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A NONLINEAR OSCILLATOR MODELING THE TRANSVERSE VIBRATIONS OF A ROD

Abstract. This paper has been originated by the motion analysis of a flexible rod whose extreme (pivoted) sections cannot undergo any displacement, but only rotate. First an historical outline of the rod tranverse vibration analysis is provided. The dynamical elastica of such a system can be easy expressed through a linear fourth order PDE solved by a Fourier series: the rod resulting movement is then known by armonical superposition, but the overall vibration rod's period is analytically unexpressed.

The authors propose an alternative analytic treatment designing a discrete model, made of a single degree of freedom oscillator, ruled by a nonlinear second order ODE, here solved through the elliptic integrals of first and third kind. This greater complexity is balanced by having found a closed formula for computing the overall vibration period by means of a suitable punctual oscillator.

1. Introduction

Physical systems are characterized by their state variable(s) depending both on time and space.

In Mechanics these models are obtained whenever the first mechanical approach of rigid bodies or mass points (*discrete* models controlled by ordinary differential equations, ODE) are replaced by the more realistic *continuous* elastic bodies, ruled by partial differential equations (PDE).

Nevertheless we will follow the reverse path: this paper will start introducing first a continuous deformable body formed by a doubly hinged flexible rod whose extreme (pivoted) sections cannot undergo any displacement, but only rotate. The dynamical elastica of such a system can be easy found (initial/boundary value problem) through a Fourier series, allowing to know the shape and the period of each (i^{th}) of the infinite armonics. We will know the rod resulting movement - due to the equation's linearity - by armonical superposition, *but the overall period of the rod's transverse vibration would be analytically unexpressed*. Therefore an alternative analytic treatment is proposed here, designing a discrete model of the continuous phenomenology. Whilst this latter involves a linear fourth order elementary integrable PDE, the former will imply

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a nonlinear second order ODE, exactly solved first time by us through the elliptic integrals. This greater complexity is balanced by having obtained a closed formula for the overall oscillation period via algebraic expressions including higher transcendental functions. The new problem which we arrive at, is more tractable as far as it concerns the period, which we succeed to close in a formula. The use of the special functions of Mathematical Physics for solving some nonlinear differential equations of dynamics is at the moment a stable research interest of the authors' activity [6, 7]. Notice that the elliptic integrals of the third kind, frequently met in planar Mathematical Physics, are seldom seen in problems of particle motion.

The authors are indebted with their friend professor Aldo Scimone who drew the first three pictures of this paper: they take here the opportunity for thanking him warmly. Finally we also thank the Referee for her (or his) fruitful advice.

2. Historical account of the rod transverse vibration analysis

The motion of the flexible or elastic bodies was studied before 1750 on the basis of special hypotheses (of load and/or geometry) by B. Taylor (1685–1731), Jakob Bernoulli (1654–1705), Daniel Bernoulli (1700–1782), L. Euler (1707-1783) without knowing neither the vibrating string PDE, nor that governing the *transverse* vibrations of a rod. This prevented any mathematical treatment of initial value problems in elasticity before that date. Thus, when J. D'Alembert (1717-1783) established his PDE for the vibrating string,* this had a sensational effect, because he applied the Newton's second law to a continuous body, and the Taylor's truncated formula for the restoring force.

Euler's *Decouverte d'un nouveau principe de Mécanique*† sets down the newtonian principle in the (now familiar) triple scalar form: $m\ddot{x}_i = F_{xi}$, $i = 1, 2, 3$ for the particles. And there it is stated explicitly that, passing to a continuous body, m and F have to be replaced by the differential elements dm and dF.

Let us go now to the transverse oscillations of a flexible rod: even if most of Euler's immediate effort was directed to the rigid bodies and the perfect fluids, nevertheless (from his notebooks) he is believed to have known the PDE ruling the small transverse vibrations $y(x, t)$ of an elastic rod:

$$\frac{1}{c^4}\frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} = 0 \tag{1}$$

which was to be published by himself several years later. All this is based upon a well known ‡ Truesdell's authoritative study; but the authors of this article did not succeed to see any edition of the notebooks. After a careful screening of Euler's collected works,

*J. Le Rond D'Alembert: *Recherches sur la courbe que forme une corde tendue mise en vibration*, Histoire de l'Academie de Berlin (III), 1747, published in 1749.

†The paper was presented -and presumably read- on sept. 3rd, 1750, at Berlin Academy and is nowadays in vol 5th series 2, of Euler's Collected Works.

‡C. A. Truesdell: *The Rational Mechanics of flexible or elastic bodies, 1638-1788*, (embodied in Euler's Collected Works, introduction to the vol. 10 and 11 of series 2), Zürich 1960.

they detected the equation's first official appearance in the paper *De motu vibratorio laminarum elasticarum ubi plures novae vibrationum species hactenus non pertractatae evolvuntur*[§] . Euler, equating the effects of acceleration and forces, writes (1) under the form:

$$-\left(\frac{d^2y}{dt^2}\right)+c^4\left(\frac{d^4y}{dx^4}\right)=0,$$

finding the fundamental solution involving circular and exponential functions through six constants.

3. Free vibrations of a doubly hinged rod

Not many continuous systems are known whose vibrational behaviour may be analyzed exactly. This collection includes a string in transverse vibration, a rod in longitudinal or torsional vibration, a beam in transverse ("flexional") vibration, and -finally- certain simply shaped membranes or plates. Such systems, differently linked, have uniform cross section and material, and they vibrate according to linear PDEs in one (or two) spatial dimensions and in the time.

Let us take an uniform rod of length l, volumic mass ρ, cross section area S and $y(x,t)$ be the dynamic deflection of each $x-$point of it, in the time. By considering the motion of a thin slice of the beam, and neglecting the shearing force (which gives a very small contribution to y) as compared to the bending moment, the Euler equation (1) can be written:

$$\frac{\partial^2 y}{\partial t^2}+a^2\frac{\partial^4 y}{\partial x^4}=0,\quad a^2=\frac{EJ}{S\rho}. \tag{2}$$

E is the Young modulus, and J the second moment of S about the neutral axis through the centroid. The independent variables are the time t and the coordinate x along the rod axis, while the state dependent variable is the transverse displacement:

$$y=y(t,x):[0,T]\times[0,l]\to\mathbb{R}.$$

It is common practice, [10] pages 137-138, to search the stationary vibrations, namely those not moving along the rod: in such a way the nodes -where the displacement is zero- are fixed. The same happens for those points (antinodes) having the highest displacements. This implies we are looking for a *separable* solution:

$$y(x,t)=X(x)T(t),$$

leading the equation (2) to split into two different ODEs. A fundamental system of

[§]The paper presented in 1772, but published in 17$^{\text{th}}$ volume of Novi Commentarii Academiae Scientiarum Petropolitanae of 1773, can now be read in Euler Collected Works, series 2, vol. 11, first part, pages 112/141.

solutions of (2), accordingly, will be:

$$\begin{aligned} y(x,t) &= \Big(C_1\cos(hx)+C_2\sin(hx)+C_3\cosh(hx)+C_4\sinh(hx)\Big)\times \\ &\times \Big(A\cos(ft)+B\sin(ft)\Big), \end{aligned} \tag{3}$$

where h and f are two constants homogeneous to the inverse of a length and of a time, respectively. The boundary conditions of a doubly hinged cantilever for $t \geq 0$, are usually found as a consequence of how the rod is linked. In our case the displacement and bending moment shall be zero at both the pivoted points:

$$\begin{cases} x=0, & y=0, \\ x=0, & \dfrac{\partial^2 y}{\partial x^2}=0, \\ x=l, & y=0, \\ x=l, & \dfrac{\partial^2 y}{\partial x^2}=0. \end{cases} \tag{4}$$

Applying (4) to (3), one obtains $C_1 = C_3 = C_4 = 0$, and then y will be given by the series:

$$y(x,t)=\sum_{i=1}^{\infty} y_i=\sum_{i=1}^{\infty}\sin(p_i x)\left[A_i\cos(\omega_i t)+B_i\sin(\omega_i t)\right], \tag{5}$$

with:

$$\omega_i = a\frac{\pi^2 i^2}{l^2} \quad \text{and} \quad a=\sqrt{\frac{EJ}{S\rho}},$$

where $p_i = i\,\pi\,l^{-1}$, $i = 1, 2, 3, \ldots$ and x marks whichever point along the rod. The constants A_i and B_i shall be determined by the initial conditions:

$$\begin{cases} y(x,0)=y_0, \\ \dfrac{\partial y}{\partial t}(x,0)=v_0. \end{cases}$$

And then:

$$y_0=\sum_{i=1}^{\infty}A_i\sin(p_i x); \qquad v_0=\sum_{i=1}^{\infty}p_i B_i\sin(p_i x).$$

In such a way the search of A_i and B_i has been driven to detect the coefficients of the Fourier sine expansions of the constant quantities y_0 and v_0, as it will be done soon.

The Fourier series uniform convergency entitles the term by term integration. Let us start from:

$$y(x,0)=\sum_{i=1}^{\infty}A_i\sin(p_i x). \tag{6}$$

It shall be:

$$\begin{cases} y(x,0) = y_0, & \frac{l}{2} - \frac{s}{2} \le x \le \frac{l}{2} + \frac{s}{2}, \\ y(x,0) = 0, & \text{elsewhere}, \end{cases}$$

where s is the partial length of the rod, centered on $x = \frac{l}{2}$, modeling the rod's mean point, see the attached figure.

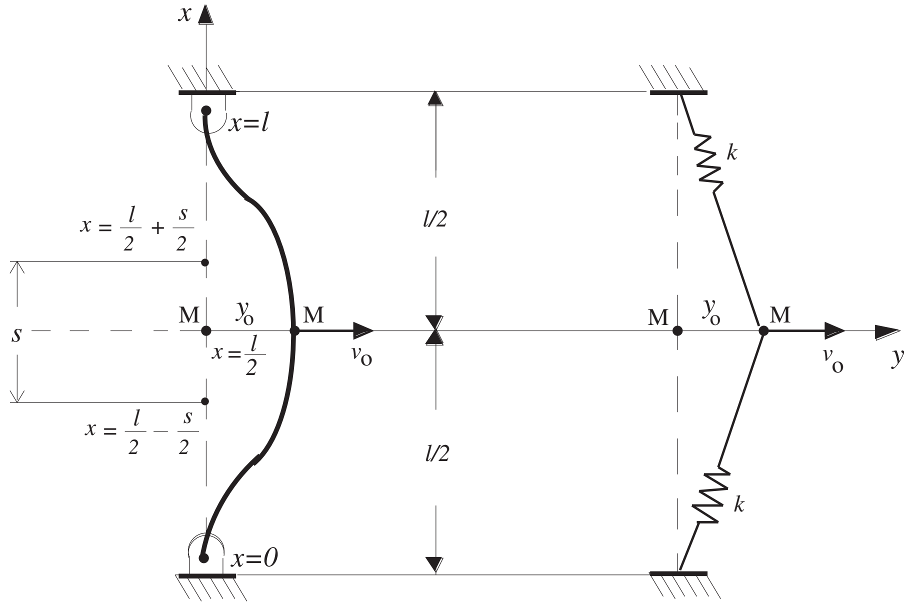

Figure 1: The rod and the oscillator.

Let us multiply both sides of (6) to $\sin\left(\frac{i\pi x}{l}\right)$ and integrate respect to x from 0 to l. One will find:

$$y_0 \int_{\frac{l}{2}-\frac{s}{2}}^{\frac{l}{2}+\frac{s}{2}} \sin\left(\frac{i\pi x}{s}\right) dx = A_i \int_0^l \sin^2\left(\frac{i\pi x}{l}\right) dx,$$

and A_i is then available. Making the same with the v_0 sine expansion, we will find, for any $i = 1, 2, \ldots$:

$$\begin{aligned} A_i &= (-1)^i \frac{4sy_0}{(2i+1)\pi} \sin\left[\frac{(2i+1)\pi l}{2s}\right], \\ B_i &= (-1)^i \frac{4sv_0}{(2i+1)\pi p_i} \sin\left[\frac{(2i+1)\pi l}{2s}\right], \\ p_i &= \frac{i\pi}{l}. \end{aligned}$$

Obviously the rod piece of length s, involved in the initial motion, is a very small portion of the rod. For example, s can be assumed to be $\frac{2l}{100}$, and so on. We are interested in the rod's mean point M motion: then, putting $x = \frac{l}{2}$ in (5), one will get it through the series:

$$y\left(\frac{l}{2}, t\right) = \sum_{i=1}^{\infty} (-1)^{2i} \left[A_i \cos(\omega_i t) + B_i \sin(\omega_i t)\right]. \tag{7}$$

4. From a continuous back to a discrete moving system

At the previous section the problem has been solved of computing by (5) the motion of each point of a continuous body made of a doubly hinged rod, loaded by an imposed motion, namely the *initial* displacement and speed of its middle point. The motion consists of infinite armonics, each having an i^{th} period T_i given by:

$$T_i = \frac{2\pi}{\omega_i} = \frac{2l^2}{\pi i^2}\sqrt{\frac{S\rho}{EJ}}, \tag{8}$$

which is going down for increasing i. But there is no way for appreciating by (8) the *overall* vibration period, and this gives a (quite strong) motivation for trying an approximate model of the continuous body, via a lumped parameter approach.

D. Bernoulli and Euler passed without hesitation from finite systems of particles to continuous systems, by thinking the latter made of infinitely many particles and believing that a proposition, holding for every finite n, would obviously keep its validity for $n \to \infty$.

We are doing the reverse: wishing the motion of one point only of the continuous body, we will focus it, whereas the remaining part of the rod will be simulated through a linear (but inclined) elastic spring without mass.

The mean point is only subjected to an elastic traction T_E given by:

$$T_E = k(l - b)$$

where k is the spring elastic (unknown) characteristic and b the initial length of the strained (half) rod:

$$\left(\frac{l}{2}\right)^2 + y_0^2 = b^2. \tag{9}$$

But the Hooke's law:

$$T_E = S\sigma = SE\varepsilon = \frac{SE(l - b)}{b},$$

where σ is the axial pressure, allows to know the equivalent spring characteristic as $k = \frac{SE}{b}$, with b given by (9). The point mass will be taken as that of the whole rod:

$$m = Sl\rho.$$

Then we have translated the previous model in a formulation ruled by (see fig. 2):

$$a = \frac{l}{2}, \qquad b = \sqrt{\left(\frac{l}{2}\right)^2 + y_0^2},$$

$$m = Sl\rho, \qquad k = \frac{SE}{\sqrt{\left(\frac{l}{2}\right)^2 + y_0^2}}.$$

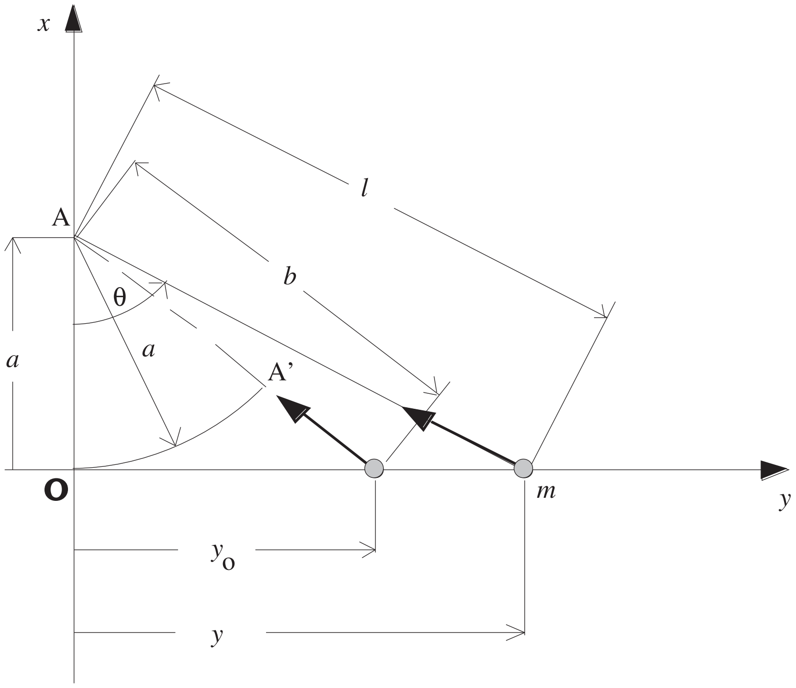

Figure 2: The oscillator's mathematical scheme.

The motion of the $m-$point along the straight line assumed as y axis, is then described by:

$$\begin{cases} m\ddot{y} + T_E \sin\vartheta = 0, \\ y(0) = y_0, \\ \dot{y}(0) = v_0. \end{cases}$$

For any y we have:

$$T_E = k\left(\sqrt{a^2 + y^2} - b\right), \qquad \sin\vartheta = \frac{y}{\sqrt{a^2 + y^2}},$$

then we obtain a second order, nonlinear, initial value problem:

$$\begin{cases} m\ddot{y} + ky\left(1 - \dfrac{b}{\sqrt{a^2+y^2}}\right) = 0, \\ y(0) = y_0, \\ \dot{y}(0) = v_0. \end{cases} \tag{10}$$

The discrete model (10) approaches the continuous phenomenology which, treated as of infinite degrees of freedom, had previously led us to a Fourier series solution.

The discrete model (10) is a continuous one! It shouldn't be confused with those computational methods developed in order to obtain a discretization of a continuous model. They last, properly termed *discretized models*, are designed for being treated numerically, and are out of the authors' interest. The discrete model now introduced, will allow an analytical treatment of the period of the solution (5) .

5. Transverse oscillator equation's integration

The (10) is a second order, nonlinear ODE, modeling *an oscillator* named *transverse* for being its motion directed differently from the elastic force, namely a one-dimensional motion of a particle under a linear elastic force, *but whose direction and intensity are both continuously varying* as the motion itself. Then the problem is a nonlinear one, as a consequence of the nonlinearity in y due to both T_E and $\sin\vartheta$. Of course if $b\left(a^2+y^2\right)^{-1/2} << 1$, one would obtain back the elementary armonic case. The structure of (10) enables us to refer to the Weierstrass theorem, [11]. We are faced with a second order autonomous ODE:

$$\begin{cases} \ddot{y} = f(y), \\ y(0) = y_0, \\ \dot{y}(0) = v_0. \end{cases}$$

By the standard Weierstrass method we can write the time equation as:

$$t = \int_{y_0}^{y} \frac{ds}{\sqrt{\Phi(s)}}, \tag{11}$$

with:

$$\Phi(y) = 2\int_{y_0}^{y} f(s)ds + v_0^2,$$

and where the sign either + or - has to be taken, according to the sign of the initial speed v_0, or, if $v_0 = 0$ accordingly with $f(y_0)$ sign, as it is well known, see e.g. [8] page 114 or [1] pages 287-292. The $\Phi = 0$ roots' existence and nature, marks completely the motion, deciding its periodic or aperiodic nature. Furthermore the reality condition $\Phi \geq 0$ must be met. We have:

$$f(s) = -\frac{k}{m}s\left(1 - \frac{\sqrt{a^2+y_0^2}}{\sqrt{a^2+s^2}}\right),$$

and then:

$$\Phi(y) = 2\,\frac{k}{m}\sqrt{a^2+y^2}\sqrt{a^2+y_0^2} - \frac{k}{m}\left(2\,a^2+y^2+y_0^2\right) + v_0^2.$$

Solving the equation $\Phi(y)=0$, we find the four roots:

$$y_\pm = \pm\sqrt{y_0^2 - \frac{2v_0\sqrt{m}\sqrt{a^2+y_0^2}}{\sqrt{k}} + \frac{m\,v_0^2}{k}}. \tag{12}$$

$$Y_\pm = \pm\sqrt{y_0^2 + \frac{2\,v_0\sqrt{m}\sqrt{a^2+y_0^2}}{\sqrt{k}} + \frac{m\,v_0^2}{k}}, \tag{13}$$

They will be real and distinct,[¶] if:

$$\frac{m\,v_0}{k} + {y_0}^2 - \frac{2\sqrt{m}\sqrt{v_0}\sqrt{a^2+{y_0}^2}}{\sqrt{k}} > 0, \tag{14}$$

then the roots marked by (12) have an absolute value less than (13). Furthermore, if, jointly with the roots' reality condition (14), we have:

$$\sqrt{y_0^2 - \frac{2v_0\sqrt{m}\sqrt{a^2+y_0^2}}{\sqrt{k}} + \frac{m\,v_0^2}{k}} < y_0 \iff \frac{\sqrt{\dfrac{m\,v_0^2-4\,a^2}{k}}}{2} < y_0, \tag{15}$$

then the particle will oscillate bounded between the limits:

$$\sqrt{y_0^2 - \frac{2v_0\sqrt{m}\sqrt{a^2+y_0^2}}{\sqrt{k}} + \frac{m\,v_0^2}{k}} \le y \le \sqrt{y_0^2 + \frac{2\,v_0\sqrt{m}\sqrt{a^2+y_0^2}}{\sqrt{k}} + \frac{m\,v_0^2}{k}},$$

namely between y_+ and Y_+. If (15) is not true, but (14) is met, then the motion would take place between y_- and y_+. If, finally, (14) is not true, the motion will take place between Y_- and Y_+.

We are going to perform the integration of (10), assuming that the initial conditions y_0, v_0 meet both (14) and (15). With some handling, (11) can be written:

$$t = \int_{y_0}^{y} \frac{ds}{\sqrt{2\,\frac{k}{m}\sqrt{a^2+s^2}\sqrt{a^2+y_0^2} - \frac{k}{m}\left(2\,a^2+s^2+y_0^2\right) + \frac{v_0^2}{m}}}, \tag{16}$$

[¶]Of course the reality test (14) ensures the reality of both the roots $y_\pm$ of (12), because the roots $Y_\pm$ of (13) are always real.

with $y_+ \leq y \leq Y_+$. The oscillation half-period $\frac{T}{2}$ will follow integrating between the rest points y_+ e Y_+ :

$$\frac{T}{2} = \int_{y_+}^{Y_+} \frac{ds}{\sqrt{2\,\frac{k}{m}\,\sqrt{a^2+s^2}\,\sqrt{a^2+y_0^2} - \frac{k}{m}\,\left(2\,a^2+s^2+y_0^2\right) + \frac{v_0^2}{m}}}. \tag{17}$$

Let us put in (17) a new variable $s = \sqrt{z^2-a^2}$:

$$T = 2\sqrt{\frac{m}{k}} \int_{z_1}^{z_2} \frac{z}{\sqrt{\left(z^2-a^2\right)\left(-z^2+2\sqrt{a^2+y_0^2}\,z+\frac{m}{k}v_0^2-y_0^2-a^2\right)}}\,dz, \tag{18}$$

where z_1 and z_2 are the roots, always real, of the polynomial $p(z) = -z^2 + 2\sqrt{a^2+y_0^2}\,z + \frac{m}{k}v_0^2 - y_0^2 - a^2$, i.e.:

$$z_1 = \frac{\sqrt{m}\,v_0 - \sqrt{k}\,\sqrt{a^2+y_0^2}}{\sqrt{k}}, \quad z_2 = \frac{\sqrt{m}\,v_0 + \sqrt{k}\,\sqrt{a^2+y_0^2}}{\sqrt{k}}. \tag{19}$$

The period integral (18) is then of the type:

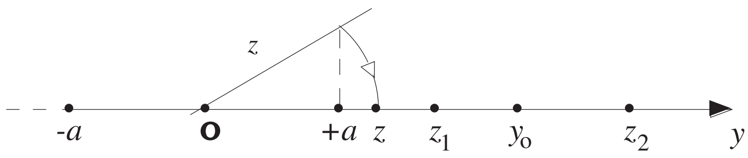

Figure 3: Relative position of z_1 and z_2.

$$\int_{z_1}^{z_2} \frac{z}{\sqrt{-\left(z-z_1\right)\left(z-z_2\right)\left(z^2-a^2\right)}}\,dz.$$

These integrals involve the elliptic functions: e.g. one can see the formulas of [2] page 125, no. 257.11 and page 205 no. 340.01. We will use those numbered as 3.148 6 and 7 of [3] at page 243, coming always from [2]. $F(k,\varphi)$ and $\Pi(k,\varphi,n)$ are the incomplete elliptic integrals of the first and third kind:

$$F(k,\varphi) = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}},$$
$$\Pi(k,\varphi,n) = \int_0^{\varphi} \frac{d\vartheta}{\left(1+n\sin^2\varphi\right)\sqrt{1-k^2\sin^2\vartheta}},$$

being $|k| < 1$ the *modulus* and φ the *amplitude* of both F and Π, and $-\infty < n < +\infty$ the *parameter* of Π.

If $\delta < \gamma < \beta < \alpha$, let us define:

$$\lambda = \arcsin\sqrt{\frac{(\alpha-\gamma)(u-\beta)}{(\alpha-\beta)(u-\gamma)}}, \qquad \mu = \arcsin\sqrt{\frac{(\beta-\delta)(\alpha-u)}{(\alpha-\beta)(u-\delta)}},$$

$$r = \sqrt{\frac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\gamma)(\beta-\delta)}},$$

then, by form. 3.148 6 and 7 [3], we get:

$$\int_\beta^u \frac{x}{\sqrt{(\alpha-x)(x-\beta)(x-\gamma)(x-\delta)}}\,dx = \tag{20}$$
$$= \frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}}\left\{(\beta-\gamma)\,\Pi\left(\lambda, \frac{\alpha-\beta}{\alpha-\gamma}, r\right) + \gamma F(\lambda, r)\right\},$$

if $\delta < \gamma < \beta \leq u < \alpha$. We will have:

$$\int_u^\alpha \frac{x}{\sqrt{(\alpha-x)(x-\beta)(x-\gamma)(x-\delta)}}\,dx = \tag{21}$$
$$= \frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}}\left\{(\alpha-\delta)\,\Pi\left(\mu, \frac{\beta-\alpha}{\beta-\delta}, r\right) + \delta F(\mu, r)\right\},$$

if $\delta < \gamma < \beta < u \leq \alpha$.

We supposed here, for simplicity, that $a < z_1 < z_2$ and $z_1 < \sqrt{a^2+y_0^2} < z_2$. The case $a = z_1$ i.e.:

$$a = \frac{k\,{y_0}^2 - m\,{v_0}^2}{2\sqrt{k}\sqrt{m}\,v_0},$$

is a very special one, and will not be treated here, being quite far from a physical meaning. Then by (20) and (21) we succeeded in integrating the ODE (10) exactly, so that the time equation (16) becomes:

$$t = \sqrt{\frac{k}{m}}\int_{\sqrt{a^2+y_0^2}}^{\sqrt{a^2+y^2}} \frac{z}{\sqrt{(z_2-z)(z-z_1)(z-a)(z+a)}}\,dz,$$

and, evaluating the integral, we finally obtain the time equation:

$$\begin{aligned} t &= \frac{2}{\sqrt{(z_2-a)(z_1+a)}}\sqrt{\frac{m}{k}} \times \\ &\times \left\{(z_2+a)\,\Pi\left(\lambda(y_0), \frac{z_1-z_2}{z_1+a}, r\right) - a\,F(\lambda(y_0), r) + \right. \\ &- \left.(z_1-a)\,\Pi\left(\lambda(y), \frac{z_1-z_2}{z_1+a}, r\right) - a\,F(\lambda(y), r)\right\}, \end{aligned} \tag{22}$$

where:

$$\lambda(y) = \arcsin\sqrt{\frac{(z_1+a)\left(z_2-\sqrt{a^2+y^2}\right)}{(z_2-z_1)\left(\sqrt{a^2+y^2}+a\right)}}, \quad r = \sqrt{\frac{2a\,(z_2-z_1)}{(z_2-a)\,(z_1+a)}},$$

with z_1 and z_2 given by (19) through the system characteristics (m, k, a) and the initial conditions y_0 and v_0.

The authors highlight that their solution (22) is completely new. For istance the excellent and recent treatise [5], page 384, when considering just the same ODE (10), judged that:

> *Exact analytic solutions for the motion of this system are essentially impossible to obtain.*

6. A sample problem

After having solved the nonlinear ODE (10), we wish to provide an example of the behaviour of the solution and its period. This has been done because the time equation (22) holds the displacement y inside the modulus $\lambda(y)$ of both the elliptic integrals, while their amplitudes are constant! So comes an intrinsic computational complexity and then a full discussion, taking into account all the parameters' ranges doesn't worth while. Accordingly, we deemed better to study a sample problem. In any case it is essential to locate carefully the singularities z_1 and z_2 (connected to the rest points) and the other singularities a and $-a$ introduced by the change of variables from s to z.

6.1. The solution

Let it be:

$$a = \frac{1}{2},\ y_0 = \frac{\sqrt{3}}{2},\ b = 1,\ m = 1,\ k = 5,\ v_0 = 1,$$

then we have:

$$f(y) = -5\,y\left(1 - \frac{1}{\sqrt{\frac{1}{4}+y^2}}\right), \qquad \Phi(y) = \frac{5+20\,y^2}{2\sqrt{\frac{1}{4}+y^2}} - \frac{21}{4} - 5\,y^2.$$

Replacing our sample values in (22) we obtain the initial value problem:

$$\begin{cases} \ddot{y} = -5y\left(1 - \dfrac{1}{\sqrt{\frac{1}{4}+y^2}}\right), \\ y(0) = \frac{\sqrt{3}}{2}, \\ \dot{y}(0) = 1. \end{cases} \tag{23}$$

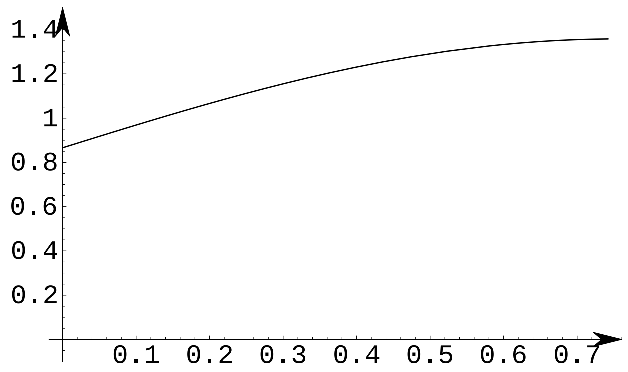

Figure 4: The plot of the exact solution.

We attached the plot of its solution $y = f(t)$ as computed analytically from (22) and inverting. The displacement values are given from the initial datum y_0 to z_2. The solution we found has been found perfectly superimposed with the numerical one as plotted by means of the Mathematica$_{®}$'s packages, VisualDSolve.m, [9] and Ode.m, [4].

To compare the numerical approximate output from (23) with the relevant exact solution (10) provided by the elliptic integrals and the inversion, it would require an analysis on the special functions software quality and implementation.

But this assessment goes outside the authors' interest, who are struggling to detect new integrable systems of Classical and Celestial Mechanics, finding their exact solutions, in terms of elementary and/or special functions.

6.2. The oscillation period

To obtain the oscillation period of our sample problem, the integration range including the poles of the integrand function, has been subdivided: in such a way both the formulas (21) and (20) have been used, with:

$$\lambda = \arcsin\sqrt{\frac{1}{2}+\frac{1}{\sqrt{5}}}, \qquad \mu = \arcsin\sqrt{\frac{1}{2}-\frac{1}{3\sqrt{5}}},$$

$$r = \sqrt{\frac{8\sqrt{5}}{11+4\sqrt{5}}}, \qquad \delta = -a = -\frac{1}{4},$$

$$\gamma = a = \frac{1}{4}, \qquad \beta = 1-\frac{1}{\sqrt{5}},$$

$$\alpha = 1+\frac{1}{\sqrt{5}}, \qquad u = 1.$$

As a consequence of the time equation (22), the oscillation period T will be given by:

$$
\begin{aligned}
5\sqrt{\frac{11}{20}+\frac{1}{\sqrt{5}}}\; T &= -2\sqrt{5}\,F(\mu,r)+2\sqrt{5}\,F(\lambda,r)+ \\
&+ \left(4+6\sqrt{5}\right)\Pi\left(\mu,\frac{4\left(2+3\sqrt{5}\right)}{41},r\right)+ \\
&+ \left(2\sqrt{5}-4\right)\Pi\left(\lambda,4\left(\sqrt{5}-2\right),r\right).
\end{aligned} \tag{24}
$$

The tables of the elliptic integrals allow to know them up to 12 exact decimal digits. We found $T \simeq 3.891179181311$.

Among the elliptic functions' tables it will be recalled that till in 1931 the Wittwer Verlag (Stuttgart) reproduced the Legendre tables (1825) at nine digits for F and E.

But the common reference is actually given by the Jahnke-Emde tables (1948) at four digits. Higher precision computations can be performed via the Spenceley tables (12 digits) issued in 1947 by the Smithsonian Institute of Washington, D.C. Then (24) holds a symbolic formulation that can be be practically accomplished with 12 digits, an accuracy going beyond each practical necessity.

Alternatively, the period could be numerically obtained from (23) by the integral:

$$
T=\frac{2}{\sqrt{5}}\int_{1-\frac{1}{\sqrt{5}}}^{1+\frac{1}{\sqrt{5}}}\frac{x}{\sqrt{-\left[x-\left(1-\frac{1}{\sqrt{5}}\right)\right]\left[x-\left(1+\frac{1}{\sqrt{5}}\right)\right]}\sqrt{x^2-\frac{1}{4}}}\,dx, \tag{25}
$$

which is quite troublesome holding an unbounded integrand.

References

[1] AGOSTINELLI C. AND PIGNEDOLI A., *Meccanica Razionale, vol. 1*, Zanichelli, Bologna 1978.

[2] BYRD P. F. AND FRIEDMAN, M. D., *Handbook of elliptic integrals for engineers and scientists*, Springer-Verlag, New York 1971.

[3] GRADSHTEYN I. S. AND RYZHIK, I. M., *Table of integrals, series and products,* Sixth edition, Academic Press, Inc., San Diego CA 2000.

[4] GRAY A., MEZZINO M. AND PINSKI M. A., *Introduction to Ordinary Differential Equations with Mathematica®*, Springer-Verlag, New York 1997.

[5] JOSÉ J. V. AND SALETAN E. J., *Classical dynamics: a contemporary approach*, Cambridge University Press, Cambridge 1998.

[6] MINGARI SCARPELLO G. AND RITELLI D., *Closed form integration of the rotating plane pendulum nonlinear equation*, Tamkang Journal of Mathematics, to appear.

[7] MINGARI SCARPELLO G. AND RITELLI D., *A nonlinear oscillation induced by two fixed gravitating centres*, International Mathematical Journal, to appear.

[8] ROY L., *Cours de Mécanique Rationelle, tome I*, Gauthier-Villars, Paris 1945.

[9] SCHWALBE D. AND WAGON S., *VisualD Solve*, Springer-Verlag, New York 1997.

[10] SETO W. W., *Theory and problems of mechanical vibrations*, Schaum P.C., New York 1964.

[11] WEIERSTRASS K., *Über eine Gattung reel periodischer Funktionen*, Math. Werke II, Mayer & Muller Berlin 1895, 1–18.

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