition $\lambda(t_\xi,\xi) = \lambda_0$. It follows from previous calculations that $\lim_{|\xi|\to\infty} t_\xi = \infty$.

Using the mean value theorem we can prove the following result.

LEMMA 7. *There exist positive constants C and δ such that*

$$|\lambda_1(t,\xi) - \lambda_1(t-d,\xi)| \le C\ d\ \frac{\tau'(t)}{\tau(t)}\ \lambda_1(t,\xi)\ , \quad |\lambda_2(t) - \lambda_2(t-d)| \le C\ \frac{\tau'(t)}{\tau(t)}$$

for any $0 \le d \le \delta\ \frac{\tau(t)}{\tau'(t)}$. *In particular, we have*

$$|\lambda(t_\xi,\xi) - \lambda(t_\xi - d,\xi)| \le Cd\ \frac{\tau'(t_\xi)}{\tau(t_\xi)}\ \lambda(t_\xi,\xi)\ , \quad 1 \le d \le \delta\ \frac{\tau(t_\xi)}{\tau'(t_\xi)}\ .$$

We have the hope that properties of solutions of $w_{tt} + b(t)^2\lambda(t,\xi)w = 0$ are not "far away" from properties of solutions of $w_{tt} + b(t)^2\lambda(t_\xi,\xi)w = 0$. For this reason let us study the ordinary differential equation $w_{tt} + \lambda_0 b(t)^2 w = 0$.

Step 2. Application of Floquet's theory
We are interested in the fundamental solution $X = X(t,t_0)$ as the solution to the Cauchy problem

$$\text{(14)} \qquad \frac{d}{dt} X = \begin{pmatrix} 0 & -\lambda_0 b(t)^2 \\ 1 & 0 \end{pmatrix} X\ , \quad X(t_0,t_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is clear that $X(t_0+1,t_0)$ is independent of $t_0 \in \mathbb{N}$.

LEMMA 8 (FLOQUET'S THEORY). *Let $b = b(t) \in C^2$, 1-periodic, positive and non-constant. Then there exists a positive real number λ_0 such that λ_0 belongs to an interval of instability for $w_{tt} + \lambda_0 b(t)^2 w = 0$, that is, $X(t_0+1, t_0)$ has eigenvalues μ_0 and μ_0^{-1} satisfying $|\mu_0| > 1$.*

Let us define for $t_\xi \in \mathbb{N}$ the matrix

$$X(t_\xi + 1, t_\xi) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} .$$

According to Lemma 8 the eigenvalues of this matrix are μ_0 and μ_0^{-1}. We suppose

$$|x_{11} - \mu_0| \geq \frac{1}{2} \, |\mu_0 - \mu_0^{-1}| \, . \tag{15}$$

Then we have $|x_{22} - \mu_0^{-1}| \geq \frac{1}{2}|\mu_0 - \mu_0^{-1}|$, too.

Step 3. A family of auxiliary problems

For every non-negative integer n we shall consider the equation

$$w_{tt} + \lambda(t_\xi - n + t, \xi) b(t_\xi + t)^2 w = 0. \tag{16}$$

It can be written as a first-order system which has the fundamental matrix $X_n = X_n(t, t_0)$ solving the Cauchy problem

$$\begin{aligned} & d_t X = A_n X \ , \ X(t_0, t_0) = I \\ & A_n = A_n(t, \xi) = \begin{pmatrix} 0 & -\lambda(t_\xi - n + t, \xi) b(t_\xi + t)^2 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{17}$$

LEMMA 9. *There exist positive constants C and δ such that*

$$\max_{t_2, t_1 \in [0,1]} \|X_n(t_2, t_1)\| \ \leq \ e^{C \lambda_0}$$

for $0 \leq n \leq \delta \, \frac{\tau(t_\xi)}{\tau'(t_\xi)}$ and t_ξ large.

Proof. The fundamental matrix X_n has the following representation:

$$X_n(t_2, t_1) = I + \sum_{j=1}^{\infty} \int_{t_1}^{t_2} A_n(r_1, \xi) \int_{t_1}^{r_1} A_n(r_2, \xi) \cdots \int_{t_1}^{r_{j-1}} A_n(r_j, \xi) dr_j \cdots dr_1 \ .$$

By Lemma 7 we have

$$\begin{aligned} \max_{t_2, t_1 \in [0,1]} \|X_n(t_2, t_1)\| \ & \leq \exp(1 + b_1^2(\lambda_1(t_\xi - n, \xi) + \sup_{t(1) \leq t} |\lambda_2(t)|)) \\ & = \exp(1 + b_1^2(\lambda_1(t_\xi - n, \xi) - \lambda_1(t_\xi, \xi) + \lambda_0 - \lambda_2(t_\xi) + \sup_{t(1) \leq t} |\lambda_2(t)|)) \\ \leq e^{C \lambda_0} & \end{aligned}$$

for large t_ξ , $0 \le n \le \delta \, \frac{\tau(t_\xi)}{\tau'(t_\xi)}$, where $b_1 = \max\limits_{[0,1]} b(t)$.

□

LEMMA 10. *Let $\eta = \eta(t)$ be a function satisfying*

$$\lim_{t\to\infty} \eta(t) \, \frac{\tau'(t)}{\tau(t)} = 0 \, . \tag{18}$$

Then there exist constants C and δ such that $\|X_n(1,0) - X(t_\xi + 1, t_\xi)\| \le C \, \lambda_0 \eta(t_\xi) \, \frac{\tau'(t_\xi)}{\tau(t_\xi)}$ for $0 \le n \le \delta \, \eta(t_\xi)$. Consequently, $\|X_n(1,0) - X(t_\xi + 1, t_\xi)\| \le \varepsilon$ for any given $\varepsilon > 0$, sufficiently large $t_\xi \in \mathbb{N}$ and $0 \le n \le \delta \, \eta(t_\xi)$.

Proof. Using the representation of $X_n(1,0)$ and of $X(t_\xi + 1, t_\xi)$, then the application of Lemma 7 to $\|X_n(1,0) - X(t_\xi + 1, t_\xi)\|$ gives

$$\begin{aligned}
\|X_n(1,0) - X(t_\xi + 1, t_\xi)\| &\le C \, \lambda_0 (n+1) \, \frac{\tau'(t_\xi)}{\tau(t_\xi)} \, \exp(C \, \lambda_0 (n+1) \, \frac{\tau'(t_\xi)}{\tau(t_\xi)}) \\
&\le C \, \lambda_0 (\delta \, \eta(t_\xi) + 1) \, \frac{\tau'(t_\xi)}{\tau(t_\xi)} \, \exp(C \, \lambda_0 (\delta \, \eta(t_\xi) + 1) \, \frac{\tau'(t_\xi)}{\tau(t_\xi)}) \to 0
\end{aligned}$$

for $t_\xi \to \infty$ and $1 \le n \le \delta \, \eta(t_\xi)$.

□

Repeating the proofs of Lemmas 9 and 10 gives the following result.

LEMMA 11. *There exist positive constants C and δ such that*

$$\|X_{n+1}(1,0) - X_n(1,0)\| \le C \, \lambda_0 \, \frac{\tau'(t_\xi - n)}{\tau(t_\xi - n)}$$

for $1 \le n \le \delta \, \eta(t_\xi)$ and large ξ.

We will later choose $\eta = \eta(t) \sim t^\alpha$ with $\alpha \in \left(\frac{1}{2} \, , \, \frac{q-1}{q}\right)$. That the interval is non-empty follows from the assumptions of our theorem. If we denote $X_n(1,0) = \begin{pmatrix} x_{11}(n) & x_{12}(n) \\ x_{21}(n) & x_{22}(n) \end{pmatrix}$, then the statements of Lemmas 8 and 10 imply

- $|\mu_n - \mu_0| \le \varepsilon$, where μ_n and μ_n^{-1} are the eigenvalues of $X_n(1,0)$;
- $|\mu_n| \ge 1 + \varepsilon$ for $\varepsilon \le (|\mu_0| - 1)/2$;
- $|x_{11}(n) - \mu_n| \ge \frac{1}{4}|\mu_0 - \mu_0^{-1}|$, $|x_{22}(n) - \mu_n^{-1}| \ge \frac{1}{4}|\mu_0 - \mu_0^{-1}|$.

From Lemma 11 we conclude

- $|x_{ij}(n+1) - x_{ij}(n)| \le C \, \lambda_0 \, \frac{\tau'(t_\xi - n)}{\tau(t_\xi - n)}$. This implies

- $|\mu_{n+1} - \mu_n| \le C\,\lambda_0\,\frac{\tau'(t_\xi - n)}{\tau(t_\xi - n)}$.

Step 4. An energy estimate from below

LEMMA 12. *Let n_0 satisfy $0 \le n_0 \le \delta\,\eta(t_\xi) \le n_0 + 1$. Then there exist positive constants C_0 and C_1 such that the solution $w = w(t,\xi)$ to*

$$\begin{aligned} &w_{tt} + b(t)^2\lambda(t,\xi)w = 0,\\ &w(t_\xi - n_0 - 1, \xi) = 1\ ,\quad w_t(t_\xi - n_0 - 1,\xi) = \frac{x_{12}(n_0)}{\mu_{n_0} - x_{11}(n_0)} \end{aligned}$$

satisfies

$$|w(t_\xi,\xi)| + |w_t(t_\xi,\xi)| \ge C_0\ \exp(C_1\,\eta(t_\xi)) \tag{19}$$

for large ξ and $\eta = \eta(t)$ fulfilling (18).

Proof. The function $w = w(t_\xi - n_0 + t, \xi)$ satisfies (16) with $n = n_0$. It follows that

$$\begin{aligned} \begin{pmatrix} \frac{d}{dt}\,w(t_\xi,\xi) \\ w(t_\xi,\xi) \end{pmatrix} &= X_1(1,0)X_2(1,0)\cdots \\ &\qquad \cdots X_{n_0-1}(1,0)X_{n_0}(1,0)\begin{pmatrix} \frac{d}{dt}\,w(t_\xi - n_0,\xi) \\ w(t_\xi - n_0,\xi) \end{pmatrix}. \end{aligned}$$

The matrix

$$B_n = \begin{pmatrix} \frac{x_{12}(n)}{\mu_n - x_{11}(n)} & 1 \\ 1 & \frac{x_{21}(n)}{\mu_n^{-1} - x_{22}(n)} \end{pmatrix}$$

is a diagonalizer for $X_n(1,0)$, that is, $X_n(1,0)B_n = B_n$ diag (μ_n, μ_n^{-1}). Since det $X_n(1,0) = 1$ and trace of $X_n(1,0)$ is $\mu_n + \mu_n^{-1}$ we get det $B_n = \frac{\mu_n - \mu_n^{-1}}{\mu_n^{-1} - x_{22}(n)}$. Using the properties of μ_n from the previous step we conclude $|\det B_n| \ge C > 0$ for all $0 < n \le \delta\,\eta(t_\xi)$. Moreover, by Lemma 9 we have $|x_{ij}(n)| \le C$, $\|B_n\| + \|B_n^{-1}\| \le C$ for all $0 < n \le \delta\,\eta(t_\xi)$. All constants C are independent of n. These estimates lead to

$$\|B_{n-1}^{-1}B_n - I\| = \|B_{n-1}^{-1}(B_n - B_{n-1})\| \le C\,\lambda_0\,\frac{\tau'(t_\xi - n)}{\tau(t_\xi - n)} \tag{20}$$

for large t_ξ. If we denote $G_n := B_{n-1}^{-1} B_n - I$, then we can write

$$\begin{aligned}
& X_1(1,0) X_2(1,0) \cdots X_{n_0-1}(1,0) X_{n_0}(1,0) \\
= \; & B_1 \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1^{-1} \end{pmatrix} B_1^{-1} B_2 \begin{pmatrix} \mu_2 & 0 \\ 0 & \mu_2^{-1} \end{pmatrix} \\
& B_2^{-1} B_3 \cdots B_{n_0-1}^{-1} B_{n_0} \begin{pmatrix} \mu_{n_0} & 0 \\ 0 & \mu_{n_0}^{-1} \end{pmatrix} B_{n_0}^{-1} \\
= \; & B_1 \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1^{-1} \end{pmatrix} (I + G_2) \begin{pmatrix} \mu_2 & 0 \\ 0 & \mu_2^{-1} \end{pmatrix} \\
& (I + G_3) \cdots (I + G_{n_0}) \begin{pmatrix} \mu_{n_0} & 0 \\ 0 & \mu_{n_0}^{-1} \end{pmatrix} B_{n_0}^{-1} .
\end{aligned}$$

We shall show that the (1, 1) element y_{11} of the matrix

$$\begin{aligned}
& \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1^{-1} \end{pmatrix} (I + G_2) \begin{pmatrix} \mu_2 & 0 \\ 0 & \mu_2^{-1} \end{pmatrix} (I + G_3) \cdots \\
& \qquad \qquad \cdots (I + G_{n_0}) \begin{pmatrix} \mu_{n_0} & 0 \\ 0 & \mu_{n_0}^{-1} \end{pmatrix}
\end{aligned}$$

can be estimated with suitable positive constants C_0 and C_1 by $C_0 \exp(C_1 \eta(t_\xi))$. It is evident from (20) that

$$|y_{11} - \prod_{n=1}^{n_0} \mu_n| \; \leq \; C \prod_{n=1}^{n_0} |\mu_n| \; \sum_{n=1}^{n_0} \; \frac{\tau'(t_\xi - n)}{\tau(t_\xi - n)}$$

for large t_ξ. We have

$$\begin{aligned}
& \sum_{n=1}^{n_0} \; \frac{\tau'(t_\xi - n)}{\tau(t_\xi - n)} \; \leq \; \int_0^{\delta\, \eta(t_\xi)} \frac{\tau'(t_\xi - t - 1)}{\tau(t_\xi - t - 1)} \; dt \; \leq \; \ln \; \frac{\tau(t_\xi - 1)}{\tau(t_\xi - \delta\, \eta(t_\xi) - 1)} \\
& \qquad \leq \ln \left(1 - \delta\, \eta(t_\xi) \quad \frac{\tau'(t_\xi - 1)}{\tau(t_\xi - 1)} \right)^{-1} \to 0 \quad \text{as} \quad t_\xi \to \infty \, .
\end{aligned}$$

Hence, we can find a positive real ν such that

$$|y_{11}| \; \geq \; (1 - \nu) \prod_{n=1}^{n_0} |\mu_n| \; \geq \; (1 - \nu)(\mu_0 - \varepsilon)^{n_0} \; \geq \; (1 - \nu)(\mu_0 - \varepsilon)^{\delta\, \eta(t_\xi) - 1} .$$

The vector of data on $t = t_\xi - n_0$ is an eigenvector of B_{n_0}. Thus the estimate for y_{11} holds for the vector $(d_t w(t_\xi, \xi), w(t_\xi, \xi))^T$ too. This proves the energy estimate from below of the lemma.

□

Step 5. Conclusion
After choosing $s_\xi = s(t_\xi) = \exp(-t_\xi^{1/q})$ for large t_ξ and taking account of $w_t(t,\xi) = \frac{1}{2}\,\tau_t(t)\tau(t)^{-\frac{1}{2}}\;v(s(t),\xi) + \tau(t)^{\frac{1}{2}}\;v_s(s(t),\xi)$ we obtain

$$\begin{aligned} & |w(t(s),\xi)| + |w_t(t(s),\xi)| \\ \le\; & \tau(t(s))^{\frac{1}{2}}\left(1 + \frac{\tau_t(t(s))}{2\tau(t(s))}\right)|v(s,\xi)| + \tau(t(s))^{-\frac{1}{2}}|v_s(s,\xi)| \\ \le\; & 2\tau(t(s))^{\frac{1}{2}}|v(s,\xi)| + \tau(t(s))^{-\frac{1}{2}}|v_s(s,\xi)| \end{aligned}$$

for large ξ. Finally, we use $\tau(t(s)) \sim |\xi|$. This follows from the definition $\lambda(t_\xi,\xi) = \lambda_0$ and $\lim\limits_{t_\xi\to\infty} \lambda_2(t_\xi) = 0$. Thus we have shown

$$|\xi|\;|\hat{u}(s_\xi,\xi)| \;+\; |\hat{u}_s(s_\xi,\xi)| \;\ge\; C_1|\xi|^{\frac{1}{2}}\;\exp(C_2\;\eta(t_\xi))\;.$$

The function $\eta(t) = t^\alpha$ satisfies (18) if $\alpha < \frac{q-1}{q}$. The function t_ξ behaves as $(\ln|\xi|)^q$.
Together these relations give

$$\begin{aligned} & |\xi|\;|\hat{u}(s_\xi,\xi)| \;+\; |\hat{u}_s(s_\xi,\xi)| \;\ge\; C_1|\xi|^{\frac{1}{2}}\;\exp(C_2\;(\ln|\xi|)^{q\alpha}) \\ & \quad \ge C_1|\xi|^{\frac{1}{2}}\;\exp(C_2\;(\ln|\xi|)^{\gamma}) \quad , \quad \text{where} \quad \gamma \in (1, q-1)\;. \end{aligned}$$

From this inequality we conclude the statement of Theorem 10.

□

REMARK 6. The idea to apply Floquet's theory to construct a counter-example goes back to [25] to study C^∞ well-posedness for weakly hyperbolic equations. This idea was employed in connection to $L_p - L_q$ decay estimates for solutions of wave equations with time-dependent coefficients in [24]. The merit of [14] is the application of Floquet's theory to strictly hyperbolic Cauchy problems with non-Lipschitz coefficients. We underline that the assumed regularity $b \in C^2$ comes from statements of Floquet's theory itself. An attempt to consider non-Lipschitz theory, weakly hyperbolic theory and theory of $L_p - L_q$ decay estimates for solutions of wave equations with a time-dependent coefficient is presented in [23].

5. How to weaken C^2 regularity to keep the classification of oscillations

There arises after the results of [6] and [7] the question whether there is something between the conditions

(21) • $a \in L^\infty[0,T] \cap C^1(0,T]\,,\; |t^\gamma a'(t)| \le C \quad \text{for}\;\; t \in (0,T]\;;$

(22) • $a \in L^\infty[0,T] \cap C^2(0,T]\,,\; |a^{(k)}(t)| \le C_k\left(\frac{1}{t}\left(\ln\frac{1}{t}\right)^\gamma\right)^k$
for $\;t \in (0,T],\; k = 1,2.$

The paper [15] is devoted to the model Cauchy problem

(23) $\quad u_{tt} - a(t,x) \triangle u = 0\,, \quad u(T,x) = \varphi(x), \quad u_t(T,x) = \psi(x)\,,$

where $a = a(t,x) \in L^\infty([0,T], B^\infty(\mathbb{R}^n))$ and $a_0 \le a(t,x)$ with a positive constant a_0.

DEFINITION 3. *Definition of* admissible space of coefficients. *Let T be a positive small constant, and let $\gamma \in [0,1]$ and $\beta \in [1,2]$ be real numbers. We define the weighted spaces of Hölder differentiable functions $\Lambda_\gamma^\beta = \Lambda_\gamma^\beta((0,T])$ in the following way:*

$$\Lambda_\gamma^\beta((0,T]) = \{a = a(t,x) \in L^\infty([0,T], B^k(\mathbb{R}^n)) : \sup_{t\in(0,T]} \|a(t)\|_{B^k(\mathbb{R}^n)}$$
$$+ \sup_{t\in(0,T]} \frac{\|\partial_t a(t)\|_{B^k(\mathbb{R}^n)}}{t^{-1}(\ln t^{-1})^\gamma} + \sup_{t\in(0,T]} \frac{\|\partial_t a\|_{M^{\beta-1}([t,T],B^k(\mathbb{R}^n))}}{(t^{-1}(\ln t^{-1})^\gamma)^\beta} \quad \textit{for all } \; k \ge 0\}\,,$$

where $\|F\|_{M^{\beta-1}(I)}$ with a closed interval I is defined by

$$\|F\|_{M^{\beta-1}(I)} = \sup_{s_1,s_2\in I, s_1\ne s_2} \frac{|F(s_1)-F(s_2)|}{|s_1-s_2|^{\beta-1}}\,.$$

- If a satisfies (21) with $\gamma = 1$, then $a \in \Lambda_0^1$.
- If a satisfies (22) with $\gamma \in [0,1]$, then $a \in \Lambda_\gamma^2$.

DEFINITION 4. Space of solutions. *Let σ and γ be non-negative real numbers. We define the exponential-logarithmic scale $H_{\gamma,\sigma}$ by the set of all functions $f \in L^2(\mathbb{R}^n)$ satisfying*

$$\|f\|_{H_{\gamma,\sigma}} := \left(\int_{\mathbb{R}^n} |\exp(\sigma(\ln\langle\xi\rangle)^\gamma)\hat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty\,.$$

In particular, we denote $H_\gamma = \bigcup_{\sigma>0} H_{\gamma,\sigma}$.

THEOREM 11. *Let $\gamma \in [0,1]$ and $\beta \in (1,2]$. If $a \in \Lambda_\gamma^\beta((0,T])$, then the Cauchy problem (23) is well-posed in H_γ on $[0,T]$, that is, there exist positive constants $C_{\gamma,\beta}$, σ and σ' with $\sigma \le \sigma'$ such that*

$$\|(\nabla u(t), u_t(t))\|_{H_{\gamma,\sigma}} \le C_{\gamma,\beta}\|(\nabla\varphi,\psi)\|_{H_{\gamma,\sigma'}} \quad \textit{for all } t \in [0,T].$$

REMARK 7. In the Cauchy problem (23) we prescribe data φ and ψ on the hyperplane $t = T$. It is clear from Theorem 4, that a unique solution of the backward Cauchy problem (23) exists for $t \in (0,T]$. The statement of Theorem 11 tells us that in the case of very slow, slow or fast oscillations ($\gamma \in [0,1]$), the solution possesses a continuous extension to $t = 0$.

OPEN PROBLEM 5. Try to prove the next statement:
If $a = a(t,x) \in \Lambda_\gamma^\beta((0,T])$ with $\gamma > 1$ and $\beta \in (1,2)$, then these oscillations are very fast oscillations!

The energy inequality from Theorem 11 yields the same connection between the type of oscillations and the loss of derivatives as Theorem 8.

THEOREM 12. *Let us consider the Cauchy problem (23), where $a \in \Lambda_\gamma^\beta((0,T])$ with $\gamma \in [0,1]$ and $\beta \in (1,2]$. The data φ, ψ belong to H^{s+1}, H^s, respectively. Then the following energy inequality holds:*

$$E(u)(t)\Big|_{H^{s-s_0}} \leq C(T)E(u)(0)\Big|_{H^s} \quad \textit{for all} \;\; t \in [0,T]\,,$$

where

- *$s_0 = 0$ if $\gamma = 0$ (very slow oscillations),*
- *s_0 is an arbitrary small positive constant if $\gamma \in (0,1)$ (slow oscillations),*
- *s_0 is a positive constant if $\gamma = 1$ (fast oscillations).*

Proof of Theorem 11. The proof follows that for Theorem 8. But now the coefficient depends on spatial variables, too. Our main goal is to present modifications to the proof of Theorem 8.
To Step 2. Symbols
To given real numbers $m_1, m_2 \geq 0$, we define $S\{m_1,m_2\}$ and T^{m_1} as follows:

$$\begin{aligned}
S\{m_1,m_2\} &= \{a = a(t,x,\xi) \in L^\infty_{loc}((0,T), C^\infty(\mathbb{R}^{2n})) : \\
&\quad |\partial_x^\tau \partial_\xi^\eta a(t,x,\xi)| \leq C_{\tau,\eta}\langle\xi\rangle^{m_1-|\eta|}\left(\frac{1}{t}\Big(\ln\frac{1}{t}\Big)^\gamma\right)^{m_2} \quad \text{in} \;\; Z_{hyp}(N)\}; \\
T^{m_1} &= \{a = a(t,x,\xi) \in L^\infty((0,T), C^\infty(\mathbb{R}^{2n})) : \\
&\quad |\partial_x^\tau \partial_\xi^\eta a(t,x,\xi)| \leq C_{\tau,\eta}\langle\xi\rangle^{m_1-|\eta|} \;\; \text{in} \;\; Z_{pd}(N)\}.
\end{aligned}$$

Regularization
Our goal is to carry out the first two steps of the diagonalization procedure because only two steps allow us to understand a refined classification of oscillations. But the coefficient $a = a(t,x)$ doesn't belong to C^2 with respect to t. For this reason we introduce a regularization a_ρ of a. Let $\chi = \chi(s) \in B^\infty(\mathbb{R})$ be an even non-negative function having its support on $(-1,1)$. Let this function satisfy $\int \chi(s)ds = 1$. Moreover, let the function $\mu = \mu(r) \in B^\infty[0,\infty)$ satisfy $0 \leq \mu(r) \leq 1$, $\mu(r) = 1$ for $r \geq 2$ and $\mu(r) = 0$ for $r \leq 1$. We define the pseudo-differential operator $a_\rho = a_\rho(t,x,D_x)$ with the symbol

$$a_\rho(t,x,\xi) = \mu\left(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\right)\underbrace{b_\rho(t,x,\xi)}_{Z_{hyp}(N)} + \left(1 - \mu\left(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\right)\right)\underbrace{a_0}_{Z_{pd}(N)}\,,$$

where

$$b_\rho(t,x,\xi) = \langle\xi\rangle \underbrace{\int_{\mathbb{R}} a(s,x)\chi((t-s)\langle\xi\rangle)\,ds}_{\text{regularization of a}} .$$

LEMMA 13. *The regularization a_ρ has the following properties:*

- $a_\rho(t,x,\xi) \geq a_0$*;*
- $a_\rho(t,x,\xi) \in S^0_{1,0}$*;*
- $\partial_t a_\rho(t,x,\xi) \in S\{0,1\} \cap T^{-\infty}$*;*
- $\partial_t^2 a_\rho(t,x,\xi) \in S\{-\beta+2,\beta\} \cap T^{-\infty}$*;*
- $a(t,x) - a_\rho(t,x,\xi) \in S\{-\beta,\beta\} \cap T^0$.

To Step 4. Two steps of diagonalization procedure
We start with $u_{tt} - a(t,x) \triangle u = 0$. The vector-valued function $U = (\sqrt{a_\rho}\langle D_x\rangle u, D_t u)^T$ is a solution of the first order system

$$\begin{aligned}
&(D_t - A_0 - B_0 - R_0)U = 0,\\
&A_0 := \begin{pmatrix} 0 & \sqrt{a_\rho}\langle D_x\rangle \\ \sqrt{a_\rho}\langle D_x\rangle & 0 \end{pmatrix},\\
&B_0 := \begin{pmatrix} \mathrm{Op}\left[\frac{D_t a_\rho}{2a_\rho}\right] & 0 \\ (a-a_\rho)\langle D_x\rangle \sqrt{a_\rho}^{\sharp} & 0 \end{pmatrix},
\end{aligned}$$

where $R_0 \in S^0$ uniformly for all $t \in [0,T]$, that is, $R_0 = R_0(t,x,\xi) \in L^\infty([0,T], S^0)$.
First step of diagonalization, diagonalization modulo $L^\infty([0,T],\ S\{0,1\} \cap T^1)$.
Using the same diagonalizer in the form of a constant matrix we obtain from the above system

$$\begin{aligned}
&(D_t - A_1 - B_1 - R_1)U_1 = 0 ,\\
&A_1 := \sqrt{a_\rho}\langle D_x\rangle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\\
&B_1 \in L^\infty([0,T], S\{0,1\} \cap T^1),\\
&R_1 \in L^\infty([0,T], S^0).
\end{aligned}$$

REMARK 8. We can split B_1 into two parts

$$\begin{aligned}
&B_{10} := \mathrm{Op}\left[\frac{D_t a_\rho}{4a_\rho}\right] \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\\
&B_{11} := \frac{1}{2}\,(a-a_\rho)\langle D_x\rangle \sqrt{a_\rho}^{\sharp} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.
\end{aligned}$$

The second part B_{11} belongs to $S\{-\beta+1,\beta\}\cap T^1$ for all $t \in [0,T]$. If $\beta > 1$, then this class is better than $S\{0,1\} \cap T^1$. We need $\beta > 1$ later, to understand that the influence of B_{11} is not essential. This is the reason we exclude in Theorem 11 the value $\beta = 1$.

Second step of diagonalization, diagonalization modulo $L^\infty([0,T],\ S\{-\beta+1,\beta\}\cap T^1) + L^\infty([0,T], S^0)$.

We define the diagonalizer $M_2 = M_2(t,x,D_x) := \begin{pmatrix} I & -p \\ p & I \end{pmatrix}$, where $p = p(t,x,\xi) = \frac{D_t a_\rho}{8a_\rho\sqrt{a_\rho}\langle\xi\rangle}$. Then a suitable transformation $U_2 := M_2 U_1$ changes the above system to

$$
\begin{aligned}
&(D_t - A_1 - A_2 - B_2 - R_2)U_2 = 0\ ,\\
&A_2 := \mathrm{Op}\left[\frac{D_t a_\rho}{4a_\rho}\right]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\\
&B_2 \in L^\infty([0,T], S\{-\beta+1,\beta\}\cap T^1),\\
&R_2 \in L^\infty([0,T], S^0).
\end{aligned}
$$

Transformation by an elliptic pseudo-differential operator.

We define $M_3 = M_3(t,x,\xi) := \exp\left(-\int\limits_t^T \frac{D_s a_\rho}{4a_\rho}\,ds\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The transformation $U_2 := M_3 U_3$ gives from the last system $(D_t - A_1 - B_3 - R_3)U_3 = 0$, where $B_3,\ R_3$ belong to the same symbol classes as $B_2,\ R_2$, respectively.

REMARK 9. The last step corresponds to the fact from the proof of Theorem 8, that $\int\limits_r^t \frac{\partial_s a(s)}{4a(s)}\,ds$ depends only on a.

Application of sharp Gårding's inequality for matrix-valued operators.

We generalize an idea from [2] to our model problem.

GOAL. Let us find a pseudo-differential operator $\theta = \theta(t, D_x)$ in such a way that after transformation $V(t,x) := e^{-\int\limits_t^T \theta(s,D_x)ds} U_3(t,x)$ the operator equation $(D_t - A_1 - B_3 - R_3)U_3 = 0$ is transformed to $(\partial_t - P_0 - P_1)V = 0$, where we can show that for the solution V of the Cauchy problem an energy estimate without loss of derivatives holds.

A simple computation leads to

$$
\begin{aligned}
P_0 + P_1 &= i(A_1 + B_3 + R_3) + \theta(t,D_x)I\\
&+ i\left[e^{-\int\limits_t^T \theta(s,D_x)ds}, A + B + R\right] e^{\int\limits_t^T \theta(s,D_x)ds}.
\end{aligned}
$$

The matrix-valued operator A_1 brings no loss of derivatives, here we feel the strict hyperbolicity. Taking account of the symbol classes for B_3, R_3 and our strategy due to Gårding's inequality that $\theta = \theta(t,\xi)$ should majorize $i(B_3(t,x,\xi) + R_3(t,x,\xi))$ the symbol of θ should consist at least of two parts:

- a positive constant K, due to $R_3 \in L^\infty([0,T], S^0)$;
- $K\,\theta_0(t,\xi) := K\,\mu\left(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\right)\frac{1}{\langle\xi\rangle^{\beta-1}}\left(\frac{1}{t}\left(\ln\frac{1}{t}\right)^\gamma\right)^\beta$
$+K\left(1-\mu\left(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\right)\right)\langle\xi\rangle$, due to $B_3 \in L^\infty([0,T], S\{-\beta+1,\beta\}\cap T^1)$.

It turns out that the symbol of the commutator doesn't belong to one of these symbol classes. For this reason we introduce a third part

- $K\,\theta_1(t,\xi) := K\,\mu\left(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\right)\left(\ln\frac{1}{t}\right)^\gamma + K\left(1-\mu\left(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\right)\right)\left(\ln\frac{1}{t_\xi}\right)^\gamma.$

Defining

- $P_0 = i(A_1+B_3+R_3) + K(1+\theta_0(t,D_x))I$,
- $P_1 = K\,\theta_1(t,D_x)I + i\left[e^{-\int_t^T\theta(s,D_x)ds}, A_1+B_3+R_3\right]e^{\int_t^T\theta(s,D_x)ds}$

one can show

$$\det\left(\frac{P_0+P_0^*}{2}\right)(t,x,\xi) \ge \theta_0(t,\xi) \in L^\infty([0,T], S^1_{1,0})\ ,$$
$$\det\left(\frac{P_1+P_1^*}{2}\right)(t,x,\xi) \ge \theta_1(t,\xi) \in L^\infty([0,T], S^\varepsilon_{\varepsilon,0})\ .$$

We use the sharp Gårding's inequality with (see [19]) with

- $c_0 = 0\ ,\quad m=1\ ,\quad \rho=1\ ,\quad \delta=0\quad$ for P_0 ,
- $c_0 = 0\ ,\quad m=\varepsilon\ ,\quad \rho=\varepsilon\ ,\quad \delta=0\quad$ for P_1 ,

thus $\mathrm{Re}(P_k u,u) \ge -C_k\ \|u\|^2_{L_2}$ for $k=1,2$. These are the main inequalities for proving the energy estimate

$$\|V(t,\cdot)\|^2_{L_2} \le e^{CT}\|V(T,\cdot)\|^2_{L_2} \qquad \text{for } t\in[0,T]\ .$$

It remains to estimate $\int_0^T\theta(s,\xi)ds$. This is more or less an exercise. A careful calculation brings $\int_0^T\theta(s,\xi)ds \le C\,(\ln\langle\xi\rangle)^\gamma$. The statements of Theorem 11 are proved.

$\square$

6. Construction of parametrix

In this section we come back to our general Cauchy problem (1) taking account of the classification of oscillations supposed in Definition 2 and (5). We assume

$$a_{kl} \in C([0,T], \mathcal{B}^\infty(\mathbb{R}^n)) \cap C^\infty((0,T], \mathcal{B}^\infty(\mathbb{R}^n)). \tag{24}$$

The non-Lipschitz behaviour of coefficients is characterized by

$$|D_t^k D_x^\beta a_{kl}(t,x)| \le C_{k,\beta} \left(\frac{1}{t}\left(\ln \frac{1}{t}\right)^\gamma\right)^k \tag{25}$$

for all k, β and $(t,x) \in (0,T] \times \mathbb{R}^n$, where T is sufficiently small and $\gamma \ge 0$. The transformation $U = (\langle D_x \rangle u, D_t u)^T$ transfers our starting Cauchy problem (1) to a Cauchy problem for $D_t U - AU = F$, where $A = A(t,x,D_x)$ is a matrix-valued pseudo-differential operator. The goal of this section is the construction of parametrix to $D_t - A$.

DEFINITION 5. *An operator $E = E(t,s)$, $0 \le s \le t \le T_0$, is said to be a parametrix to the operator $D_t - A$ if $D_t E - AE \in L^\infty([0,T_0]^2, \Psi^{-\infty}(\mathbb{R}^n))$. Here $\Psi^{-\infty}$ denotes the space of pseudo-differential operators with symbols from $S^{-\infty}$ (see [19]).*

We will prove that *E is a matrix Fourier integral operator*. The considerations of this section are based on [17], where the case $\gamma = 1$ was studied, and on [23]. We will sketch this construction of the parametrix and show how the different loss of derivatives appears. It is more or less standard to get from the parametrix to the existence of C^1 solutions in t of (1) with values in Sobolev spaces.

Step 1. Tools

With the function $t = t_\xi$ from the proof of Theorem 8 we define for $\langle \xi \rangle \ge M$ the *pseudo-differential zone $Z_{pd}(N)$, hyperbolic zone $Z_{hyp}(N)$*, respectively, by

$$Z_{pd}(N) = \{(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n} : \ t \le t_\xi\}\ , \tag{26}$$

$$Z_{hyp}(N) = \{(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n} : \ t \ge t_\xi\}. \tag{27}$$

Moreover, we divide $Z_{hyp}(N)$ into the so-called *oscillations subzone $Z_{osc}(N)$* and the *regular subzone $Z_{reg}(N)$*. These subzones are defined by

$$Z_{osc}(N) = \{(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n} : \ t_\xi \le t \le \tilde{t}_\xi\}\ , \tag{28}$$

$$Z_{reg}(N) = \{(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n} : \ \tilde{t}_\xi \le t\}\ , \tag{29}$$

where $t = \tilde{t}_\xi$ solves

$$\tilde{t}_\xi \langle \xi \rangle = 2N(\ln\langle \xi \rangle)^{2\gamma}. \tag{30}$$

In each of these zones we define its own class of symbols.

DEFINITION 6. *By T_{2N} we denote the class of all amplitudes $a = a(t, x, \xi) \in L^\infty([0, T], C^\infty(\mathbb{R}^{2n}))$ satisfying for $(t, x, \xi) \in Z_{pd}(2N)$ and all α, β the estimates*

$$\text{ess sup}_{(t,x)\in[0,t_\xi]\times\mathbb{R}^n} |\partial_x^\beta \partial_\xi^\alpha a(t, x, \xi)| \leq C_{\beta\alpha} \langle\xi\rangle^{1-|\alpha|}. \tag{31}$$

By $S^m_{\rho,\delta}(\mathbb{R}^n)$ we will denote the usual symbol spaces (see [19]).
To describe the behaviour in oscillations subzone $Z_{osc}(N)$ we need the following class of symbols.

DEFINITION 7. *By $S_N\{m_1, m_2\}$, $m_2 \geq 0$, we denote the class of all amplitudes $a = a(t, x, \xi) \in C^\infty((0, T] \times \mathbb{R}^{2n})$ satisfying*

$$|\partial_t^k \partial_x^\beta \partial_\xi^\alpha a(t, x, \xi)| \leq C_{k\beta\alpha} \langle\xi\rangle^{m_1-|\alpha|} \left(\tfrac{1}{t}(\ln \tfrac{1}{t})^\gamma\right)^{m_2+k} \tag{32}$$

for all k, α, β and $(t, x, \xi) \in Z_{hyp}(N)$.

Finally, we use symbols describing the behaviour of the solution in the regular part $Z_{reg}(N)$ of $Z_{hyp}(N)$.

DEFINITION 8. *By $S^\star_N\{m_1, m_2\}$, $m_2 \geq 0$, we denote the class of all amplitudes $a = a(t, x, \xi) \in C^\infty((0, T] \times \mathbb{R}^{2n})$ satisfying*

$$|\partial_t^k \partial_x^\beta \partial_\xi^\alpha a(t, x, \xi)| \leq C_{k\beta\alpha} \langle\xi\rangle^{m_1-|\alpha|} \left(\tfrac{1}{t}(\ln \tfrac{1}{t})^\gamma\right)^{m_2+k} \tag{33}$$

for all k, α, β and $(t, x, \xi) \in Z_{reg}(N)$.

To all these symbol classes one can define corresponding pseudo-differential operators. To get a calculus for these symbol classes it is useful to know that under assumptions on the behaviour of the symbols in $Z_{pd}(N)$ we have relations to classical parameter-dependent symbol classes.

LEMMA 14. *Assume that the symbol $a \in S_N\{m_1, m_2\}$ is constant in $Z_{pd}(N)$. Then*

$$a \in L^\infty([0, T], S_{1,0}^{\max(0,m_1+m_2)}(\mathbb{R}^n)),\ \partial_t^k a \in L^\infty([0, T], S_{1,0}^{m_1+m_2+k}(\mathbb{R}^n)) \tag{34}$$

for all $k \geq 1$.

The statements (34) allow us to apply the standard rules of classical symbolic calculus. One can show

a hierarchy of symbol classes $S_N\{m_{1,k}, m_2\}$ for $m_{1,k} \to -\infty$.

LEMMA 15. *Assume that the symbols $a_k \in S_N\{m_{1,k}, m_2\}$, $k \geq 0$, vanish in $Z_{pd}(N)$ and that $m_{1,k} \to -\infty$ as $k \to \infty$. Then there is a symbol $a \in S_N\{m_{1,0}, m_2\}$ with support in $Z_{hyp}(N)$ such that*

$$a - \sum_{l=0}^{k-1} a_l \in S_N\{m_{1,k}, m_2\} \qquad \text{for all } k \geq 1.$$

The symbol is uniquely determined modulo $C^\infty([0,T], S^{-\infty}(\mathbb{R}^n))$.

a hierarchy of symbol classes $S_N\{m_1-k, m_2+k\}$ for $k \geq 0$.

LEMMA 16. *Assume that the symbols $a_k \in S_N\{m_1-k, m_2+k\}$, $k \geq 0$, vanish in $Z_{pd}(N)$. Then there is a symbol $a \in S_N\{m_1, m_2\}$ with support in $Z_{hyp}(N)$ such that*

$$a - \sum_{l=0}^{k-1} a_l \in S_N\{m_1-k, m_2+k\} \quad \textit{for all} \;\; k \geq 1 \, .$$

The symbol is uniquely determined modulo $\bigcap_{l \geq 0} S_N\{m_1-l, m_2+l\}$.

Asymptotic representations of symbols vanishing in $Z_{pd}(N)$ by using these hierarchies.

A composition formula of pseudo-differential operators whose symbols are constant in $Z_{pd}(N)$.

LEMMA 17. *Let A and B be pseudo-differential operators with symbols $a := \sigma(A)$ and $b := \sigma(B)$ from $S_N\{m_1, m_2\}$ and $S_N\{k_1, k_2\}$, where we use the representations*

$$A(t,x,D_x)u = \frac{1}{(2\pi)^n}\text{Os-}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\xi} a(t,x,\xi)u(x+y)d\xi dy\,;$$
$$B(t,x,D_x)u = \frac{1}{(2\pi)^n}\text{Os-}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\xi} b(t,x,\xi)u(x+y)d\xi dy.$$

Let us suppose that both symbols a and b are constant in $Z_{pd}(N)$. Then the operator $A \circ B$ has a symbol $c = c(x,t,\xi)$ which belongs to $S_N\{m_1+k_1, m_2+k_2\}$ and satisfies

$$c(t,x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} D_\xi^\alpha a(t,x,\xi)\partial_x^\alpha b(t,x,\xi)$$

modulo a regularizing symbol from $C^\infty([0,T], S^{-\infty})$.

The existence of parametrix to elliptic matrix pseudo-differential operators belonging to $S_N\{0,0\}$ and which are constant in $Z_{pd}(N)$.

LEMMA 18. *Assume that the symbol $a := \sigma(A)$ of the matrix pseudo-differential operator A belongs to $S_N\{0,0\}$ and is a constant matrix in $Z_{pd}(N)$. If A is elliptic, this means $|\det a(t,x,\xi)| \geq C > 0$ for all $(t,x,\xi) \in [0,T]\times\mathbb{R}^{2n}$, then there exists a parametrix $A^\sharp$, where $a^\sharp := \sigma(A^\sharp) \in S_N\{0,0\}$ is a constant matrix in $Z_{pd}(N)$, too.*

Proof. We set $a_0^\sharp(t,x,\xi) := a(t,x,\xi)^{-1}$. The symbol $a_0^\sharp$ belongs to $S_N\{0,0\}$. Using Lemma 14 we can recursively define symbols $a_k^\sharp$ by

$$\sum_{|\alpha|=1}^{k} \frac{1}{\alpha!}\Big(D_\xi^\alpha a(t,x,\xi)\Big)\Big(\partial_x^\alpha a_{k-|\alpha|}^\sharp(t,x,\xi)\Big) =: -a(t,x,\xi)a_k^\sharp(t,x,\xi).$$

It is clear that $a_k^\sharp(t,x,\xi) \equiv 0$ in $Z_{pd}(N)$ and $a_k^\sharp \in S_N\{-k,0\}$.
The application of Lemma 15 gives a symbol $a_R^\sharp \in S_N\{0,0\}$ and a right parametrix $A_R^\sharp$ with symbol $\sigma(A_R^\sharp) =: a_R^\sharp$ and

$$\begin{aligned}
&a_R^\sharp - \sum_{l=0}^{k-1} a_l^\sharp \in S_N\{-k,0\},\\
&a_R^\sharp(t,x,\xi) = a_0^\sharp(t,x,\xi) \text{ in } Z_{pd}(N),\\
&AA_R^\sharp - I \in C^\infty([0,T],\Psi^{-\infty}),
\end{aligned}$$

where I denotes the identity operator. In the same way we can show the existence of a left parametrix $A_L^\sharp$ with $A_L^\sharp A - I \in C^\infty([0,T],\Psi^{-\infty})$. As usual one can show that $A_L^\sharp$ and $A_R^\sharp$ coincide modulo $C^\infty([0,T],\Psi^{-\infty})$. This gives the existence of a parametrix with symbol belonging to $S_N\{0,0\}$. It is uniquely determined modulo $C^\infty([0,T],\Psi^{-\infty})$.

□

Step 2. Diagonalization procedure
We have to carry out perfect diagonalization. The main problem is to understand what the perfect diagonalization procedure means. Here we follow the following strategy:

- The first step of perfect diafonalization we carry out in all zones.
- The second step of perfect diagonalization we only carry out in $Z_{hyp}(N)$.
- The perfect diagonalization we only carry out in $Z_{reg}(N)$.

Perfect diagonalization means diagonalization modulo $T_{2N} \cap (S_{2N}\{0,0\} + S_{2N}\{-1,2\}) \cap \Big\{ \bigcap_{r\ge 0} S^\star_{2N}\{-r, r+1\}\Big\}$.

Let us explain these steps more in detail. We start with

$$\begin{aligned}
&Lu := D_t^2 u - \sum_{k,l=1}^n a_{kl}(t,x) D^2_{x_k x_l} u = g,\\
&u(0,x) = \varphi(x),\\
&D_t u(0,x) = -i\psi(x),
\end{aligned}$$

where $g := -f$ from (1). The transformation $U = (U_1,U_2)^T = \Big(\langle D_x\rangle u, D_t u\Big)^T$ transfers this Cauchy problem to

$$D_t U - AU = G\ ,\ U(0,x) = \begin{pmatrix} \langle D_x\rangle \varphi(x) \\ -i\psi(x) \end{pmatrix}, \tag{35}$$

where

$$A := \begin{pmatrix} 0 & \langle D_x\rangle \\ \sum\limits_{k,l=1}^n a_{kl}(t,x) D^2_{x_k x_l} \langle D_x\rangle^{-1} & 0 \end{pmatrix}\ ,\ G := \begin{pmatrix} 0 \\ g \end{pmatrix}.$$

LEMMA 19. *Symbol $\sigma(A)$ belongs to $T_{2N} \cap S_N\{1,0\}$.*

Now we care for the main step of diagonalization, this means, for the step which transforms A to a *diagonal matrix pseudo-differential operator* modulo an operator with symbol from $T_{2N} \cap S_N\{0,1\}$. Therefore we define the pseudo-differential operators of first order $\Phi_k = \Phi_k(t,x,D_x)$, $k = 1,2$, having symbols

$$\varphi_k(t,x,\xi) = d_k\langle\xi\rangle\, \chi\Big(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\Big) + \tau_k(t,x,\xi)\Big(1-\chi\Big(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\Big)\Big). \tag{36}$$

Here $d_2 = -d_1$ is a positive constant and

$$\tau_k(t,x,\xi) = (-1)^k\sqrt{a(t,x,\xi)}\,,\ a(t,x,\xi) := \sum_{k,l=1}^{n} a_{kl}(t,x)\xi_k\xi_l. \tag{37}$$

The function $\chi = \chi(s)$ is from $C_0^\infty(\mathbb{R})$, $\chi(s) \equiv 1$ for $|s| \le 1$, $\chi(s) \equiv 0$ for $|s| \ge 2$ and $0 \le \chi(s) \le 1$.

LEMMA 20. *a) The non-vanishing symbols $\varphi_k = \varphi_k(t,x,\xi)$, $k = 1,2$, belong to $T_{2N} \cap S_N\{1,0\}$.*

b) The special choice of d_k, $k = 1,2$, yields $\varphi_2 - \varphi_1 = 2\varphi_2$.

To start the diagonalization procedure we define the matrix pseudo-differential operator ($h(D_x) = \langle D_x\rangle$)

$$M(t,x,D_x) = \begin{pmatrix} I & I \\ \Phi_1(t,x,D_x)h^{-1}(D_x) & \Phi_2(t,x,D_x)h^{-1}(D_x) \end{pmatrix}.$$

Due to Lemma 18 we have the existence of $M^\sharp$. This follows from the analysis of

$$\sigma(M) = \begin{pmatrix} 1 & 1 \\ \frac{\varphi_1(t,x,\xi)}{h(\xi)} & \frac{\varphi_2(t,x,\xi)}{h(\xi)} \end{pmatrix},$$

that by (36) and (37) the symbol $\sigma(M)$ is a constant matrix in $Z_{pd}(N)$, $\det\sigma(M) = \frac{2\varphi_2(t,x,\xi)}{\langle\xi\rangle} \ge C > 0$ for $(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n}$. Hence, M is elliptic with a symbol belonging to $S_N\{0,0\}$. The parametrix $M^\sharp$ belongs to $S_N\{0,0\}$, too. We will later apply Duhamel's principle to find a representation of the solution to (35). Therefore we devote to find a fundamental solution to (35), this is a solution $E = E(t,s)$ satisfying

$$D_tE - AE = 0\,,\ E(s,s) = I. \tag{38}$$

Setting $E_0 = M^\sharp E$ leads to

$$\begin{aligned} D_tE_0 &= M^\sharp D_tE + D_tM^\sharp E = M^\sharp AE + D_tM^\sharp E \\ &= M^\sharp AME_0 + D_tM^\sharp ME_0 + R_\infty E, \end{aligned}$$

where $R_\infty \in C^\infty([0,T],\Psi^{-\infty})$. The symbols $\sigma(M^\sharp)$, $\sigma(M)$ are constant in $Z_{pd}(N)$. Consequently,

$$\sigma(M^\sharp AM) = \sigma(M^\sharp)\sigma(A)\sigma(M) + f_0(t,x,\xi) + r_\infty(t,x,\xi),$$

where

$$f_0(t,x,\xi) = \begin{cases} 0 \quad \text{in} \quad Z_{pd}(N) \\ \in T_{2N} \cap S_N\{0,0\}, \end{cases} \tag{39}$$

and $r_\infty \in C^\infty([0,T], S^{-\infty}(\mathbb{R}^n))$. Straightforward calculations yield

$$\sigma(M^\sharp)\sigma(A)\sigma(M) = \begin{cases} d(t,x,\xi) \quad \text{in} \quad Z_{hyp}(2N) \\ \in T_{2N}, \end{cases} \tag{40}$$

where

$$d(t,x,\xi) = \begin{pmatrix} \tau_1(t,x,\xi) & 0 \\ 0 & \tau_2(t,x,\xi) \end{pmatrix}$$

and

$$\sigma(M^\sharp)\sigma(A)\sigma(M) = \begin{pmatrix} \frac{\tau_1^2+\varphi_1^2}{2\varphi_1} & \frac{\varphi_2^2-\tau_2^2}{2\varphi_2} \\ \frac{\varphi_1^2-\tau_1^2}{2\varphi_1} & \frac{\tau_2^2+\varphi_2^2}{2\varphi_2} \end{pmatrix}(t,x,\xi) \quad \text{in} \quad Z_{pd}(2N).$$

Consequently, the following identity holds in $Z_{pd}(2N)$:

$$\sigma(M^\sharp)\sigma(A)\sigma(M) = \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} + \sigma(Q),$$

where the symbol $\sigma(Q) \in T_{2N} \cap S_N\{1,0\}$ and $\sigma(Q) \equiv 0$ in $Z_{hyp}(2N)$. Finally, let us devote to $D_t M^\sharp M = -M^\sharp D_t M + R_\infty$. We have

$$\sigma(M^\sharp D_t M) = \sigma(M^\sharp)\sigma(D_t M) + f_0(t,x,\xi) + r_\infty(t,x,\xi)\ ,$$

where

$$f_0(t,x,\xi) = \begin{cases} 0 \quad \text{in} \quad Z_{pd}(N) \\ \in T_{2N} \cap S_N\{-1,1\}, \end{cases} \tag{41}$$

and $r_\infty \in C^\infty([0,T], S^{-\infty}(\mathbb{R}^n))$. Using

$$\begin{aligned} \sigma(M^\sharp)\sigma(D_t M) &= \left(\tfrac{\varphi_2-\varphi_1}{h} \right)^{-1} \begin{pmatrix} \frac{\varphi_2}{h} & -1 \\ -\frac{\varphi_1}{h} & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ D_t\frac{\varphi_1}{h} & D_t\frac{\varphi_2}{h} \end{pmatrix} \\ &= \left(\tfrac{\varphi_2-\varphi_1}{h} \right)^{-1} \begin{pmatrix} -D_t\frac{\varphi_1}{h} & -D_t\frac{\varphi_2}{h} \\ D_t\frac{\varphi_1}{h} & D_t\frac{\varphi_2}{h} \end{pmatrix} \end{aligned}$$

and (38) to (41) we arrive at the next result. In the formulation of this result we use due to the influence of Q some symbols in $Z_{hyp}(2N)$ and take into consideration that symbols from $S_N\{1,0\}$ supported in the transition zone $Z_{pd}(2N) \setminus Z_{pd}(N)$ belong to T_{2N}.

LEMMA 21. *The fundamental solution $E = E(t,s)$ solving (38) can be represented in the form $E(t,s) = M(t)E_0(t,s)M^\sharp(s)$, where M is an elliptic operator with symbol $\sigma(M) \in S_N\{0,0\}$ and $E_0 = E_0(t,s)$ solves*

$$D_t E_0 - \mathcal{D}E_0 + P_1 E_0 + P_2 E_0 + QE_0 + R_\infty E = 0. \tag{42}$$

The matrix pseudo-differential operators $\mathcal{D}$, P_1, P_2, Q, R_∞ possess the following properties:

- $\mathcal{D}$: $\sigma(\mathcal{D}) \in T_{2N} \cap S_N\{1,0\}$,
$$\sigma(\mathcal{D}) = \begin{pmatrix} \varphi_1 + \frac{h}{2\varphi_2} D_t \frac{\varphi_2}{h} & 0 \\ 0 & \varphi_2 + \frac{h}{2\varphi_2} D_t \frac{\varphi_2}{h} \end{pmatrix};$$
- P_1: *diagonal,* $\sigma(P_1) \in T_{2N} \cap S_N\{0,0\}$, $\sigma(P_1) \equiv 0$ *in* $Z_{pd}(N)$;
- P_2: *antidiagonal,* $\sigma(P_2) \in T_{2N} \cap S_N\{0,1\}$, $\sigma(P_2) \equiv 0$ *in* $Z_{pd}(N)$;
- Q: $\sigma(Q) \in T_{2N}$, $\sigma(Q) \equiv 0$ *in* $Z_{hyp}(2N)$;
- R_∞: $\sigma(R_\infty) \in C^\infty([0,T], S^{-\infty}(\mathbb{R}^n))$.

This finishes the first step of perfect diagonalization, this step yields a diagonalization modulo $T_{2N} \cap S_{2N}\{0,1\}$.

In the next step of perfect diagonalization our goal consists in the *diagonalization of the antidiagonal matrix operator* P_2 with symbol $\sigma(P_2)$ modulo $S_{2N}\{-1,2\}$. In the hierarchy of symbols described in Lemma 16 the corresponding pseudo-differential operator has a better smoothing property than pseudo-differential operators with symbols from $S_{2N}\{0,1\}$. We restrict ourselves to

$$D_t E_0 - \mathcal{D}E_0 + P_1 E_0 + P_2 E_0 + QE_0 = 0. \tag{43}$$

LEMMA 22. *There exist an elliptic pseudo-differential operator N_1 with $\sigma(N_1) \in S_N\{0,0\}$,*
$\sigma(N_1) \equiv I$ in $Z_{pd}(N)$, and pseudo-differential operators F_1 of diagonal structure and P_3 with $\sigma(F_1) \in T_{2N} \cap S_N\{0,0\}$, $\sigma(F_1) \equiv 0$ in $Z_{pd}(N)$, and $\sigma(P_3) \in T_{2N} \cap S_{2N}\{-1,2\}$ such that

$$(D_t - \mathcal{D} + P_1 + P_2 + Q)N_1 = N_1(D_t - \mathcal{D} + F_1 + P_3) \tag{44}$$

holds modulo an regularizing operator R_∞ with symbol $\sigma(R_\infty)$ belonging to $C^\infty([0,T], S^{-\infty}(\mathbb{R}^n))$.

Proof. We localize our considerations to $Z_{hyp}(N)$ by using the pseudo-differential operator $I - \chi(t, D_x)$ with symbol $I\Big(1 - \chi\Big(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\Big)\Big)$. We define F_1 with the symbol $\sigma(F_1)(t,x,\xi) = \Big(1 - \chi\Big(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\Big)\Big)\sigma(P_1)(t,x,\xi)$, which belongs to $T_{2N} \cap S_N\{0,0\}$.

Moreover, we introduce

$$n_1^{(1)}(t,x,\xi) := \begin{pmatrix} 0 & \frac{p_{12}}{\varphi_1-\varphi_2} \\ \frac{p_{21}}{\varphi_2-\varphi_1} & 0 \end{pmatrix} \Big(1-\chi\Big(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\Big)\Big) \in T_{2N}\cap S_N\{-1,1\},$$

where

$$\sigma(P_2) = \begin{pmatrix} 0 & p_{12} \\ p_{21} & 0 \end{pmatrix} \in T_{2N}\cap S_N\{0,1\}.$$

Setting $N_1 = I + N_1^{(1)}$, $\sigma(N_1^{(1)}) = n_1^{(1)}$, we are able to conclude that the symbol $\sigma(B^{(1)})$ of

$$\begin{aligned} B^{(1)} &:= (D_t - \mathcal{D} + P_1 + P_2 + Q)(I + N_1^{(1)}) - (I + N_1^{(1)})(D_t - \mathcal{D} + F_1) \\ &= P_1 + P_2 + Q - [\mathcal{D}, N_1^{(1)}] - F_1 + D_t N_1^{(1)} + (P_1 + P_2 + Q)N_1^{(1)} \\ &\quad - N_1^{(1)} F_1 \end{aligned}$$

belongs to $T_{2N}\cap S_{2N}\{-1,2\}$. This follows from

- $\sigma(D_t N_1^{(1)}) \in T_{2N}\cap S_N\{-1,2\}$, $\sigma(D_t N_1^{(1)}) \equiv 0$ in $Z_{pd}(N)$;
- $\sigma((P_1+P_2)N_1^{(1)} - N_1^{(1)}F_1) \in T_{2N}\cap S_N\{-1,2\}$, $\sigma((P_1+P_2)N_1^{(1)} - N_1^{(1)}F_1) \equiv 0$ in $Z_{pd}(N)$;
- $\sigma((1-\chi)P_2 - [\mathcal{D}, N_1^{(1)}]) \in T_{2N}\cap S_N\{-1,2\}$, $\sigma((1-\chi)P_2 - [D, N_1^{(1)}]) \equiv 0$ in $Z_{pd}(N)$.

The last relation is a conclusion from

$$\begin{aligned} \sigma((1-\chi)P_2 - [\mathcal{D}, N_1^{(1)}]) &= \begin{pmatrix} 0 & (1-\chi)p_{12} \\ (1-\chi)p_{21} & 0 \end{pmatrix} \\ &- \begin{pmatrix} \varphi_1 + \frac{h}{2\varphi_2} D_t \frac{\varphi_2}{h} & 0 \\ 0 & \varphi_2 + \frac{h}{2\varphi_2} D_t \frac{\varphi_2}{h} \end{pmatrix} \times \begin{pmatrix} 0 & \frac{(1-\chi)p_{12}}{\varphi_1-\varphi_2} \\ \frac{(1-\chi)p_{21}}{\varphi_2-\varphi_1} & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & \frac{(1-\chi)p_{12}}{\varphi_1-\varphi_2} \\ \frac{(1-\chi)p_{21}}{\varphi_2-\varphi_1} & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 + \frac{h}{2\varphi_2} D_t \frac{\varphi_2}{h} & 0 \\ 0 & \varphi_2 + \frac{h}{2\varphi_2} D_t \frac{\varphi_2}{h} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \bmod\ S_N\{-1,2\}. \end{aligned}$$

The symbol $\sigma((1-\chi)P_2 - [\mathcal{D}, N_1^{(1)}])$ vanishes in $Z_{pd}(N)$ because of $\sigma(P_2) = \sigma(N_1^{(1)}) \equiv 0$ and belongs to T_{2N}. The remainder $R_1 := (P_1 + P_2)\chi + QN_1$ belongs to T_{2N} and vanishes in $Z_{hyp}(2N)$. Summarizing these observations we see that $B^{(1)} = \tilde{B}^{(1)} + R_1$, where $\sigma(\tilde{B}^{(1)}) \in T_{2N}\cap S_N\{-1,2\}$, $\equiv 0$ in $Z_{pd}(N)$ and $\sigma(R_1) \in T_{2N}$, $\equiv 0$ in $Z_{hyp}(2N)$. Now let us show that a sufficiently large N in (30) guarantees that N_1 is an elliptic pseudo-differential operator with symbol belonging to $S_N\{0,0\}$. Due to our construction $\sigma(N_1) \equiv I$ in $Z_{pd}(N)$. We know that

$$|n_1^{(1)}(x,t,\xi)| \le \frac{C}{\langle\xi\rangle}\frac{1}{t}(\ln\frac{1}{t})^\gamma \le \frac{C_1}{N} \text{ in } Z_{hyp}(N).$$

Consequently, a large N yields $|\sigma(N_1)| \ge 1/2$ in $[0,T]\times\mathbb{R}^{2n}$. Using $\sigma(N_1) = I$ in $Z_{pd}(N)$ gives together with Lemma 18 the existence of $N_1^\sharp$ with $\sigma(N_1^\sharp) \in S_N\{0,0\}$. It is clear that the symbol of

$$P_3 := N_1^\sharp B^{(1)} = N_1^\sharp(\tilde{B}^{(1)} + R_1) \tag{45}$$

belongs to $T_{2N} \cap S_{2N}\{-1,2\}$ modulo a regularizing operator R_∞ with symbol $\sigma(R_\infty)$ belonging to $C^\infty([0,T], S^{-\infty}(\mathbb{R}^n))$.

□

This finishes the second step of perfect diagonalization, this step yields a diagonalization modulo $T_{2N} \cap S_{2N}\{-1,2\}$.

Summarizing we have proved the next result.

LEMMA 23. *The fundamental solution $E_0 = E_0(t,s)$ solving (43) can be represented in the form $E_0(t,s) = N_1(t)E_1(t,s)N_1^\sharp(s)$, where $N_1^\sharp$ and N_1 are elliptic pseudo-differential operators with symbols $\sigma(N_1^\sharp),\ \sigma(N_1) \in S_N\{0,0\}$, both symbols are constant in $Z_{pd}(N)$. The matrix operator $E_1 = E_1(t,s)$ solves*

$$D_tE_1 - \mathcal{D}E_1 + F_1E_1 + P_3E_1 + R_\infty E_1 = 0,$$

where the matrix pseudo-differential operators $\mathcal{D},\ F_1,\ P_3,\ R_\infty$ possess the following properties:

- $\mathcal{D}$: $\sigma(\mathcal{D}) \in T_{2N} \cap S_N\{1,0\}$, $\sigma(\mathcal{D}) = \begin{pmatrix} \varphi_1 + \frac{\langle\xi\rangle}{2\varphi_2}D_t\frac{\varphi_2}{\langle\xi\rangle} & 0 \\ 0 & \varphi_2 + \frac{\langle\xi\rangle}{2\varphi_2}D_t\frac{\varphi_2}{\langle\xi\rangle} \end{pmatrix}$;
- F_1: *diagonal,* $\sigma(F_1) \in T_{2N} \cap S_N\{0,0\}$, $\sigma(F_1) \equiv 0$ in $Z_{pd}(N)$;
- P_3: $\sigma(P_3) \in T_{2N} \cap S_{2N}\{-1,2\}$;
- R_∞: $\sigma(R_\infty) \in C^\infty([0,T], S^{-\infty}(\mathbb{R}^n))$.

Now let us sketch the perfect diagonalization.

REMARK 10. Let us explain our philosophy to carry out further steps of perfect diagonalization. We will localize further steps of diagonalization to $Z_{reg}(N)$. In this part of $Z_{hyp}(N)$ we get the improvement of smoothness of the remainder P_{p+2}. This improvement of smoothness can be understood after calculating for $\gamma \in (0,1]$

$$\begin{aligned}&\int_{\tilde{t}_\xi}^t \Big|\sigma(P_{p+2})(\tau,x,\xi)\Big|d\tau \\ &\le \int_{\tilde{t}_\xi}^t \frac{C_p}{\langle\xi\rangle^p}\Big(\frac{1}{\tau}\Big(\ln\frac{1}{\tau}\Big)^\gamma\Big)^{p+1}d\tau \le \frac{C_p(\ln\langle\xi\rangle)^{\gamma(p+1)}}{(\langle\xi\rangle\tilde{t}_\xi)^p} = \frac{C_p}{(2N)^p}(\ln\langle\xi\rangle)^{\gamma(p+1)-2\gamma p},\end{aligned}$$

where $\tilde{t}_\xi$ is defined as in formula (30). In the oscillations subzone we use for the construction of parametrix a behaviour of the symbol of remainder like $S_{2N}\{0,0\} + S_{2N}\{-1,2\}$. It turns out that the perfect diagonalization means diagonalization modulo operators with symbols from $T_{2N} \cap (S_{2N}\{0,0\} + S_{2N}\{-1,2\}) \cap \Big\{ \bigcap_{p\geq 0} S^{\star}_{2N}\{-p, p+1\} \Big\}$.

LEMMA 24. *There exist a matrix elliptic operator N_2 with $\sigma(N_2) \in S_N\{0,0\}$, $\sigma(N_2) \equiv I$ in $Z_{pd}(N) \cup Z_{osc}(N)$, a diagonal matrix pseudo-differential operator F_2 with $\sigma(F_2) \in (S^{\star}_N\{0,0\} + S^{\star}_N\{-1,2\})$, $\sigma(F_2) \equiv 0$ in $Z_{pd}(N) \cup Z_{osc}(N)$, and a matrix pseudo-differential operator P_∞ with $\sigma(P_\infty)(t,x,\xi) \in T_{2N} \cap (S_{2N}\{0,0\} + S_{2N}\{-1,2\}) \cap \Big\{ \bigcap_{p\geq 0} S^{\star}_{2N}\{-p, p+1\} \Big\}$ such that*

$$(D_t - \mathcal{D} + F_1 + P_3)N_2 = N_2(D_t - \mathcal{D} + F_2 + P_\infty). \tag{46}$$

This identity holds modulo a regularizing operator R_∞ with symbol $\sigma(R_\infty)$ belonging to $C^\infty([0,T], S^{-\infty}(\mathbb{R}^n))$.

Proof. We choose the representation $N_2 \sim I + \sum_{r\geq 1} N_2^{(r)}$ and $F_2 \sim \sum_{r\geq 0} F_2^{(r)}$. Our goal is to show the relation

$$(D_t - \mathcal{D} + F_1 + P_3)(I + \sum_{r\geq 1} N_2^{(r)}) \sim (I + \sum_{r\geq 1} N_2^{(r)})(D_t - \mathcal{D} + \sum_{r\geq 0} F_2^{(r)} + P_\infty).$$

For further constructions we use $P_3 = P_{3,1} + P_{3,2}$, where $P_{3,1}$ denotes the *diagonal part* of P_3 and $P_{3,2}$ denotes the *antidiagonal part*.
We localize our considerations to $Z_{reg}(N)$ by using the pseudo-differential operator $I - \chi I$ with symbol $I\Big(1 - \chi\Big(\frac{t\langle\xi\rangle}{2N(\ln\langle\xi\rangle)^{2\gamma}}\Big)\Big)$. We define $F_2^{(0)}$ with the symbol $\sigma(F_2^{(0)})(t,x,\xi) = \Big(1 - \chi\Big(\frac{t\langle\xi\rangle}{2N(\ln\langle\xi\rangle)^{2\gamma}}\Big)\Big)\sigma(F_1 + P_{3,1})(t,x,\xi)$, which belongs to $S^{\star}_N\{0,0\} + S^{\star}_N\{-1,2\}$. Moreover, we introduce

$$n_2^{(1)}(t,x,\xi) := \begin{pmatrix} 0 & \frac{p_{13}}{\varphi_1 - \varphi_2} \\ \frac{p_{31}}{\varphi_2 - \varphi_1} & 0 \end{pmatrix} \Big(1 - \chi\Big(\frac{t\langle\xi\rangle}{2N(\ln\langle\xi\rangle)^{2\gamma}}\Big)\Big) \in S^{\star}_N\{-2,2\},$$

where

$$\sigma(P_{3,2}) = \begin{pmatrix} 0 & p_{13} \\ p_{31} & 0 \end{pmatrix} \in T_{2N} \cap S_{2N}\{-1,2\}.$$

Setting $N_2 = I + N_2^{(1)}$, $\sigma(N_2^{(1)}) = n_2^{(1)}$, we get similar as in the proof of Lemma 22 that the symbol $\sigma(B^{(1)})$ of

$$B^{(1)} : = (D_t - \mathcal{D} + F_1 + P_3)(I + N_2^{(1)}) - (I + N_2^{(1)})(D_t - \mathcal{D} + F_2^{(0)})$$

belongs to $T_{2N} \cap (S_{2N}\{0,0\} + S_{2N}\{-1,2\}) \cap S^{\star}_{2N}\{-2,3\}$. Moreover, we can show that $B^{(1)} = \tilde{B}^{(1)} + R_1$, where $\sigma(\tilde{B}^{(1)}) \in S^{\star}_N\{-2,3\}$, $\equiv 0$ in $Z_{pd}(N) \cup Z_{osc}(N)$ and

$\sigma(R_1) \in T_{2N} \cap (S_{2N}\{0,0\} + S_{2N}\{-1,2\})$, $\equiv 0$ in $Z_{reg}(2N)$. Now we are able to start an induction procedure. Let us suppose that $\tilde{B}^{(r)}$ is already constructed and its symbol $\sigma(\tilde{B}^{(r)}) \in S_N^\star\{-(r+1), r+2\}$, $\equiv 0$ in $Z_{pd}(N) \cup Z_{osc}(N)$. Then $F_2^{(r)} := \tilde{B}_1^{(r)}$ has the same properties, where $\tilde{B}_1^{(r)}$, $\tilde{B}_2^{(r)}$ denote the diagonal part and antidiagonal part of $\tilde{B}^{(r)}$, respectively. We introduce

$$n_2^{(r+1)}(t,x,\xi) := \begin{pmatrix} 0 & \frac{p_{1(r+3)}}{\varphi_1-\varphi_2} \\ \frac{p_{(r+3)1}}{\varphi_2-\varphi_1} & 0 \end{pmatrix}$$

as the symbol of $N^{(r+1)}$, where

$$\sigma(\tilde{B}_2^{(r)}) = \begin{pmatrix} 0 & p_{1(r+3)} \\ p_{(r+3)1} & 0 \end{pmatrix} \in S_N^\star\{-(r+1), r+2\}, \equiv 0 \text{ in } Z_{pd}(N) \cup Z_{osc}(N).$$

Then we have to check the operator

$$B^{(r+1)} := (D_t - \mathcal{D} + F_1 + P_3)(I + \sum_{l=1}^{r+1} N_2^{(l)}) - (I + \sum_{l=1}^{r+1} N_2^{(l)})(D_t - \mathcal{D} + \sum_{l=0}^{r} F_2^{(l)})$$

and can show that $B^{(r+1)} = \tilde{B}^{(r+1)} + R_1$, where $\sigma(\tilde{B}^{(r+1)}) \in S_N^\star\{-(r+2), r+3\}$, $\equiv 0$ in $Z_{pd}(N) \cup Z_{osc}(N)$ and R_1 is as above. By Lemma 16 we find a symbol $n_2 = n_2(t,x,\xi) \sim I + \sum_{r\geq 1} \sigma(N_2^{(r)})(t,x,\xi)$, $n_2 \in S_N^\star\{0,0\}$ modulo $\bigcap_{r\geq 0} S_N^\star\{-r,r\}$, and $n_2 \equiv I$ in $Z_{pd}(N) \cup Z_{osc}(N)$, and a symbol $f_2 = f_2(t,x,\xi) \sim \sum_{r\geq 0} \sigma(F_2^{(r)})(t,x,\xi)$, $f_2 \in (S_N^\star\{0,0\} + S_N^\star\{-1,2\})$ modulo $\bigcap_{r\geq 0} S_N^\star\{-r,r+1\}$, $f_2 \equiv 0$ in $Z_{pd}(N) \cup Z_{osc}(N)$. Then the above operator identity holds with $\sigma(N_2) := n_2$ and $\sigma(F_2) := f_2$, where P_∞ can be represented in the form $P_\infty = F_\infty + R$, where $\sigma(R) = \sigma(F_1 + P_3)\chi\Big(\frac{t\langle\xi\rangle}{2N(\ln\langle\xi\rangle)^{2\gamma}}\Big)$.

The first pseudo-differential operator F_∞ has a symbol $\sigma(F_\infty)$ from $\Big\{\bigcap_{r\geq 0} S_N^\star\{-r, r+1\}\Big\}$, $\sigma(F_\infty) \equiv 0$ in $Z_{pd}(N) \cup Z_{osc}(N)$. The second pseudo-differential operator R has a symbol $\sigma(R)$ belonging to $T_{2N} \cap (S_{2N}\{0,0\} + S_{2N}\{-1,2\})$. Moreover, $\sigma(R)$ vanishes in $Z_{reg}(2N)$.

$\square$

Thus we finished our perfect diagonalization modulo $T_{2N} \cap (S_{2N}\{0,0\} + S_{2N}\{-1,2\}) \cap \Big\{\bigcap_{r\geq 0} S_{2N}^\star\{-r, r+1\}\Big\}$.

To complete the perfect diagonalization it remains to understand that a parametrix $N_2^\sharp$ to N_2 exists. From the construction we know that $\sigma(N_2 - I) \in S_N^\star\{-1,1\}$ and vanishes in $Z_{pd}(N) \cup Z_{osc}(N)$. A suitable large constant N in the definition of zones guarantees that N_2 is elliptic and its symbol is equal to I in $Z_{pd}(N) \cup Z_{osc}(N)$. Hence, the statement of Lemma 18 gives the existence of $N_2^\sharp$ with symbol from $S_N^\star\{0,0\}$ and equal to I in $Z_{pd}(N) \cup Z_{osc}(N)$. Thus we can formulate the next result.

LEMMA 25. *The fundamental solution $E_0 = E_0(t,s)$ solving (43) can be represented in the form $E_0(t,s) = N_1(t)N_2(t)E_1(t,s)N_2^\sharp(s)N_1^\sharp(s)$, where $N_1^\sharp$, N_1 and $N_2^\sharp$, N_2 are elliptic operators with symbols $\sigma(N_1^\sharp), \sigma(N_1) \in S_N\{0,0\}$, both symbols are $\equiv I$ in $Z_{pd}(N)$ and $\sigma(N_2^\sharp)$, $\sigma(N_2) \in S_N^\star\{0,0\}$, both symbols are $\equiv I$ in $Z_{pd}(N) \cup Z_{osc}(N)$. The matrix operator $E_1 = E_1(t,s)$ solves*

$$D_t E_1 - \mathcal{D}E_1 + F_2 E_1 + P_\infty E_1 + R_\infty E_1 = 0,$$

where the matrix pseudo-differential operators $\mathcal{D}$, F_2, P_∞, R_∞ possess the following properties:

- $\mathcal{D}$: $\sigma(\mathcal{D}) \in T_{2N} \cap S_N\{1,0\}$,

$$\sigma(\mathcal{D}) = \begin{pmatrix} \varphi_1 + \frac{h}{2\varphi_2} D_t \frac{\varphi_2}{h} & 0 \\ 0 & \varphi_2 + \frac{h}{2\varphi_2} D_t \frac{\varphi_2}{h} \end{pmatrix};$$

- F_2: *diagonal,* $\sigma(F_2) \in (S_N^\star\{0,0\} + S_N^\star\{-1,2\})$, $\sigma(F_2) \equiv 0$ in $Z_{pd}(N) \cup Z_{osc}(N)$;

- P_∞: $\sigma(P_\infty) \in T_{2N} \cap (S_{2N}\{0,0\} + S_{2N}\{-1,2\}) \cap \Big\{ \bigcap_{p\geq 0} S_{2N}^\star\{-p, p+1\}\Big\}$;

- R_∞: $\sigma(R_\infty) \in C^\infty([0,T], S^{-\infty}(\mathbb{R}^n))$.

All the statements together yield the following result.

LEMMA 26. *The determination of a parametrix to the matrix pseudo-differential operator $D_t - A$ can be reduced, after transformations by elliptic matrix pseudo-differential operators (corresponding to perfect diagonalization), to the determination of a parametrix to the matrix pseudo-differential operator $D_t - \mathcal{D} + F_2 + P_\infty$, where the matrix pseudo-differential operators $\mathcal{D}$, F_2, P_∞, possess the following properties:*

- $\mathcal{D}$: $\sigma(\mathcal{D}) \in T_{2N} \cap S_N\{1,0\}$,

$$\sigma(\mathcal{D}) = \begin{pmatrix} \varphi_1 + \frac{\langle\xi\rangle}{2\varphi_2} D_t \frac{\varphi_2}{\langle\xi\rangle} & 0 \\ 0 & \varphi_2 + \frac{\langle\xi\rangle}{2\varphi_2} D_t \frac{\varphi_2}{\langle\xi\rangle} \end{pmatrix};$$

- F_2: *diagonal,* $\sigma(F_2) \in (S_N^\star\{0,0\} + S_N^\star\{-1,2\})$, $\sigma(F_2) \equiv 0$ in $Z_{pd}(N) \cup Z_{osc}(N)$;

- P_∞: $\sigma(P_\infty) \in T_{2N} \cap (S_{2N}\{0,0\} + S_{2N}\{-1,2\}) \cap \Big\{ \bigcap_{p\geq 0} S_{2N}^\star\{-p, p+1\}\Big\}$.

Here we use

$$\varphi_k(t,x,\xi) = d_k\langle\xi\rangle\ \chi\Big(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\Big) + \tau_k(t,x,\xi)\Big(1-\chi\Big(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\Big)\Big),$$

where $d_2 = -d_1$ *is a positive constant and*

$$\tau_k(t,x,\xi) = (-1)^k\sqrt{a(t,x,\xi)}\ ,\ a(t,x,\xi) := \sum_{k,l=1}^{n} a_{k,l}(t,x)\xi_k\xi_l .$$

The function $\chi = \chi(s)$ *is from* $C_0^\infty(\mathbb{R})$*,* $\chi(s)\equiv 1$ *for* $|s|\le 1$*,* $\chi(s)\equiv 0$ *for* $|s|\ge 2$ *and* $0\le\chi(s)\le 1$.

Step 3. Construction of parametrix

We need four steps for the construction of the parametrix.

Transformation by an elliptic pseudo-differential operator.

Let K be the diagonal elliptic pseudo-differential operator with symbol

$$\sigma(K) = \begin{pmatrix} \sqrt{\frac{\varphi_2}{\langle\xi\rangle}} & 0 \\ 0 & \sqrt{\frac{\varphi_2}{\langle\xi\rangle}} \end{pmatrix} .$$

This symbol is constant in $Z_{pd}(N)$, $\sigma(K)\in S_N\{0,0\}$. Then the following operator-valued identity holds modulo a regularizing operator:

$$(D_t - \mathcal{D} + F_2)K = K(D_t - \mathcal{D}_1 + F_3), \tag{47}$$

where

$$\sigma(\mathcal{D}_1) := \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}\ ,\ \sigma(F_3)\equiv 0 \ \text{ in } \ Z_{pd}(N),$$
$$\sigma(F_3)\in T_{2N}\cap(S_N\{0,0\}+S_N^\star\{-1,2\}).$$

REMARK 11. This transformation corresponds to the special structure of our starting operator and explains that we have no contribution to the loss of derivatives from $\mathcal{D}$. This we already observed in Section 3 during the proof of Theorem 8. In the representation of V_1 from (10) there appears $E_2 = E_2(t,t_\xi,\xi)$. Although in E_2 there appears the term $\frac{1}{2}\frac{D_s a}{a}$ which belongs to $S_1\{0,1\}$ (see Definition 7), this term has no contribution to the loss of derivatives.

Parametrix to $D_t - \mathcal{D}_1$.

LEMMA 27. *The parametrix* $E_2(t,s) =$ diag$(E_2^-(t,s), E_2^+(t,s))$ *to* $D_t - \mathcal{D}_1$ *is a diagonal Fourier integral operator with*

$$E_2^\mp(t,s)w(x) = \int_{\mathbb{R}^n} e^{i\phi^\mp(t,s,x,\xi)}e_2^\mp(t,s,x,\xi)\hat{w}(\xi)d\xi\ ,$$

$$\phi^\mp(s,s,x,\xi) = x\cdot\xi\ ,\ e_2^\mp(s,s,x,\xi) = 1\ .$$

The phase functions $\phi^\mp$ *satisfy*

- $\phi^{\mp}(t,s,x,\xi) = x\cdot\xi + d_k\langle\xi\rangle(t-s),\ k=1$ *for* ϕ^-, $k=2$ *for* ϕ^+ *if* $0 \le s,t \le t_\xi$;
- $|\partial_\xi^\alpha \partial_x^\beta(\phi^{\mp}(t,s,x,\xi) - x\cdot\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{1-|\alpha|}\max(s,t)$ *if* $\max(s,t) \ge t_\xi$.

The amplitude functions $e_2^{\mp}$ *satisfy*

- $e_2^{\mp}(t,s,x,\xi) = 1$ *if* $0 \le s,t \le t_\xi$;
- $e_2^{\mp} \in C([0,T_0]^2, S^0_{1,0}(\mathbb{R}^n))$.

To prove this result we follow the following steps:
Study of the Hamiltonian flow generated by $\varphi_1 = \varphi_1(t,x,\xi)$ *and* $\varphi_2 = \varphi_2(t,x,\xi)$.
Construction of phase functions
Let us denote by $\lambda = \lambda(t,x,\xi)$ one of the functions $\varphi_k = \varphi_k(t,x,\xi),\ k=1,2$. The *Hamiltonian flow* $(q,p) = (q,p)(t,s,y,\eta) =: H_{s,t}(y,\eta)$ is the solution to

$$\frac{dq}{dt} = \nabla_\xi\lambda(t,q,p),\quad q(s,s,y,\eta) = y\ ;\ \frac{dp}{dt} = -\nabla_x\lambda(t,q,p),\quad p(s,s,y,\eta) = \eta.$$

Using $\sigma(\lambda) \in T_{2N} \cap S_N\{1,0\}$ we know that the growth of λ with respect to q and p is at most linear. Thus the solution (q,p) exists globally in time, $t \in [0,T]$, for all (y,η). For the following considerations we need suitable estimates for $q = q(t,s,y,\eta)$ and $p = p(t,s,y,\eta)$. Following the approach of [12] and [26] one can prove the next results.

LEMMA 28. *There exists a (in general small) positive constant* T_0 *such that*

$$\tfrac{q(t,s)-y}{t-s}\ ,\ \partial_t q(t,s),\ \partial_s q(t,s) \in L^\infty([0,T_0]^2, S^0_{1,0}(\mathbb{R}^n_y \times \mathbb{R}^n_\eta));$$
$$\tfrac{p(t,s)-\xi}{t-s}\ ,\ \partial_t p(t,s),\ \partial_s p(t,s) \in L^\infty([0,T_0]^2, S^1_{1,0}(\mathbb{R}^n_y \times \mathbb{R}^n_\eta)).$$

LEMMA 29. *If* T_0 *is small, then the inverse function* $y = y(t,s,x,\eta)$ *to* $x = q(t,s,y,\eta)$ *exists and satisfies*

$$\tfrac{y(t,s)-x}{t-s}\ ,\ \partial_t y(t,s),\ \partial_s y(t,s) \in L^\infty([0,T_0]^2,\ S^0_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\eta)).$$

Construction of phase functions $\phi^{\mp}$ *solving the eikonal equations.*

Let us construct the phase function $\phi = \phi(t,s,x,\xi)$ solving the *eikonal equation* $\partial_t\phi(t,s,x,\xi) - \lambda(t,x,\nabla_x\phi(t,s,x,\xi)) = 0,\ \phi(s,s,x,\xi) = x\cdot\xi$.

LEMMA 30. *The phase function* $\phi = \phi(t,s,x,\xi)$ *is defined as follows:*
$\phi(t,s,x,\xi) := v(t,s,y(t,s,x,\xi),\xi)$, *where*

$$v(t,s,y,\xi) = y\cdot\xi - \int_s^t \Big(p\cdot\nabla_\xi\lambda - \lambda\Big)\Big(\tau, q(\tau,s,y,\xi), p(\tau,s,y,\xi)\Big)d\tau.$$

Construction of amplitudes $e_2^{\mp}$ by solving the transport equations and by using the asymptotic representation theorem.

Following our representation

$$E_2^{\mp}(t,s)w(x) = \int\limits_{\mathbb{R}^n} e^{i\phi^{\mp}(t,s,x,\xi)} e_2^{\mp}(t,s,x,\xi)\hat{w}(\xi)d\xi$$

with $\phi^{\mp}(s,s,x,\xi) = x\cdot\xi$, $e_2^{\mp}(s,s,x,\xi) = 1$, as usual, the asymptotic representation

$$e_2^{\mp}(t,s,x,\xi) \sim \sum_{j=0}^{\infty} e_{2,j}^{\mp}(t,s,x,\xi) \quad \text{modulo } \; C([0,T_0]^2, S^{-\infty}(\mathbb{R}^n)),$$
$$e_{2,0}^{\mp}(s,s,x,\xi) = 1\ ,\ e_{2,j}^{\mp}(s,s,x,\xi) = 0 \quad \text{for } \; j \geq 1,$$

allows us to derive so-called *transport equations*.
We have to study the action of $D_t - \varphi_1(t,x,D_x)$, $D_t - \varphi_2(t,x,D_x)$ respectively on E_2^- , E_2^+. We consider $(D_t - \varphi_1)E_2^-$ and suppose that all assumptions are satisfied for the action of the pseudo-differential operator $D_t - \varphi_1(t,x,D_x)$ on the Fourier integral operator E_2^-. On the one hand we get formally

$$D_t E_2^-(t,s)w(x) = \int\limits_{\mathbb{R}^n} e^{i\phi^-(t,s,x,\xi)}\Big(\partial_t\phi^- \sum_{j=0}^{\infty} e_{2,j}^- + \tfrac{1}{i}\partial_t \sum_{j=0}^{\infty} e_{2,j}^-\Big)(t,s,x,\xi)\hat{w}(\xi)d\xi;$$

on the other hand we use formally

$$\begin{aligned}\varphi_1(t,x,D_x)E_2^-(t,s)w(x) = \int\limits_{\mathbb{R}^n} e^{i\phi^-(t,s,x,\xi)}\Big[\varphi_1\Big(t,x,\nabla_x\phi^-(t,s,x,\xi)\Big)\\ \sum_{j=0}^{\infty} e_{2,j}^-(t,s,x,\xi) + \nabla_\xi\varphi_1\Big(t,x,\nabla_x\phi^-(t,s,x,\xi)\Big)\cdot\tfrac{1}{i}\sum_{j=0}^{\infty}\nabla_x e_{2,j}^-(t,s,x,\xi)\\ -\tfrac{i}{2}\sum_{k,l=1}^{n}\partial^2_{\xi_k\xi_l}\varphi_1\Big(t,x,\nabla_x\phi^-(t,s,x,\xi)\Big)\Big(\partial^2_{x_kx_l}\phi^-\sum_{j=0}^{\infty} e_{2,j}^-\Big)(t,s,x,\xi)\\ +r_2(t,s,x,\xi)\Big]\hat{w}(\xi)d\xi,\end{aligned}$$

where

$$\begin{aligned} r_2(t,s,x,\xi) \quad \sim \quad \sum_{|\alpha|=2}^{\infty}\tfrac{1}{\alpha!}D_y^\alpha\Big(\partial_\xi^\alpha\varphi_1\Big(t,x,\int_0^1\nabla_x\phi^-(t,s,y+r(x-y),\xi)dr\Big)\\ \sum_{j=0}^{\infty} e_{2,j}^-(t,s,y,\xi)\Big)_{y=x}.\end{aligned}$$

Supposing that all series converge uniformly and using that ϕ^- solves the eikonal equation with $\lambda = \varphi_1$ we arrive at the *transport equations* to determine $e_{2,j}^{\mp}$ for $j \geq 0$. Finally we arrive at the statements of Lemma 27.

Parametrix to $D_t - \mathcal{D}_1 + F_3$.

LEMMA 31. *The parametrix $E_4 = E_4(t,s)$ to the operator $D_t - \mathcal{D}_1 + F_3$ can be written as $E_4(t,s) = E_2(t,s)Q_4(t,s)$, where $E_2 = E_2(t,s)$ is the diagonal Fourier integral operator from Lemma 27 and $Q_4 = Q_4(t,s)$ is a diagonal pseudo-differential operator with symbol belonging to $W^{1,\infty}([0,T_0]^2, S^0_{1,0}(\mathbb{R}^n))$.*

To prove this result we follow the following steps:

Application of Egorov's theorem, that is, conjugation of F_3 by Fourier integral operators, here we use the diagonal structure.

We will construct the parametrix to $D_t - \mathcal{D}_1 + F_3$. Using $E_2 = E_2(t,s)$ from the previous step we choose the representation

$$E_4(t,s) = E_2(t,s)Q_4(t,s), \quad Q_4(s,s) \sim I.$$

This implies the Cauchy problem

$$D_t Q_4 + E_2(s,t)F_3(t)E_2(t,s)Q_4 \sim 0\ , \quad Q_4(s,s) \sim I.$$

According to Egorov's theorem [26] (here we can use the diagonal structure of $D_t - \mathcal{D}_1 + F_3$) the matrix operator $R_4(t,s) := E_2(s,t)F_3(t)E_2(t,s)$ is a pseudo-differential operator whose symbol is $r_4 = r_4(t,s,x,\xi) = f_3(t, H_{s,t}(x,\xi)),\ f_3 := \sigma(F_3)$, modulo a symbol from $S_N\{-1,0\} + S^\star_N\{-2,2\}$, where $H_{s,t}(x,\xi)$ denotes the Hamiltonian flow starting at (x,ξ) and generated by the symbols $\varphi_k = \varphi_k(t,x,\xi),\ k = 1,2$.

For $t \in [0,T_0]$ with a sufficiently small T_0 we understand to which zone the Hamiltonian flow belongs to.

We can write $f_3(t,x,\xi) = f_{3,0}(t,x,\xi) + f_{3,1}(t,x,\xi)$, where $f_{3,0} \in S_N\{0,0\}$, $f_{3,1} \in S^\star_N\{-1,2\}$,
$f_{3,0} \equiv 0$ in $Z_{pd}(N)$, $f_{3,1} \equiv 0$ in $Z_{pd}(N) \cup Z_{osc}(N)$.

LEMMA 32. *Let us denote by $\lambda = \lambda(t,x,\xi)$ one of the functions $\varphi_k = \varphi_k(t,x,\xi),\ k = 1,2$. The Hamiltonian flow $(q,p) = (q,p)(t,s,y,\eta) =: H_{s,t}(y,\eta)$ is the solution to*

$$\frac{dq}{dt} = \nabla_\xi \lambda(t,q,p), \quad q(s,s,y,\eta) = y\ ;\ \frac{dp}{dt} = -\nabla_x \lambda(t,q,p), \quad p(s,s,y,\eta) = \eta.$$

Then the symbols $f_{3,0}$ and $f_{3,1}$ satisfy

$$\left|\partial_x^\beta \partial_\xi^\alpha f_{3,0}(t, H_{s,t}(x,\xi))\right| \le C_{\alpha,\beta}\langle\xi\rangle^{-|\alpha|},$$
$$\left|\partial_x^\beta \partial_\xi^\alpha f_{3,1}(t, H_{s,t}(x,\xi))\right| \le C_{\alpha,\beta}\Big(\tfrac{1}{t}(\ln \tfrac{1}{t})^\gamma\Big)^2\langle\xi\rangle^{-1-|\alpha|}$$

for all $(t,x,\xi) \in [0,T_0] \times \mathbb{R}^{2N}$.

The statement of this lemma makes it clear that the following representation is reasonable for $Q_4 = Q_4(t,s)$:

$$Q_4(t,s)w(x) = \int_{\mathbb{R}^n} e^{i\,x\cdot\xi} q_4(t,s,x,\xi)\hat{w}(\xi)d\xi, \quad q_4(s,s,x,\xi) = I.$$

We determine the matrix amplitude q_4 by equivalence to a series, that is $q_4(t,s,x,\xi) \sim \sum_{j=0}^{\infty} q_{4,j}(t,s,x,\xi)$. After determination of $q_{4,j} = q_{4,j}(t,s,x,\xi)$ for $j \geq 0$ we obtain the statement of Lemma 31.

Parametrix to $D_t - \mathcal{D} + F_2$.

LEMMA 33. *The parametrix $E_3 = E_3(t,s)$ to the operator $D_t - \mathcal{D} + F_2$ can be written as $E_3(t,s) = K(t)E_2(t,s)Q_4(t,s)K^{\sharp}(s)$, where K and its parametrix $K^{\sharp}$ having symbols from $L^{\infty}([0,T_0], S^0_{1,0}(\mathbb{R}^n)) \cap C^{\infty}((0,T_0]^2, S^0_{1,0}(\mathbb{R}^n))$ are the elliptic pseudo-differential operators from the above transformation.*

REMARK 12. From Lemma 33 we conclude that the parametrix to $D_t - \mathcal{D} + F_2$ gives no loss of derivatives of the solution to (1). In the next point we will see that this loss comes from P_{∞}.

Parametrix to $D_t - \mathcal{D} + F_2 + P_{\infty}$.

LEMMA 34. *The parametrix $E_1 = E_1(t,s)$ to the operator $D_t - \mathcal{D} + F_2 + P_{\infty}$ can be written as $E_1(t,s) = E_3(t,s)Q_1(t,s)$, where $Q_1 = Q_1(t,s)$ is a matrix pseudo-differential operator with symbol from $L^{\infty}([0,T_0]^2, S^{K_0}_{1-\varepsilon,\varepsilon}(\mathbb{R}^n)) \cap W^{1,\infty}([0,T_0]^2, S^{K_0+1+\varepsilon}_{1-\varepsilon,\varepsilon}(\mathbb{R}^n))$ for every small $\varepsilon > 0$. Here the constant K_0 describes the loss of derivatives coming from the pseudo-differential zone $Z_{pd}(2N)$ and the oscillations subzone $Z_{osc}(2N)$.*

To prove this result we use the next observations and ideas:

- Egorov's theorem is not applicable because P_{∞} has no diagonal structure.
- We have to use the properties of P_{∞} after perfect diagonalization.
- The next result is a base to get a relation between the type of oscillations and the loss of derivatives.

LEMMA 35. *The Fourier integral operator $P_{\infty}(t)E_3^{\mp}(t,s)$ is a pseudo-differential operator with the representation*

$$P_{\infty}(t)E_3^{\mp}(t,s)w(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\tilde{r}^{\mp}(t,s,x,\xi)\hat{w}(\xi)d\xi,$$

where the symbol satisfies the estimates

$$\left|\partial_x^{\beta}\partial_{\xi}^{\alpha}\tilde{r}^{\mp}(t,s,x,\xi)\right| \leq \begin{cases} C_{\alpha\beta\varepsilon p}\left(\frac{1}{t}(\ln\frac{1}{t})^{\gamma}\right)^{p+1}\langle\xi\rangle^{-p+\varepsilon|\beta|-(1-\varepsilon)|\alpha|} \text{ in } Z_{reg}(2N), \\ C_{\alpha\beta\varepsilon}\left(1+\left(\frac{1}{t}(\ln\frac{1}{t})^{\gamma}\right)^2\langle\xi\rangle^{-1}\right)\langle\xi\rangle^{\varepsilon|\beta|-(1-\varepsilon)|\alpha|} \text{ in } Z_{osc}(2N), \\ C_{\alpha\beta\varepsilon}\langle\xi\rangle^{1+\varepsilon|\beta|-(1-\varepsilon)|\alpha|} \text{ in } Z_{pd}(2N), \end{cases}$$

for every $p \geq 0$, small $\varepsilon > 0$ and all $s \in [0,t]$.

Step 4. Conclusion

Using Lemma 34 and the backward transformation (from the steps of perfect diagonalization) we obtain the parametrix for $D_t - A$. The backward transformation doesn't bring an additional loss of derivatives. Therefore we can conclude the following result.

THEOREM 13. *Let us consider*

$$u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t,x)u_{x_k x_l} = 0, \quad u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),$$

where the coefficients satisfy the conditions (24) and (25). The data φ, ψ belong to H^{s+1}, H^s, respectively. Then the following energy inequality holds:

$$E(u)|_{H^{s-s_0}}(t) \le C(T)E(u)|_{H^s}(0) \text{ for all } t \in (0,T], \tag{48}$$

where
- $s_0 = 0$ *if* $\gamma = 0$,
- s_0 *is an arbitrary small positive constant if* $\gamma \in (0,1)$,
- s_0 *is a positive constant if* $\gamma = 1$,
- *there doesn't exist a positive constant* s_0 *satisfying (48) if* $\gamma > 1$, *that is, we have an infinite loss of derivatives.*

It seems to be remarkable that we can prove the same relation between types of oscillations and loss of derivatives as in Theorem 8.

7. Concluding remarks

Let us mention further results which are obtained for model problems with non-Lipschitz behaviour and more problems which could be of interest.

REMARK 13. *Lower regularity with respect to x.* The results and the approach from [15] motivate the study of the question of how to weaken the regularity with respect to x (compare with [9]). From this paper we understand to which class the remainder should belong after diagonalization. Thus pseudo-differential operators with symbols of finite smoothness or maybe paradifferential operators should be used.

REMARK 14. *Quasi-linear models.* Quasi-linear models with behaviour of suitable derivatives as $O(\frac{1}{t})$ were studied in [3] and [18]. Here the log-effect from (5) could not be observed.

REMARK 15. *Applications to Kirchhoff type equations.* A nice application of non-Lipschitz theory with behaviour $a'(t) = O((T-t)^{-1})$ for $t \to T-0$ to Kirchhoff equations was described in [16]. The assumed regularity of data could be weakened in [13] by proving that these very slow oscillations (in the language of Definition 2) produce no loss of derivatives (see Theorem 8).

REMARK 16. *p-Evolution equations.* The paper [1] is devoted to the Cauchy problem for p-evolution equations with LogLip coefficients. The paper [4] is devoted among other things to p-evolution equations of higher order with non-Lipschitz coefficients. Concerning our starting model this means p-evolution equations of second order with respect to t with coefficients behaving like $|ta'(t)| \leq C$ on $(0, T]$. An interesting question is to find p-evolution models with log-effect from (5).

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Microlocal Analysis

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RIEMANN–HILBERT PROBLEM AND SOLVABILITY OF DIFFERENTIAL EQUATIONS

Abstract. In this paper Riemann–Hilbert problem is applied to the solvability of a mixed type Monge-Ampère equation and the index formula of ordinary differential equations. Blowing up onto the torus turns mixed type equations into elliptic equations, to which R-H problem is applied.

1. Introduction

This paper is concerned with the Riemann–Hilbert problem and the (unique) solvability of differential equations. The Riemann–Hilbert problem has many applications in mathematics and physics. In this paper we are interested in the solvability of a mixed type Monge-Ampère equation, a homology equation appearing in a normal form theory of singular vector fields and the index formula of ordinary differential equations. These equations have a singularity at some point, say at the origin. We handle these singular nature of the equations by a kind of blowing up and the Riemann–Hilbert problem.

Our idea is as follows. When we want to solve these degenerate mixed type equations in a class of analytic functions, we transform the equation onto the torus embedded at the origin. This is done by a change of variables similar to a blowing up procedure. Although we transform the local problem for a mixed type equation to a global one on tori, it turns out that, in many cases the transformed equations are elliptic on the torus. This enables us to apply a Riemann–Hilbert problem with respect to tori. Once we can solve the lifted problems we extend the solution on the torus inside the torus analytically by a harmonic (analytic) extension. The extended function is holomorphic in a domain whose Silov boundary is a torus. Moreover, by the maximal principle, the extended function is a solution of a given nonlinear equation since it satisfies the same equation on its Silov boundary, i.e., on tori. The uniqueness on the boundary and the maximal principle also implies the uniqueness of the solution.

This paper is organized as follows. In Section 2 we give examples and a general class of mixed type equations for which the blowing up procedure turns the mixed type equations into elliptic equations on tori. In Section 3 we discuss the relation of the blowing up procedure with a resolution of singularities. In Sections 4 and 5 we study the solvability of ordinary differential equations via blowing up procedure and the Riemann-Hilbert problem. In Section 6 we study the index formula of a system of

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singular ordinary differential operators from the viewpoint of the blowing up procedure and the Riemann-Hilbert problem. In Sections 7 and 8 we apply the R–H problem with respect to $\mathbb{T}^2$ to a construction of a parametrix. In Section 9 we apply the results of Sections 7 and 8 to the unique solvability of a mixed type Monge–Ampère equation of two variables. In Section 10 we study the solvability of a mixed type Monge-Ampère equation of general independent variables. In Section 11 we apply our argument to a system of nonlinear singular partial differential equations arising from the normal form theory of a singular vector field. In Section 12 we extend our argument to the solvability of equations containing a large parameter.

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2. Blowing up and mixed type operators

Let us consider the following Monge-Ampère equation

$$M(u) := \det(u_{x_i x_j}) = f(x),\ u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad i, j = 1, \ldots, n,$$

where $x = (x_1, \ldots, x_n) \in \Omega \subset \mathbb{R}^n$ (resp. in $\mathbb{C}^n$) for some domain Ω. Let $u_0(x)$ be a smooth (resp. holomorphic) function in Ω, and set

$$f_0(x) = \det((u_0)_{x_i x_j}).$$

Then $u_0(x)$ is a solution of the above equation with $f = f_0$. (f_0 is a so-called Gauss curvature of u_0). Consider a solution $u = u_0 + v$ which is a perturbation of $u_0(x)$, namely

$$(MA) \qquad \det(v_{x_i x_j} + (u_0)_{x_i x_j}) = f_0(x) + g(x) \quad \text{in } \Omega,$$

where g is smooth in Ω(resp. analytic in Ω).

Define

$$W_R(D_R) := \{u = \sum_\eta u_\eta x^\eta;\ \|u\|_R := \sum_\eta |u_\eta| R^\eta < \infty\}.$$

We want to solve (MA) for $g \in W_R(D_R)$.

We shall lift (MA) onto the torus $\mathbb{T}^n$. The function space $W_R(D_R)$ is transformed to $W_R(\mathbb{T}^n)$,

$$W_R(\mathbb{T}^n) = \{u = \sum_\eta u_\eta R^\eta e^{i\eta\theta};\ \|u\|_R := \sum_\eta |u_\eta| R^\eta < \infty\},$$

where $R = (R_1, \ldots, R_n)$, $R^\eta = R_1^{\eta_1} \cdots R_n^{\eta_n}$. In order to calculate the operator on the torus we make the substitution

$$\partial_{x_j} \mapsto \frac{1}{R_j e^{i\theta_j}} \frac{1}{i} \frac{\partial}{\partial \theta_j} \equiv \frac{1}{R_j e^{i\theta_j}} D_j, \quad x_j \mapsto R_j e^{i\theta_j} \equiv z_j.$$

The reduced operator on the torus is given by

$$\det\left(z_j^{-1}z_k^{-1}D_jD_kv+(u_0)_{x_jx_k}(z)\right)=f_0+g.$$

REMARK 1. The above transformation onto the torus is related with a Cauchy-Riemann equation as follows. For the sake of simplicity we consider the one dimensional case. The same things hold in the general case. We recall the following formula, for $t=re^{i\theta}$

$$t\partial=t\frac{\partial}{\partial t}=\frac{1}{2}\left(r\frac{\partial}{\partial r}-i\frac{\partial}{\partial\theta}\right),\quad \overline{t\partial}=\bar{t}\overline{\frac{\partial}{\partial t}}=\frac{1}{2}\left(r\frac{\partial}{\partial r}+i\frac{\partial}{\partial\theta}\right),$$

where $\overline{\partial}$ be a Cauchy-Riemann operator. Assume that $\overline{\partial}u=0$. Then, by the above formula we obtain

$$r\frac{\partial}{\partial r}u=-i\frac{\partial}{\partial\theta}u,\quad t\partial_tu=-i\frac{\partial}{\partial\theta}u=D_\theta u.$$

Note that the second relation is the one which we used in the above.

REMARK 2. (*Relation to Langer's transformation*) The transformation used in the above is essentially $x_j=e^{i\theta_j}$. Similar transformation $x=e^y$ was used by Langer in the study of asymptotic analysis of Schrödinger operator for a potential with pole of degree 2 at $x=0$

$$-\frac{d^2}{dx^2}+\lambda^2\left(V(x)+\frac{k(k+1)}{x^2\lambda^2}\right)u=Eu,$$

where E is an energy and $V(x)$ is a regular function.

Some examples

Let $n=2$, and set $x_1=x$, $x_2=y$. Consider the Monge-Ampère equation

$$(MA)\qquad M(u)+c(x,y)u_{xy}=f_0(x,y)+g(x,y),$$

where

$$M(u)=u_{xx}u_{yy}-u_{xy}^2,\quad f_0=M(u_0)+c(x,y)(u_0)_{xy},$$

with $c(x,y)$ and u_0 being analytic in x and y. Let $Pv:=M'_{u_0}v=\frac{d}{d\varepsilon}M(u_0+\varepsilon v)|_{\varepsilon=0}$ be a linearization of $M(u)$ at $u=u_0$. By simple calculations we obtain

$$M'_{u_0}v:=(u_0)_{xx}\partial_y^2v+(u_0)_{yy}\partial_x^2v-2(u_0)_{xy}\partial_x\partial_yv.$$

EXAMPLE 1. Consider the equation (MA) for

$$u_0=x^2y^2,\quad c(x,y)=kxy\quad k\in\mathbb{R}.$$

We have $f_0 = 4(k-3)x^2y^2$. The linearized operator is given by

$$P = 2x^2\partial_x^2 + 2y^2\partial_y^2 + (k-8)xy\partial_x\partial_y, \quad \partial_x = \partial/\partial x, \partial_y = \partial/\partial y.$$

The characteristic polynomial is given by (with the standard notation) $-2x^2\xi_1^2 - 2y^2\xi_2^2 - (k-8)xy\xi_1\xi_2$. The discriminant is given by

$$D \equiv (k-8)^2x^2y^2 - 16x^2y^2 = (k-4)(k-12)x^2y^2.$$

It follows that (MA) is (degenerate) hyperbolic if and only if $k < 4$ or $k > 12$, while (MA) is (degenerate) elliptic if and only if $4 < k < 12$. In either case, (MA) degenerates on the lines $xy = 0$, namely the characteristic polynomial vanishes.

By lifting P onto the torus we obtain

$$2D_1(D_1 - 1) + 2D_2(D_2 - 1) + (k-8)D_1D_2.$$

Here, for the sake of simplicity we assume $R_j = 1$. The symbol is given by

$$\sigma(\eta) = 2(\eta_1(\eta_1 - 1) + \eta_2(\eta_2 - 1)) + (k-8)\eta_1\eta_2,$$

where η_j is the covariable of θ_j. Consider now the homogeneous part of degree 2. If this does not vanish on $|\eta| = 1$ we obtain the following

$$2 + (k-8)\eta_1\eta_2 \neq 0 \ \text{ for all } \eta \in \mathbb{R}^2, \ |\eta| = 1.$$

The condition is clearly satisfied if $k = 8$. If $k \neq 8$, noting that $-1/2 \leq \eta_1\eta_2 \leq 1/2$ we obtain $-1/2 \leq -2/(k-8) \leq 1/2$. By simple calculation we obtain $4 < k < 12$. Namely, if the given operator is degenerate elliptic the operator on the torus is an elliptic operator.

EXAMPLE 2. Consider (MA) under the following condition

$$u_0 = x^4 + kx^2y^2 + y^4, \ \ k \in R, \ c \equiv 0.$$

Then we have

$$f_0 = M(u_0) = 12(2kx^4 + 2ky^4 + (12 - k^2)x^2y^2).$$

The linearized operator is given by

$$P = 12y^2\partial_x^2 + 12x^2\partial_y^2 + 2k(x^2\partial_x^2 + y^2\partial_y^2) - 8xy\partial_x\partial_y.$$

The characteristic polynomial is given by

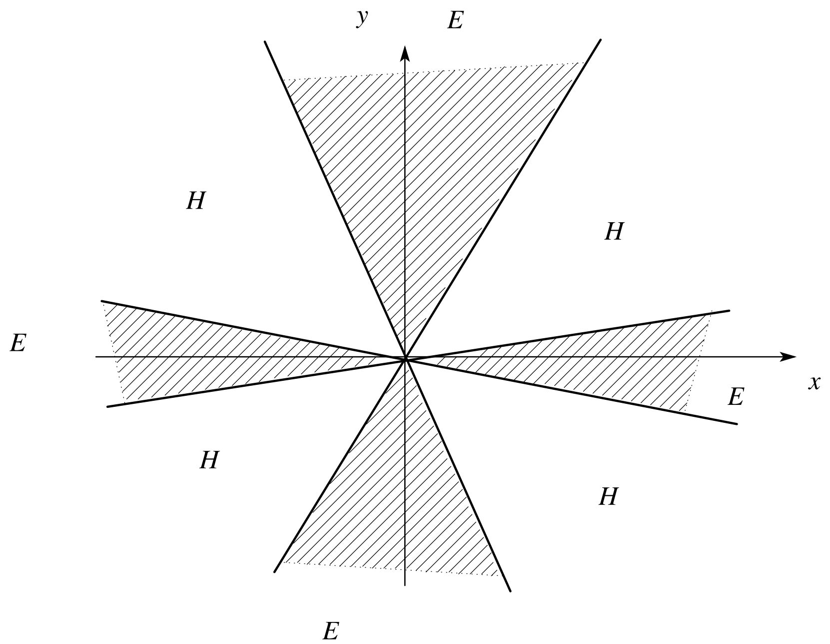
$$-12y^2\xi_1^2 - 12x^2\xi_2^2 - 2k(x^2\xi_1^2 + y^2\xi_2^2) + 8xy\xi_1\xi_2.$$

Since the discriminant is equal to $-f_0$, we study the signature of f_0. The following facts are easy to verify :

$$f_0/12 = 2k\left(x^2 + \frac{12-k^2}{4k}y^2\right)^2 - \frac{D}{8k}y^4, \quad D = (k^2 - 12)^2 - 16k^2.$$

It follows that $D < 0$ iff $-6 < k < -2$ or $2 < k < 6$, and $D > 0$ iff $k < -6$, $k > 6$ or $-2 < k < 2$. Hence, by the signature of f_0 we obtain:
if $k < -2$ it is hyperbolic and degenerates at the origin,
if $k = -2$ it is hyperbolic and degenerates on the line $x = \pm y$,
if $-2 < k < 0$ it is of mixed type,
if $k = 0$ it is elliptic and degenerates on the lines $x = 0$ and $y = 0$,
if $0 < k < 6$ it is elliptic and degenerates at the origin,
if $k = 6$ it is elliptic and degenerates on the lines $x = \pm y$,
if $k > 6$ it is of mixed type.

More precisely, in the mixed case the set $\{f_0 = 0\} \subset R^2$ consists of four lines intersecting at the origin. The equation changes its type from elliptic to hyperbolic or vice versa when crossed one of these lines. The equation degenerates on this line. (See the following figure of the case $k > 6$, where H and E denote the hyerbolic and elliptic region, respectively.)



In the case $-2 < k < 0$, a similar structure appears. The elliptic and hyperbolic regions are interchanged.

The operator on the torus is given by

$$\begin{aligned}\hat{P} \;&:=\; 12(e^{2i\theta_2-2i\theta_1}D_1(D_1-1)+e^{2i\theta_1-2i\theta_2}D_2(D_2-1))\\ &+\; 2k(D_1(D_1-1)+D_2(D_2-1))-8D_1D_2.\end{aligned}$$

Here we assume $R_j = 1$ as before. Setting $z_j = e^{i\theta_j}$, the principal symbol is given by

$$\sigma(z, \eta) := 2k(\eta_1^2 + \eta_2^2) - 8\eta_1\eta_2 + 12(z_1^{-2}z_2^2\eta_1^2 + z_1^2 z_2^{-2}\eta_2^2).$$

Hence the condition $\sigma(z, \eta) \neq 0$ on $\mathbb{T}^2$ reads:

$$k - 4\eta_1\eta_2 + 6(\eta_1^2 t^2 + \eta_2^2 t^{-2}) \neq 0 \qquad \forall t \in \mathbb{C},\ |t| = 1\ \forall \eta \in \mathbb{R}^2,\ |\eta| = 1.$$

If $\eta_1 = \eta_2$ we have $\eta_1 = \eta_2 = \pm 1/\sqrt{2}$ in view of $|\eta| = 1$. By substituting this into the above equation we have, for $t = e^{i\theta}$

$$k - 2 + 6\cos 2\theta \neq 0 \quad 0 \leq \theta \leq 2\pi.$$

It follows that $k \notin [-4, 8]$. Similarly, if $\eta_1 = -\eta_2$ it follows that $k \notin [-8, 4]$. In case $\eta_1 \neq \pm\eta_2$ we have

$$2i\ Im\,(\eta_1^2 t^2 + \eta_2^2 t^{-2}) = (\eta_1^2 - \eta_2^2)(t^2 - t^{-2}) \neq 0, \quad \text{if } t^2 \neq \pm 1.$$

Hence the imaginary part of $k - 4\eta_1\eta_2 + 6(\eta_1^2 t^2 + \eta_2^2 t^{-2})$ does not vanish.

If $t^2 = \pm 1$, our condition can be written in $k \neq 4\eta_1\eta_2 \pm 6$. Because $-1/2 \leq \eta_1\eta_2 \leq 1/2$ it follows that $k \notin [-8, -4]$ and $k \notin [4, 8]$. Summing up the above we obtain $k < -8$ or $k > 8$. Under the condition the operator on the torus is elliptic. Especially, we remark that the same property holds in the mixed case $k > 8$.

We will extend these examples to more general equations. Because the problem is an essentially linear problem we study a linear equation. We consider a Grushin type operator

$$P = \sum_{|\alpha| \geq m, |\beta| \leq m} a_{\alpha\beta} x^\alpha \left(\frac{\partial}{\partial x}\right)^\beta,$$

where $a_{\alpha\beta} \in \mathbb{R}$ and $m \geq 1$. For the sake of simplicity we assume $R_j = 1$ $(j = 1, \ldots, n)$. The principal symbol of the lifted operator of P on $\mathbb{T}^n$ is given by, with $e^{i\theta} = (e^{i\theta_1}, \ldots, e^{i\theta_n}) \in \mathbb{T}^n$,

$$p(\theta, \xi) = \sum_{|\alpha| \geq m, |\beta| = m} a_{\alpha\beta} e^{i(\alpha - \beta)\theta} \xi^\beta.$$

Let $p_0(\xi)$ be the averaging of $p(\theta, \xi)$ on $\mathbb{T}^n$

$$p_0(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} p(\theta, \xi) d\theta = \sum_\alpha a_{\alpha\alpha} \xi^\alpha,$$

and define

$$Q(\theta, \xi) = p(\theta, \xi) - p_0(\xi).$$

We assume that $p_0(\xi)$ is elliptic: there exist $C > 0$ and $N > 0$ such that

$$|p_0(\xi)| \geq C|\xi|^m, \text{ for all } \xi \in \mathbb{R}^n, |\xi| \geq N.$$

We define the norm of $\|Q\|$ as the sum of absolute values of all Fourier coefficients of Q. We note that if $\|Q\|$ is sufficiently small compared with C the lifted operator P on the torus is elliptic.

We will show that P may be of mixed type in some neighborhood of the origin for any $C > 0$. We assume that P is hyperbolic with respect to x_1 at the point $x = r(1, 0, \ldots, 0)$ for some small $r > 0$ chosen later. We note that this condition is consistent with the ellipticity assumption. Indeed, in P all terms satisfying $\alpha = \beta$ vanish at $x = r(1, 0, \ldots, 0)$ except for the term $r^m \partial_{x_1}^m$. On the other hand there appears the term

$$\sum_{|\beta|\le m, \alpha=(\alpha_1,0,\ldots,0), \alpha_1>m} a_{\alpha\beta} r^{\alpha_1} \partial_x^\beta$$

from P corresponding to $\alpha \neq \beta$. We note that $\partial_{x_1}^m$ does not appear in the sum. Therefore, by an appropriate choice of the sign of coefficients in the averaging part the hyperbolicity condition is satisfied. This is possible for any large C. Same argument is valid if we consider near the other cordinate axis x_j.

Next we study the type of P near $x = r(1, \ldots, 1)$. We can write the principal symbol of $i^{-m}P$ as follows.

$$\begin{aligned} \sum_{|\alpha|\ge m, |\beta|=m} a_{\alpha\beta} r^{|\alpha|} \xi^\beta &= r^m \sum_{|\alpha|=m, |\beta|=m} a_{\alpha\beta} \xi^\beta + \sum_{|\alpha|>m, |\beta|=m} a_{\alpha\beta} r^{|\alpha|} \xi^\beta \\ &= r^m \left(\sum_{|\alpha|=m} a_{\alpha\alpha} \xi^\alpha + \sum_{|\alpha|=m, \alpha\neq\beta} a_{\alpha\beta} \xi^\beta \right) + \sum_{|\alpha|>m, |\beta|=m} a_{\alpha\beta} r^{|\alpha|} \xi^\beta. \end{aligned}$$

The averaging part in the bracket in the right-hand side dominates the second term if $|a_{\alpha\beta}|$ is sufficiently small for $\alpha \neq \beta$, namely if $\|Q\|$ is sufficiently small. The terms corresponding to $|\alpha| > m$, $|\beta| = m$ can be absorbed to the first term if $r > 0$ is sufficiently small. Therefore we see that P is elliptic near $x = r(1, \ldots, 1)$ for sufficiently small $r > 0$. Hence P is of mixed type in some neighborhood of the origin, while its blow up to the torus is elliptic. Summing up the above we have

THEOREM 1. *Under the above assumptions, if $\|Q\|$ is sufficiently small and if P is hyperbolic with respect to x_1 at the point $x = r(1, 0, \ldots, 0)$ for small $r > 0$ the operator P is of mixed type near the origin, while its blowing up to the torus is elliptic.*

In the following sections we will construct a parametrix for such operators.

3. Relation to a resolution

We will show that the transformation in the previous section can be introduced directly via a resolution of singularities as follows. First we give a definition of a resolution in a special case.

Let $\mathbb{C}\mathbf{P}^1$ be a complex projective space and let $p : \mathbb{C}^2 \setminus O \to \mathbb{C}\mathbf{P}^1$ be a fibration of a projective space. Denote the graph of p by $\Gamma \subset (\mathbb{C}^2 \setminus O) \times \mathbb{C}\mathbf{P}^1$. The set Γ can be regarded as a smooth surface in $\mathbb{C}^2 \times \mathbb{C}\mathbf{P}^1$. The projection $\pi_1 : \mathbb{C}^2 \times \mathbb{C}\mathbf{P}^1 \to \mathbb{C}^2$ maps Γ onto $\mathbb{C}^2 \setminus O$ homeomorphically. The closure of the graph Γ of the map p in $\mathbb{C}^2 \times \mathbb{C}\mathbf{P}^1$ is the surface $\Gamma_1 = \Gamma \cup (O \times \mathbb{C}\mathbf{P}^1)$.

Indeed, let (x, y) be the coordinate in $\mathbb{C}^2$, and let $u = y/x$ be the local coordinate of $\mathbb{C}\mathbf{P}^1$. Then (x, y, u) is a local coordinate of $\mathbb{C}^2 \times \mathbb{C}\mathbf{P}^1$. Γ is given by $y = ux$, $x \neq 0$, and Γ_1 is given by $y = ux$. This is obtained by adding $O \times \mathbb{C}\mathbf{P}^1$ to Γ.

We can show the smoothness of Γ_1 by considering the second coordinate $(x, y, v), x = vy$. The projection $\pi_2 : \mathbb{C}^2 \times \mathbb{C}\mathbf{P}^1 \to \mathbb{C}\mathbf{P}^1$ foliate Γ_1 with a family of lines.

DEFINITION 1. *The procedure from $\mathbb{C}^2$ to Γ_1 is called the blowing up to $O \times \mathbb{C}\mathbf{P}^1$.*

EXAMPLE 3. Consider three lines intersecting at the origin O, $y = \alpha x$, $y = \beta x$, $y = \gamma x$. By $y = ux$, these lines are given by $x = 0, u = \alpha, u = \beta, u = \gamma$. In Γ_1 they intersect with $\mathbb{C}\mathbf{P}^1$ at different points.

We cosider the case $y = x^2$, $y = 0$. By blowing up we see that $u = x$, $u = 0$, $x = 0$ on Γ_1. Indeed, $y = 0$ is $0 = ux$, and $y = x^2$ is $ux = x^2$. Hence we are lead to the above case.

In the case $x^2 = y^3$, by setting $x = vy$ we have $v^2 = y$ and $y = 0$. Hence we are reduced the above case.

Grushin type operators

Let us consider a Grushin type operator.

$$P = \sum_{|\alpha|=|\beta|} a_{\alpha\beta} y^\alpha \left(\frac{\partial}{\partial y}\right)^\beta .$$

For the sake of simplicity we assume that $a_{\alpha\beta}$ are constants. We make the blowing up

$$y_j = z_j t, \quad j = 1, \dots, n$$

where t is a variable which tends to zero and z_j $(j = 1, 2, \dots, n)$ are variables which remain non zero when $t \to 0$. By introducing these variables we study the properties of P.

EXAMPLE 4. In the case of an Euler operator $\sum_{j=1}^n y_j \frac{\partial}{\partial y_j}$, we obtain

$$\sum_{j=1}^n y_j \frac{\partial}{\partial y_j} = t\frac{\partial}{\partial t} = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}.$$

If we introduce $z_j = \exp(i\theta_j)$, the right hand side is elliptic on a Hardy space on the torus. On the other hand in the radial direction t, it behaves like a Fuchsian operator.

If we assume that t is a parameter we have

$$\frac{\partial}{\partial z_j} = \frac{\partial y_j}{\partial z_j}\frac{\partial}{\partial y_j} = t\frac{\partial}{\partial y_j}.$$

Noting that $|\alpha| = |\beta|$ we obtain

$$y^\alpha \partial_y^\beta = z^\alpha t^{|\alpha|} t^{-|\beta|} \partial_z^\beta = z^\alpha \partial_z^\beta.$$

Hence P is transformed to the following operator on the torus

$$\hat{P} = \sum_{|\alpha|=|\beta|} a_{\alpha\beta} z^\alpha \left(\frac{\partial}{\partial z}\right)^\beta.$$

This is identical with the operator introduced in the previous section if we set $z_j = e^{i\theta_j}$.

4. Ordinary differential operators

Consider the following ordinary differential operator

$$p(t, \partial_t) = \sum_{k=0}^{m} a_k(t) \partial_t^k,$$

where $\partial_t = \partial/\partial t$ and $a_k(t)$ is holomorphic in $\Omega \subset \mathbb{C}$. For the sake of simplicity, we assume $\Omega = \{|t| < r\}$, where $(r > 0)$ is a small constant. We consider the following map

$$p : \mathcal{O}(\Omega) \mapsto \mathcal{O}(\Omega).$$

The operator p is singular at $t = 0$. Therefore, instead of considering at the origin directly we lift p onto the torus $\mathbb{T} = \{|t| = r\}$. In the following we assume that $r = 1$ for the sake of simplicity. The case $r \neq 1$ can be treated similarly if we consider the weighted space.

Let $L^2(\mathbb{T})$ be the set of square integrable functions on the torus, and define the Hardy space $H^2(\mathbb{T})$ by

$$H^2(\mathbb{T}) := \{u = \sum_{-\infty}^{\infty} u_n e^{in\theta} \in L^2; u_n = 0 \text{ for } n < 0\}.$$

$H^2(\mathbb{T})$ is closed subspace of $L^2(\mathbb{T})$. Let π be the projection on $L^2(\mathbb{T})$ to $H^2(\mathbb{T})$. Namely,

$$\pi\left(\sum_{-\infty}^{\infty} u_n e^{in\theta}\right) = \sum_{0}^{\infty} u_n e^{in\theta}.$$

In this situation, the correspondence between functions on the torus and holomorphic functions in the disk is given by

$$\mathcal{O}(\Omega) \ni \sum_{0}^{\infty} u_n z^n \longleftrightarrow \sum_{0}^{\infty} u_n e^{in\theta} \in H^2(\mathbb{T}).$$

By the relation $t\partial_t \mapsto D_\theta$ the lifted operator on the torus is given by

$$\hat{p} = \sum_k a_k(e^{i\theta})e^{-ik\theta} D_\theta(D_\theta - 1)\cdots(D_\theta - k + 1),$$

where we used $t^k\partial_t^k = t\partial_t(t\partial_t - 1)\cdots(t\partial_t - k + 1)$. By definition we can easily see that $\pi\hat{p} = \hat{p}$.

For a given equation $Pu = f$ in some neighborhood of the origin we consider $\hat{p}\hat{u} = \hat{f}$ on the torus, where $\hat{f}(\theta) = f(e^{i\theta})$. If we obtain a solution $\hat{u} = \sum_0^\infty u_n e^{in\theta} \in H^2(\mathbb{T})$ of $\hat{p}\hat{u} = \hat{f}$, $u := \sum_0^\infty u_n t^n$ is a holomorphic extension of $\hat{u}$ into $|t| \le 1$. The function $Pu - f$ is holomorphic in the disk $|t| \le 1$, and vanishes on its boundary since $\hat{p}\hat{u} = \hat{f}$. Maximal principle implies that $Pu = f$ in the disk, i.e, u is a solution of a given equation. Clearly, the maximal principle also implies that if the solution on the torus is unique, the analytic solution inside is also unique. Hence it is sufficient to study the solvability of the equation on the torus.

Reduced equation on the torus

Define $\langle D_\theta \rangle$ by the following

$$\langle D_\theta \rangle u := \sum_n u_n \langle n \rangle e^{in\theta}, \ \langle n \rangle = (1 + n^2)^{1/2}.$$

This operator also operates on the set of holomorphic functions in the following way

$$\langle t\partial_t \rangle u := (1 + (t\partial/\partial t)^2)^{1/2} u = \sum u_n \langle n \rangle z^n.$$

We can easily see that

$$D_\theta(D_\theta - 1)\cdots(D_\theta - k + 1)\langle D_\theta \rangle^{-k} = Id + K,$$

where K is a compact operator on H^2.

It follows that since $\langle D_\theta \rangle^{-m}$ is an invertible operator we may consider $\hat{p}\langle D_\theta \rangle^{-m}$ instead of $\hat{p}$. Note that $\hat{p}\langle D_\theta \rangle^{-m} = \pi\hat{p}\langle D_\theta \rangle^{-m}$, and the principal part of $\hat{p}\langle D_\theta \rangle^{-m}$ is $a_m(e^{i\theta})e^{-im\theta}$. Hence, modulo compact operators we are lead to the following operator

$$(*) \qquad \pi a_m(e^{i\theta})e^{-im\theta} : H^2 \mapsto H^2.$$

Indeed, the part with order $< m$ is a compact operator if $\langle D_\theta \rangle^{-m}$ is multiplied.

The last operator contains no differentiation, and the coefficients are smooth. It should be noted that although $a_m(t)$ vanishes at $t = 0$, $a_m(e^{i\theta})$ does not vanish on the torus.

DEFINITION 2. *We call the operator* $(*)$ *on* $H^2(\mathbb{T})$ *a Toeplitz operator. The function* $a_m(e^{i\theta})$ *is called the symbol of a Toeplitz operator.*

5. Riemann-Hilbert problem and solvability

DEFINITION 3. *A rational function $p(z) := a(z)z^{-m}$ is said to be Riemann-Hilbert factorizable with respect to $|z| = 1$ if the following factorization*

$$p(z) = p_-(z)p_+(z),$$

holds, where $p_+(z)$, being holomorphic in $|z| < 1$ and continuous up to the boundary, does not vanish in $|z| \leq 1$, and $p_-(z)$, being holomorphic in $|z| > 1$ and continuous up to the boundary, does not vanish in $|z| \geq 1$.

The factorizability is equivalent to saying that the R–H problem for the jump function p and the circle has a solution.

EXAMPLE 5. We consider $p(z) := a(z)z^{-m}$ $(a(0) \neq 0)$ $(m \geq 1)$. Let $a(z)$ be a polynomial of order $m + n$ $(n \geq 1)$. Then we have

$$\begin{aligned} p(z) &= c(z-\lambda_1)\cdots(z-\lambda_m)(z-\lambda_{m+1})\cdots(z-\lambda_{m+n})z^{-m} \\ &= c(1-\frac{\lambda_1}{z})\cdots(1-\frac{\lambda_m}{z})(z-\lambda_{m+1})\cdots(z-\lambda_{m+n}), \end{aligned}$$

where $\lambda_j \in \mathbb{C}$. We can easily see that p is Riemann-Hilbert factorizable with respect to the unit circle if and only if

$$(RH) \qquad |\lambda_1| \leq \cdots \leq |\lambda_m| < 1 < |\lambda_{m+1}| \leq \cdots \leq |\lambda_{m+n}|.$$

THEOREM 2. *Suppose that (RH) is satisfied. Then the kernel and the cokernel of the map $(*)$ vanishes.*

Proof. We consider the kernel of $(*)$. By definition, $\pi pu = 0$ is equivalent to

$$p(e^{i\theta})u(e^{i\theta}) = g(e^{i\theta}),$$

where g consists of negative powers of $e^{i\theta}$. If $|\lambda_j| < 1$ the series $(1 - \lambda_j e^{-i\theta})^{-1}$ consists of only negative powers of $e^{i\theta}$. Hence, if $(1 - \lambda_j e^{-i\theta})U(e^{i\theta}) = F(e^{i\theta})$ for some F consisting of negative powers it follows that $U(e^{i\theta}) = (1 - \lambda_j e^{-i\theta})^{-1}F(e^{i\theta})$ consists of negative powers. By repeating this argument we see that

$$(z-\lambda_{m+1})\cdots(z-\lambda_{m+n})u(z), \quad z = e^{i\theta}$$

consists of only negative powers. On the other hand, since this is a polynomial of z we obtain $u = 0$.

Next we study the cokernel. Let $f \in H^2(\mathbb{T})$ be given. For the sake of simplicity we want to solve

$$\left(1-\lambda_1 e^{-i\theta}\right)\left(e^{i\theta}-\lambda_2\right)u(e^{i\theta}) \equiv f(e^{i\theta}) \quad \text{modulo negative powers,}$$

where $|\lambda_1| < 1 < |\lambda_2|$. Hence we have

$$\left(e^{i\theta} - \lambda_2\right) u(e^{i\theta}) \equiv \left(1 - \lambda_1 e^{-i\theta}\right)^{-1} f = f_+ + f_- \equiv f_+$$

modulo negative powers. Here f_+ (resp. f_-) consists of Fourier coefficients of non-negative (resp. negative) part. Hence, we have

$$\left(e^{i\theta} - \lambda_2\right) u(e^{i\theta}) = f_+.$$

The solution is given by $u(e^{i\theta}) = (e^{i\theta} - \lambda_2)^{-1} f_+$. Hence the cokernel vanishes. This ends the proof.

$\square$

6. Index formula of an ordinary differential operator

We will give an elementary proof of an index formula. (Cf. Malgrange, Komatsu, Ramis). Let $\Omega \subset C$ be a bounded domain satisfying the following condition.

(A.1) There exists a conformal map $\psi : D_w = \{|z| < w\} \mapsto \Omega$ such that ψ can be extented in some neighborhood of $\overline{D_w} = \{|z| \leq w\}$ holomorphically.

Let $w > 0$, $\mu \geq 0$, and define

$$G_w(\mu) = \{u = \sum_n u_n x^n;\ \|u\|^2 := \sum_n (|u_n| \frac{w^n n!}{(n-\mu)!})^2 < \infty\},$$

where $(n-\mu)! = 1$ if $n - \mu \leq 0$. Clearly, $G_w(\mu)$ is a Hilbert space. Define $\mathcal{A}_w(\mu)$ as the totality of holomorphic functions $u(x)$ on Ω such that $u(\psi(z)) \in G_w(\mu)$.

Consider an $N \times N$ ($N \geq 1$) matrix-valued differential operator

$$P(x, \partial_x) = (p_{ij}(x, \partial_x)),$$

where p_{ij} is holomorphic ordinary differential operator on $\overline{\Omega}$. For simplicity, we assume that there exist real numbers ν_i, μ_j $(i, j = 1, \ldots, N)$ such that

$$ord\ p_{ij} \leq \mu_j - \nu_i, \quad ord\ p_{ii} = \mu_i - \nu_i.$$

Hence

$$P(x, \partial_x) : \prod_{j=1}^{N} \mathcal{A}_w(-\mu_j) \longrightarrow \prod_{j=1}^{N} \mathcal{A}_w(-\nu_j). \tag{1}$$

If we write

$$p_{ij}(x, \partial_x) = \sum_{k=0}^{\mu_j - \nu_i} a_k(x) \partial_x^k, \quad a_k(x) \in \mathcal{O}(\overline{\Omega})$$

we obtain, by the substitution $x = \psi(z)$

$$\tilde{p}_{ij}(z, \partial_z) = \sum_{k=\mu_j-\nu_i} a_k(\psi(z))\psi'(z)^{-k}\partial_z^k + \cdots .$$

Here the dots denotes terms of order $< \mu_j - \nu_i$, which are compact operators.

Define a Toeplitz symbol $Q^\Omega(z)$ by $Q^\Omega(z) := (q_{ij}^\Omega(z))$. Here

$$q_{ij}^\Omega(z) = a_{\mu_j-\nu_i}(\psi(z))(z\psi'(z))^{\nu_i-\mu_j}. \tag{2}$$

Then we have

THEOREM 3. *Suppose* (A.1). *Then the map* (1) *is a Fredholm operator if and only if*

$$\det Q^\Omega(z) \neq 0 \qquad \text{for } \forall z \in \mathbb{C}, |z| = w. \tag{3}$$

If (3) *holds the Fredholm index of (1),* $\chi(:= dim_{\mathbb{C}}\,\mathrm{Ker}\,P - codim_{\mathbb{C}}\mathrm{Im}\,P)$ *is given by the following formula*

$$-\chi = \frac{1}{2\pi}\oint_{|z|=w} d(\log\det Q^\Omega(z)), \tag{4}$$

where the integral is taken in counterclockwise direction.

Proof. Suppose (3). We want to show the Fredholmness of (1). For the sake of simplicity, we suppose that $\mu_j - \nu_i = m$, i.e., $ord\ p_{ij} = m$. If we lift P onto the torus and we multiply the lifted operator on torus with $\langle D_\theta\rangle^{-m}$ we obtain an operator πQ^Ω on H^2 modulo compact operators. It is easy to show that πQ^Ω on H^2 is a Fredholm operator. (cf. [3]). Because the difference of these operators are compact operators the lifted operator is a Fredholm operator.

In order to see the Fredholmness of (1) we note that the kernel of the operator on the boundary coincides with that of the operator inside (under trivial analytic extension) because of a maximal principle. The same property holds for a cokernel. Therefore the Fredholmness of the lifted operator implies the Fredholmness of (1).

Conversely, assume that (1) is a Fredholm operator. We want to show (3). By the argument in the above we may assume that the operator πQ^Ω on H^2 is a Fredholm operator. For the sake of simplicity, we prove in the case $N = 1$, a single case.

We denote πQ^Ω by T. Let K be a finite dimensional projection onto $Ker\ T$. Then there exists a constant $c > 0$ such that

$$\|Tf\| + \|Kf\| \geq c\|f\|, \qquad \forall f \in H^2.$$

It follows that

$$\|\pi Q^\Omega \pi g\| + \|\pi K \pi g\| + c\|(1-\pi)g\| \geq c\|g\|, \qquad \forall g \in L^2.$$

Let U be a multiplication operator by $e^{i\theta}$. Then we have

$$\|\pi Q^{\Omega}\pi U^n g\| + \|\pi K\pi U^n g\| + c\|(1-\pi)U^n g\| \geq c\|U^n g\|, \qquad \forall g \in L^2.$$

Because U preserves the distance we have

$$\|U^{-n}\pi Q^{\Omega}\pi U^n g\| + \|\pi K\pi U^n g\| + c\|U^{-n}(1-\pi)U^n g\| \geq c\|g\|, \qquad \forall g \in L^2.$$

The operator $U^{-n}\pi U^n$ is strongly bounded in L^2 uniformly in n. We have

$$U^{-n}\pi U^n g \to g$$

strongly in L^2 for every trigonometric polynomial g. Therefore it follows that $U^{-n}\pi U^n g \to g$ strongly in L^2. Thus $U^{-n}(1-\pi)U^n g$ converges to 0 strongly, and

$$U^{-n}\pi Q^{\Omega}\pi U^n g = U^{-n}\pi U^n Q^{\Omega} U^{-n}\pi U^n g \to Q^{\Omega}$$

in the strong sense. On the other hand, because U^n converges to 0 weakly $\pi K\pi U^n g$ tends to 0 strongly by the compactness of K. It follows that

$$\|Q^{\Omega} g\| \geq c\|g\|$$

for every $g \in L^2$. If Q^{Ω} vanishes at some point t_0, there exists g with support in some neighborhood of t_0 with norm equal to 1. This contradicts the above inequality. Hence we have proved the assertion.

Next we will show the index formula (4). For the sake of simplicity, we assume that $w = 1$ and $Q^{\Omega}(z)$ is a rational polynomial of z, namely

$$Q^{\Omega}(z) = c(z-\lambda_1)\cdots(z-\lambda_m)(z-\lambda_{m+1})\cdots(z-\lambda_{m+n})z^{-k}.$$

Here

$$|\lambda_1| \leq \cdots \leq |\lambda_m| < 1 < |\lambda_{m+1}| \leq \cdots \leq |\lambda_{m+n}|.$$

We can easily see that the right-hand side of (4) is equal to $m-k$. We will show that the Fredholm index of the operator

$$\pi Q^{\Omega} : H^2 \to H^2$$

is equal to $k-m$. Because $(z-\lambda_{m+1})\cdots(z-\lambda_{m+n})$ does not vanish on the unit disk the multiplication operator with this function is one-to-one on H^2. We may assume that $Q^{\Omega}(z) = (z-\lambda_1)\cdots(z-\lambda_m)z^{-k}$.

We can calculate the kernel and the cokernel of this operator by constructing a recurrence relation. Let us first consider the case $Q^{\Omega}(z) = (z-\lambda)z^{-k}$ ($|\lambda| < 1$). By substituting $u = \sum_{n=0}^{\infty} u_n z^n$ into

$$\pi(z-\lambda)z^{-k}u = 0$$

we obtain

$$(z-\lambda)z^{-k}\sum_{n=0}^{\infty}u_n z^n = \sum_{n=0}^{\infty}(u_{n-1}-\lambda u_n)z^{n-k} \equiv 0,$$

modulo negative powers of z. By comparing the coefficients we obtain the following recurrence relation

$$u_{k-1} - u_k\lambda = 0, \quad u_k - \lambda u_{k+1} = 0, \dots$$

Here $u_0, u_1, \dots u_{k-2}$ are arbitrary. Suppose that $u_{k-1} = c \neq 0$. Then we have

$$u_k = c/\lambda, u_{k+1} = c/\lambda^2, \dots$$

Because the radius of convergence of the function u constructed from this series is < 1, u is not in the kernel. Therefore, the kernel is $k-1$ dimensional.

Next we want to show that the cokernel is trivial, namely the map is surjective. Consider the following equation

$$\pi(z-\lambda)z^{-k}u = f = \sum_{n=0}^{\infty} f_n z^n.$$

By the same arguement as in the above we obtain

$$u_{k-1} - u_k\lambda = f_0, \quad u_k - \lambda u_{k+1} = f_1, \quad u_{k+1} - \lambda u_{k+2} = f_2, \dots$$

By setting

$$u_0 = u_1 = \dots = u_{k-2} = 0,$$

we obtain, from the above recurrence relations

$$u_{k-1} = \lambda u_k + f_0 = f_0 + \lambda f_1 + \lambda^2 u_{k+1} = f_0 + \lambda f_1 + \lambda^2 f_2 + \lambda^3 u_{k+2} + \cdots$$
$$= f_0 + \lambda f_1 + \lambda^2 f_2 + \lambda^3 f_3 + \cdots$$

The series in the right-hand side converges because $|\lambda| < 1$. Similarly we have

$$u_k = \lambda u_{k+1} + f_1 = f_1 + \lambda f_2 + \lambda^2 u_{k+2} = f_1 + \lambda f_2 + \lambda^2 f_3 + \lambda^3 u_{k+3} + \cdots$$
$$= f_1 + \lambda f_2 + \lambda^2 f_3 + \lambda^3 f_4 + \cdots.$$

The series also converges. In the same way we can show that u_j $(j = k-1, k, k+1, \dots)$ can be determined uniquely. Hence the map is surjective. It follows that Ind $= k-1$. This proves the index formula. The general case can be treated in the same way by solving a recurrence relation.

We give an alternative proof of this fact. We recall the following facts.

The operator πz^{-k} has exactly k dimensional kernel given by the basis $1, z, \dots, z^{k-1}$. The map $\pi(z-\lambda)$ $(|\lambda| < 1)$ has one dimensional cokernel. Indeed, the equation $(z-\lambda)\sum u_n z^n = 1$ does not have a solution in H^2 because we have

$u_0 = -1/\lambda$, $u_1 = (-1/\lambda)^2$, $u_2 = (-1/\lambda)^3$, ..., which does not converge on the torus. These facts show the index formula for particular symbols.

In order to show the index formula for general symbols we recall the following theorems.

THEOREM 4 (ATKINSON). *If $A : H^2 \to H^2$ and $B : H^2 \to H^2$ are Fredholm operators BA is a Fredholm operator with the index*

$$Ind\, BA = Ind\, B + Ind\, A.$$

THEOREM 5. *For the Toeplitz operators $\pi q : H^2 \to H^2$ and $\pi p : H^2 \to H^2$ the operator $\pi(pq) - (\pi p)(\pi q)$ is a compact operator.*

These theorems show that the index formula for Q^Ω is reduced to the one with symbols given by every factor of the factorization of Q^Ω.

7. Riemann-Hilbert problem - Case of 2 variables

We start with

DEFINITION 4. *A function $a(\theta_1, \theta_2) = \sum_\eta a_\eta e^{i\eta\theta}$ on $T^2 := S \times S$, $S = \{|z| = 1\}$ is Riemann-Hilbert factorizable with respect to T^2 if there exist nonvanishing functions a_{++}, a_{-+}, a_{--}, a_{+-} on T^2 with (Fourier) supports contained repectively in*

$$I := \{\eta_1 \geq 0, \eta_2 \geq 0\}, \quad II := \{\eta_1 \leq 0, \eta_2 \geq 0\},$$

$$III := \{\eta_1 \leq 0, \eta_2 \leq 0\}, \quad IV := \{\eta_1 \geq 0, \eta_2 \leq 0\}$$

such that

$$a(\theta_1, \theta_2) = a_{++} a_{-+} a_{--} a_{+-}.$$

THEOREM 6. *Suppose that the following conditions are verified.*

$$(A.1) \qquad \sigma(z, \xi) \neq 0 \quad \forall z \in \mathbb{T}^2, \forall \xi \in \mathbb{R}^2_+, |\xi| = 1,$$

$$(A.2) \qquad ind_1\, \sigma = ind_2\, \sigma = 0,$$

where

$$ind_1\, \sigma = \frac{1}{2\pi i} \oint_{|\zeta|=1} d_{z_1} \log \sigma(\zeta, z_2, \xi),$$

and $ind_2\, \sigma$ is similarly defined. Then $\sigma(z, \xi)$ is R–H factorizable.

Here the integral is an integer-valued continuous function of z_2 and ξ, which is constant on the connected set $\mathbb{T}^2 \times \{|\xi| = 1\}$. Hence it is constant.

Proof. Suppose that (A1) and (A.2) are verified. Then the function $\log a(\theta)$ is well defined on $\mathbb{T}^2$ and smooth. By Fourier expansion we have

$$\log a(\theta) = b_{++} + b_{-+} + b_{--} + b_{+-}$$

where the supports of $b_{++}, b_{-+}, b_{--}, b_{+-}$ are contained in I, II, III, IV, respectively. The factorization

$$a(\theta) = \exp(b_{++}) \exp(b_{-+}) \exp(b_{--}) \exp(b_{+-})$$

is the desired one. This ends the proof.

□

REMARK 3. The above definition can be extended to a symbol of a pseudodifferential operator $a = a(\theta_1, \theta_2, \xi_1, \xi_2)$. We assume that the factors a_{++}, a_{-+}, a_{--}, a_{+-} are smooth functions of ξ, in addition.

8. Riemann-Hilbert problem and construction of a parametrix

In this section we give a rather concrete construction of a parametrix of an operator reduced on the tori under the R–H factorizability.

Let $L^2(\mathbb{T}^2)$ be a set of square integrable functions, and let us define subspaces H_1, H_2 of $L^2(\mathbb{T}^2)$ by

$$H_1 := \left\{ u \in L^2; u = \sum_{\zeta_1 \geq 0} u_\zeta e^{i\zeta\theta} \right\}, \ H_2 := \left\{ u \in L^2; u = \sum_{\zeta_2 \geq 0} u_\zeta e^{i\zeta\theta} \right\}.$$

We note that $H^2(\mathbb{T}^2) = H_1 \cap H_2$. We define the projections π_1 and π_2 by

$$\pi_1 : L^2(\mathbb{T}^2) \longrightarrow H_1, \quad \pi_2 : L^2(\mathbb{T}^2) \longrightarrow H_2.$$

Then the projection $\pi : L^2(\mathbb{T}^2) \to H^2(\mathbb{T}^2)$ is, by definition, equal to $\pi_1\pi_2$. We define a Toeplitz operator $T_{+\cdot}$ and $T_{\cdot+}$ by

$$T_{+\cdot} := \pi_1 a(\theta, D) : H_1 \longrightarrow H_1, \ T_{\cdot+} := \pi_2 a(\theta, D) : H_2 \longrightarrow H_2.$$

If the Toeplitz symbols of these operators are Riemann-Hilbert factorizable it follows that $T_{+\cdot}$ and $T_{\cdot+}$ are invertible modulo compact operators, and their inverses (modulo compact operators) are given by

$$T_{+\cdot}^{-1} = \pi_1 a_{++}^{-1} a_{+-}^{-1} \pi_1 a_{-+}^{-1} a_{--}^{-1} \pi_1, \ T_{\cdot+}^{-1} = \pi_2 a_{++}^{-1} a_{-+}^{-1} \pi_2 a_{+-}^{-1} a_{--}^{-1} \pi_2, \tag{5}$$

where the equality means the one modulo compact operators.

THEOREM 7. *Let $a(\theta, D)$ be a pseudodifferential operator on the torus. Suppose that $a(\theta, D)$ is R–H factorizable. Then the parametrix R of $\pi a(\theta, D)$ is given by*

$$R = \pi(T_{+\cdot}^{-1} + T_{\cdot+}^{-1} - a(\theta, D)^{-1}), \tag{6}$$

where $a(\theta, D)^{-1}$ is a pseudodifferential operator with symbol given by $a(\theta, \xi)^{-1}$.

These facts are essentially proved in [9] under slightly different situation. We give the proof for the reader's convenience. In the following $A \equiv B$ means that A and B are equal modulo compact operators.

Proof of (5). By comparing the principal symbol of both sides we obtain $a(\theta, D) \equiv a_{++}a_{-+}a_{--}a_{+-}$.

$$T_{+\cdot}\pi_1 a_{++}^{-1}a_{+-}^{-1}\pi_1 a_{-+}^{-1}a_{--}^{-1}\pi_1 \equiv \pi_1 a_{++}a_{-+}a_{--}a_{+-}\pi_1 a_{++}^{-1}a_{+-}^{-1}\pi_1 a_{-+}^{-1}a_{--}^{-1}\pi_1$$

$$\equiv \pi_1 a_{-+}a_{--}a_{++}a_{+-}a_{++}^{-1}a_{+-}^{-1}\pi_1 a_{-+}^{-1}a_{--}^{-1}\pi_1$$

$$+\pi_1 a_{-+}a_{--}a_{++}a_{+-}(I-\pi_1)a_{++}^{-1}a_{+-}^{-1}\pi_1 a_{-+}^{-1}a_{--}^{-1}\pi_1 \equiv \pi_1 a_{-+}a_{--}\pi_1 a_{-+}^{-1}a_{--}^{-1}\pi_1,$$

where we used

$$(I-\pi_1)a_{++}^{-1}a_{+-}^{-1}\pi_1 = 0.$$

Therefore, the right-hand side is equal to

$$\pi_1 a_{-+}a_{--}a_{-+}^{-1}a_{--}^{-1}\pi_1 + \pi_1 a_{-+}a_{--}(I-\pi_1)a_{-+}^{-1}a_{-+}^{-1}\pi_1$$

and hence $\equiv \pi_1$. Here we used $\pi_1 a_{-+}a_{--}(I-\pi_1) = 0$. Similarly, we can show

$$\pi_1 a_{++}^{-1}a_{+-}^{-1}\pi_1 a_{-+}^{-1}a_{--}^{-1}\pi_1 T_{+\cdot} \equiv \pi_1.$$

This ends the proof. ◻

Proof of (6). Noting that $\pi = \pi_1\pi_2$ we have

$$\pi T_{+\cdot}^{-1}\pi a\pi = \pi T_{+\cdot}^{-1}\pi_1\pi_2 a\pi = \pi T_{+\cdot}^{-1}\pi_1 a\pi - \pi T_{+\cdot}^{-1}\pi_1(I-\pi_2)a\pi$$

$$\equiv \pi - \pi a_{++}^{-1}a_{+-}^{-1}\pi_1 a_{-+}^{-1}a_{--}^{-1}\pi_1(I-\pi_2)a\pi$$

$$= \pi - \pi a_{++}^{-1}a_{+-}^{-1}(\pi_1\pi_2 + \pi_1(I-\pi_2))a_{-+}^{-1}a_{--}^{-1}\pi_1(I-\pi_2)a\pi.$$

Similarly, we have

$$\pi T_{\cdot+}^{-1}\pi a\pi = \pi T_{\cdot+}^{-1}\pi_1\pi_2 a\pi = \pi T_{\cdot+}^{-1}\pi_2 a\pi - \pi T_{\cdot+}^{-1}\pi_2(I-\pi_1)a\pi$$

$$\equiv \pi - \pi a_{++}^{-1}a_{-+}^{-1}\pi_2 a_{+-}^{-1}a_{--}^{-1}\pi_2(I-\pi_1)a\pi$$

$$= \pi - \pi a_{++}^{-1}a_{-+}^{-1}(\pi_1\pi_2 + \pi_2(I-\pi_1))a_{+-}^{-1}a_{--}^{-1}\pi_2(I-\pi_1)a\pi.$$

On the othe hand, since $a^{-1}a \equiv I$ we have

$$-\pi a^{-1}\pi a\pi = -\pi a^{-1}\pi_1\pi_2 a\pi \equiv -\pi - \pi a^{-1}(\pi_1\pi_2 - I)a\pi.$$

By using

$$\pi_1\pi_2 - I = \pi_1(\pi_2 - I) + (\pi_1 - I)\pi_2 - (\pi_1 - I)(\pi_2 - I)$$

we have

$$-\pi a^{-1}\pi a\pi \equiv -\pi_1\pi_2 - \pi_1\pi_2 a^{-1}\pi_1(\pi_2 - I)a\pi$$
$$-\pi a^{-1}(\pi_1 - I)\pi_2 a\pi + \pi a^{-1}(\pi_1 - I)(\pi_2 - I)a\pi.$$

Combining these relations

$$RT \equiv \pi - \pi a_{++}^{-1}a_{+-}^{-1}(\pi + \pi_1(I - \pi_2))a_{-+}^{-1}a_{--}^{-1}\pi_1(I - \pi_2)a\pi$$
$$-\pi a_{++}^{-1}a_{-+}^{-1}(\pi + \pi_2(I - \pi_1))a_{+-}^{-1}a_{--}^{-1}\pi_2(I - \pi_1)a\pi$$
$$-\pi a^{-1}\pi_1(\pi_2 - I)a\pi - \pi a^{-1}(\pi_1 - I)\pi_2 a\pi + \pi a^{-1}(\pi_1 - I)(\pi_2 - I)a\pi.$$

We note

$$\pi + \pi_1(I - \pi_2) = I - (\pi_1 - I)(\pi_2 - I) - \pi_2(I - \pi_1),$$
$$\pi + \pi_2(I - \pi_1) = I - (\pi_1 - I)(\pi_2 - I) - \pi_1(I - \pi_2).$$

It follows that

$$RT - \pi \equiv \pi a_{++}^{-1}a_{+-}^{-1}((\pi_1 - I)(\pi_2 - I)$$
$$+\pi_2(I - \pi_1))a_{-+}^{-1}a_{--}^{-1}\pi_1(I - \pi_2)a\pi + \pi a^{-1}(\pi_1 - I)(\pi_2 - I)a\pi$$
$$+\pi a_{++}^{-1}a_{-+}^{-1}\left((\pi_1 - I)(\pi_2 - I) + \pi_1(I - \pi_2)\right)a_{+-}^{-1}a_{--}^{-1}\pi_2(I - \pi_1)a\pi.$$

In order to show that the right-hand side operators are compact operators we will show that the operators

$$\pi\varphi(\pi_1 - I)(\pi_2 - I),\ \pi_2(I - \pi_1)\varphi\pi_1(I - \pi_2),\ \pi_1(I - \pi_2)\varphi\pi_2(I - \pi_1)$$

are compact. Here φ is an appropriately chosen smooth function. In order to show this let

$$u = \sum_\alpha u_\alpha e^{i\alpha\theta} \in L^2,\ \varphi(\xi) = \sum_\beta \varphi_\beta(\xi)e^{i\beta\theta}$$

be the Fourier expansion of $u \in L^2$ and $\varphi \in C^\infty$, respectively. Because $\varphi(\theta, D)$ is order zero pseudodifferential operator the Fourier coefficients of $\varphi_\beta(\xi)$ is rapidly decreasing in ξ when $|\beta| \to \infty$. Therefore

$$\pi\varphi(\pi_1 - I)(\pi_2 - I)u = \sum_{\mu=\alpha+\beta\in I}\left(\sum_{\alpha+\beta=\mu,\alpha\in III}\varphi_\beta(\mu)u_\alpha\right)e^{i\mu\theta}.$$

Because $\mu \in I$ and $-\alpha \in I$ by the definition of I and III, β satisfies that $|\beta| = |\mu - \alpha| \geq |\mu|$. It follows that, for all $n \geq 1$ and μ

$$|\mu|^n \sum_{\alpha+\beta=\mu,\alpha\in III}|\varphi_\beta(\mu)||u_\alpha| \leq \sum|\beta|^n|\varphi_\beta(\mu)||u_\alpha| < \infty.$$

Indeed, $|\varphi_\beta(\mu)||\beta|^n$ is bounded in μ and β. It follows that the Fourier coefficients converge uniformly in $u \in L^2$. Thus $\pi\varphi(\pi_1 - I)(\pi_2 - I)$ is a compact operator. The compactness of other operators are proved similarly. Hence R is a left regularizer. We can similarly show that R is a right regularizer. This ends the proof.

□

9. Solvability in two dimensional case

Let f be a formal power series, and $k = \operatorname{ord} f$ be the order of f, namely the least degree of monomials which constitute f. Hence it follows that $\partial_x^\alpha f(0) \neq 0$ for some $|\alpha| = k$ and $\partial_x^\beta f(0) = 0$ for all $|\beta| \leq k-1$. For a polynomial u_0 of $\operatorname{ord} u_0 = 4$ we define $f_0 = M(u_0)$. Then we have

THEOREM 8. *Let $n = 2$. Suppose that (A.1) and (A.2) are verified. Then there exist $r > 0$ and an integer $N \geq 4$ depending only on u_0 and the equation such that, for every $g \in W_R$ satisfying $\|g\|_R < r$, $ord\, g \geq N$ the equation* (MA)

$$(MA) \qquad M(v + u_0) := \det(v_{x_i x_j} + (u_0)_{x_i x_j}) = f_0(x) + g(x) \quad in\ \Omega,$$

has a unique solution $v \in W_R$ such that $ord\, v \geq N$.

REMARK 4. The conditions (A.1) and (A.2) are invariant if we replace R with $R\rho$ $(0 < \rho < 1)$. By taking R small, if necessary, we may assume $\|g\|_R < r$. Hence the solution exists in some neighborhood of the origin.

Proof. We linearize M

$$M(u_0 + v) = M(u_0) + \pi P v + R(v),$$

where $R(v)$ is a remainder. It follows that

$$(*) \qquad \pi P v + R(v) = g \quad \text{on} \quad W_R(\mathbb{T}^n).$$

By the argument in the preceeding section there exists a parametrix S of πP. Indeed, we have $S\pi P = \pi + R$, where R is an operator of negative order. It follows that the norm of R on the subspace of W_R with order greater than N can be made arbitrarily small if N is sufficiently large. It follows that $S\pi P = \pi + R$ is invertible on the subspace of W_R with order greater than N for sufficiently large N. Therefore if N is sufficiently large and if the order of g is greater than N we can solve $(*)$ by a standard iteration. Hence, if $\|g\|_R$ is sufficiently small $(*)$ has a unique solution v.

Let $\hat{v}$ be an analytic extension of v to D_R. The function

$$M(u_0 + \hat{v}) - f_0 - g$$

is holomorphic in D_R, and vanishes on the Silov boundary of D_R. By the maximal principle, we have

$$M(u_0 + \hat{v}) = f_0 + g \quad \text{in } D_R.$$

Hence we have the solvability.

Uniqueness. Suppose that there exist two solutions w_1 and w_2 to (MA) such that $\|w_j\| \leq \varepsilon$ for small ε. We blow up the equation to $\mathbb{T}^n$. By the uniqueness of the operator on the boundary we have $w_1 = w_2$ on $\mathbb{T}^n$. By the maximal principle we have $w_1 = w_2$ in D_R.

□

We consider two examples in Section 2. We use the same notations as in Section 2.

EXAMPLE 6. The condition (A.1) reads

$$2 + (k - 8)\eta_1\eta_2 \neq 0 \ \text{ for all } \eta \in \mathbb{R}^2_+, \ |\eta| = 1.$$

This is equivalent to $k > 4$. We can easily see that (A.2) holds if $k > 4$. The condition is weaker than the ellipticity condition in Section 2 because we work on a Hardy space. The same is true in the next example.

EXAMPLE 7. By the same argument as before we can verify that (A.1) is equivalent to $k < -6$ or $k > 8$. We can easily verify (A.2) for $\xi = (0, 1)$ under these conditions.

Convergence of all formal power series solutions We give an application of Theorem 8. Kashiwara-Kawai-Sjöstrand ([5]) gave a subclass of linear Grushin operators for which all formal power series solutions converge. Here we give a class of nonlinear operators for which all formal power series solutions converge.

THEOREM 9. *Assume* (A.1) *and* (A.2). *Then, for every g holomorphic in some neighborhood of the origin such that* $\operatorname{ord} g > 4$ *all formal power series solutions of* (MA) *of the form $u = u_0 + w$, $\operatorname{ord} w > 4$ converge in some neighborhood of the origin.*

Proof. Let $w = \sum_{j=5}^{\infty} w_j$ be any formal solution of (MA) for $\operatorname{ord} g \geq 5$, where w_j is a polynomial of homogenous degree j. Let k be an integer determined later, and set $w = w_0 + U$, where $w_0 = \sum_{j=5}^{k} w_j$, $\operatorname{ord} U \geq k + 1$. Determine h by $M(u_0 + w_0) = f_0 + h$, and write the equation in the form

$$M(u_0 + w_0 + U) = f_0 + h + g - h.$$

The order of $g-h$ can be made arbitrarily large if k is sufficiently large. It follows from Theorem 8 that, if k is sufficiently large the formal power series solution U is uniquely determined by $g - h$. The condition (A.1) and (A.2) are invariant if we replace u_0 with $u_0 + w_0$. By Remark 4 the above equation has a unique analytic solution. By the uniqueness of a formal solution U converges.

□

10. Solvability in general independent variables

For a given $u_0(x)$ holomorphic in some neighborhood of the origin such that $\operatorname{ord} u_0 = 4$ we set $f_0(x) = M(u_0) := \det((u_0)_{x_i x_j})$. For an analytic $g(x)$ ($\operatorname{ord} g \geq 5$) we study the equation

$$(MA) \qquad M(u_0 + v) = f_0(x) + g(x).$$

By the argument in Section 2 M may be of mixed type at $u = u_0$, while its blow up onto the torus is elliptic. If we can construct a parametrix of the reduced operator on the torus of the linearized operator of (MA) the argument in the case of two independent variables can be applied to the case of general independent variables. Therefore, in order to show the solvability we construct a parametrix.

Let P be the linearized operator of $M(u)$ at $u = u_0$

$$P := M_{u_0} = \sum_{|\alpha|\le m} (\partial M/\partial z_\alpha)(x, u_0)\partial_x^\alpha \equiv \sum_{\alpha, |\alpha|\le m} a_\alpha(x)\partial_x^\alpha,$$

where $m \in \mathbf{N}$, and $a_\alpha(x)$ is holomorphic in some neighborhood of the origin. We define the symbol $\sigma(z, \xi)$ of the reduced operator on tori by

$$\sigma(z, \xi) := \sum_{|\alpha|\le m} a_\alpha(z) z^{-\alpha} p_\alpha(\xi)\langle\xi\rangle^{-m},$$

where $z_j = R_j e^{i\theta_j}$, $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ and $p_\alpha(\xi) = \prod_{j=1}^n \xi_j(\xi_j - 1)\cdots(\xi_j - \alpha_j + 1)$.

REMARK 5. By elementary calculations we can show that

$$\sigma(z, \xi)\langle\xi\rangle^m = (z_1\cdots z_n)^{-2} \det\Big(\xi_j\xi_k + z_j z_k u^0_{x_j x_k}(z)\Big) \ - f_0(z).$$

We will not use the concrete expression in the following argument.

We decompose $\sigma(z, \xi)$ as follows

$$\sigma(z, \xi) = \sigma'(\xi) + \sigma''(z, \xi),$$

where $\sigma'(\xi) = \int_{\mathbb{T}^n} \sigma(Re^{i\theta}, \xi)d\theta$ is the average over $\mathbb{T}^n$. We assume

(B.1) there exist constant $c \in \mathbb{C}$, $|c| = 1$ and $K > 0$ such that

$$Re\, c\sigma'(\xi) \ge K > 0 \quad \text{for all} \quad \forall\xi \in \mathbb{Z}^n_+.$$

Then we have

THEOREM 10. *Assume* (B.1). *Then there exists K_0 such that for every $K \ge K_0$ the reduced operator of P on $\mathbb{T}^n$ has a parametrix on $W_R(\mathbb{T}^n)$.*

Proof. We lift the operator $P < D_x >^{-m}$ to the torus. Its symbol is given by $\sigma(z, \xi)$. We have, for $u \in W_R(\mathbb{T}^n)$

$$\|(I - \varepsilon c\pi\sigma)u\|_R = \|\pi(1 - \varepsilon c\sigma)u\|_R \le \|(1 - \varepsilon c\sigma)u\|_{\ell^1_R}.$$

Here we used the boundedness of $\pi : \ell^1_R \to \ell^1_{R,+}$. If we can prove that

$$\|(1 - \varepsilon c\sigma)u\|_{\ell^1_R} < \|u\|_{\ell^1_R} = \|u\|_R$$

we have $\|(I - \varepsilon c\pi\sigma)u\|_R < \|u\|_R$. Thus $\varepsilon c\pi\sigma = I - (I - \varepsilon c\pi\sigma)$ is invertible on $W_R(\mathbb{T}^n)$, and $\pi\sigma$ is invertible. Indeed, it follows from (B.1) that there exists $K_1 > 0$ such that if $K > K_1$ we have

$$Re\, c\sigma(z,\xi) = Re\, c\sigma'(\xi) + Re\, c\sigma''(z,\xi) > K - K_1, \quad \forall z \in \mathbb{T}^n, \forall \xi \in \mathbb{Z}^n_+.$$

Hence, if $\varepsilon > 0$ is sufficiently small we have

$$\|1 - \varepsilon c\sigma(\cdot,\xi)\|_{L^\infty} < 1 - \varepsilon(K - K_1), \quad \forall \xi \in \mathbb{Z}^n_+.$$

From this estimate we can prove the desired estimate (cf. [12]).

□

11. Solvability of a homology equation

We want to linearize an analytic singular vector field at a singular point via coordinate change. The transformation satisfies a so-called homology equation

$$\mathcal{L}u = R(x+u), \quad \mathcal{L} = \sum_{j=1}^{n} \lambda_j x_j \frac{\partial}{\partial x_j},$$

where $x = (x_1, \ldots, x_n)$, and $R(y)$ is an analytic function of y given by the vector field, and λ_j are eigenvalues of the linear part of the vector field. Here we assume that the vector field is semi-simple. We say that the Poincaré condition is satisfied if the convex hull of all λ_j in the complex plane does not contain the origin. Let us apply our arguement to this equation. By a blowing up we obtain a nonlinear equation on $H^2(\mathbb{T}^n)$. Then we have

PROPOSITION 1. *The Poincaré condition holds if and only if the lifted operator of $\mathcal{L}$ to $H^2(\mathbb{T}^n)$ is elliptic.*

Proof. The latter condition reads: $\sum_{j=1}^n \lambda_j \xi_j \neq 0\ \forall \xi \in \mathbb{R}^n_+, |\xi| = 1$. One can easily see that Poincaré condition implies the condition. Conversely, if the ellipticity hols we obtain the Poincaré condition. This ends the proof.

□

We remark that, by the solvability on tori we can prove the so-called Poincaré's theorem.

Next we think of the simultaneous reduction of a system of d vector fields $\{\mathcal{X}^\nu\}_\nu$ whose eigenvalues of the linear parts are given by λ_j^ν $(j = 1, \ldots, n)$ $(\nu = 1, \ldots, d)$. By the same way as before we are lead to the system of equations

$$\mathcal{L}_\mu u = R_\mu(x+u), \quad \mathcal{L}_\mu = \sum_{j=1}^{n} \lambda_j^\mu x_j \frac{\partial}{\partial x_j}, \quad \mu = 1, \ldots, d.$$

Define $\lambda_j := (\lambda_j^1, \ldots, \lambda_j^d), \quad j = 1, \ldots, n$ and

$$\Gamma := \left\{ \sum_{j=1}^{n} \xi_j \lambda_j ; \xi_j \geq 0, \xi_1^2 + \cdots + \xi_n^2 \neq 0 \right\}.$$

We say that a system of vector fields satisfies a simultaneous Poincaré condition if Γ does not contain the origin. Set $\xi = (\xi_1, \ldots, \xi_n)$. Then the condition can be written in

$$\forall \xi \in \mathbb{R}_+^n \setminus 0, \quad \exists k \ (1 \leq k \leq d) \text{ such that } \sum_{j=1}^{n} \lambda_j^k \xi_j \neq 0.$$

This is equivalent to saying that the lifted operator on tori is an elliptic system.

12. Analysis of equations containing a large parameter

Let $p(x, \partial_x)$ be a pseudodifferential operator of order m with polynomial coefficients, and let $q(x)$ be a rational function. For a given analytic f we consider the asymptotic behaviour when $\lambda \to \infty$ of the solution u of the equation

$$(p(x, \partial_x) + \lambda^2 q(x))u = f(x). \tag{7}$$

By the substitution $x \mapsto e^{i\theta} = (e^{i\theta_1}, \ldots, e^{i\theta_n})$ we obtain an equation on $\mathbb{T}^n$.

$$(p(e^{i\theta}, e^{-i\theta} D_\theta) + \lambda^2 q(e^{i\theta}))u = f(e^{i\theta}).$$

We consider the case $n = 1$. Set $z = e^{i\theta}$ and define

$$\sigma(z, \xi, \lambda) := p(z, z^{-1}\xi) + \lambda^2 q(z).$$

Assume the uniform R-H factorization condition

$$(URH) \qquad \sigma(z, \xi, \lambda) \neq 0 \text{ for } \forall z \in \mathbb{T}, \forall (\xi, \lambda) \in \mathbb{R}_+^2, \xi^2 + \lambda^2 = 1,$$

$$\frac{1}{2\pi i} \int_{|z|=1} d_z \log \sigma(z, \xi, \lambda) = 0 \ \exists (\xi, \lambda) \in \mathbb{R}_+^2, \xi^2 + \lambda^2 = 1.$$

Let $\| \cdot \|_s$ be a Sobolev norm. We recall that ord f is the least degree of monomials which constitute f. Then we have

THEOREM 11. *Let $s > 0$, and assume (URH). Then there exists $N \geq 1$ such that for any f satisfying ord $f \geq N$ (7) has a unique solution u. Moreover, there exists $C > 0$ such that the estimate*

$$\|u\|_{s+mq} \leq \lambda^{-2p}(C\|f\|_s + C^{-1}\|u\|_0)$$

holds for all $\lambda > 0$, where $p + q = 1$, $0 \leq p \leq 1$.

Proof. We consider the principal part and we neglect the lower order terms. Write

$$\pi\sigma(z, z^{-1}D_\theta, \lambda) = (D_\theta^m + \lambda^2)\pi(D_\theta^m + \lambda^2)^{-1}\sigma(z, z^{-1}D_\theta, \lambda).$$

Then $\pi(D_\theta^m + \lambda^2)^{-1}\sigma(z, z^{-1}D_\theta, \lambda)$ is uniformly invertible for $\lambda > 0$ by virtue of (URH). The estimate for $D_\theta^m + \lambda^2$ follows from direct computation. □

We consider the case $n = 2$. We define $\sigma(z, \xi, \lambda)$ $(z \in \mathbb{T}^2)$ as in the above, and we assume

$$(URH) \qquad \sigma(z, \xi, \lambda) \neq 0 \text{ for } \forall z \in \mathbb{T}, \forall(\xi, \lambda) \in \mathbb{R}_+^3, |\xi|^2 + \lambda^2 = 1,$$

$$\frac{1}{2\pi i}\int_{|z|=1} d_{z_j} \log\sigma(z, \xi, \lambda) = 0, \text{ for } j = 1, 2, \exists(\xi, \lambda) \in \mathbb{R}_+^3, |\xi|^2 + \lambda^2 = 1.$$

Under these conditions the operator $\pi(|D_\theta|^m + \lambda^2)^{-1}\sigma(z, D_\theta, \lambda)$ has a regularizer. Therefore, it can be transformed to $|D_\theta|^m + \lambda^2$ modulo compact operators. By solving the transformed equation via Fourier method we obtain the same estimate as $n = 1$.

REMARK 6. If λ moves in a sector, $\lambda = \rho e^{i\alpha}$ $(\theta_1 \leq \alpha \leq \theta_2)$ we can treat (7) similarly if we replace q with $e^{2i\alpha}q$ in (URH).

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