Rend. Sem. Mat. Univ. Pol. Torino
Vol. 61, 1 (2003)

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EXISTENCE AND UNIQUENESS OF PERIODIC SOLUTIONS FOR A MODEL OF CONTAMINANT FLOW IN POROUS MEDIUM

Abstract. This paper deals with the existence and uniqueness of the weak periodic solution for a model of transport of a pollutant flow in a porous medium. Our model is described by means of a nonlinear degenerate parabolic problem. To prove the existence of periodic solutions, we use as preliminary steps the Schauder fixed point theorem for the Poincaré map of a nondegenerate initial–boundary value problem associated to ours and the a–priori estimates deduced on these solutions. Our uniqueness result follows from a more general result which shows the continuous dependence of solutions with respect to the data. As another consequence of this general result we prove a comparison principle for periodic solutions.

1. Introduction

In this paper we consider a nonlinear parabolic problem which arises from a model of transport for a pollutant flow in a porous medium (see [3]).

$$(P)\left\{\begin{array}{c} u_t = div(\nabla\varphi(u) - \psi(u)\mathbf{V}(x,t)), \text{ in } Q_T := \Omega\times(0,T) \\ (\nabla\varphi(u) - \psi(u)\mathbf{V}(x,t))\cdot\mathbf{n} = g(x,t), \text{ on } S_T := \partial\Omega\times(0,T) \\ u(x,t+\omega) = u(x,t), \text{ in } Q_T,\ T\geq\omega>0 \end{array}\right\}$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, $\mathbf{n}$ denotes the outward unit normal vector on the boundary $\partial\Omega$. The increased demand for water in various parts of the word, makes very important the problem of the water quality for the development and use of water resources.

Special attention should be devoted to the pollution of groundwater in acquifers and surface water. The term pollutant shall be used to denote dissolved matter carried with water. We deal with the transport of mass of certain solute that moves with the water in the interstices of an inhomogeneous porous medium. At every point within a porous medium, we have the product $\psi(u)\mathbf{V}(x,t)$ between the liquid velocity $\mathbf{V}(x,t)$ and a nonlinear function $\psi(u)$ of the concentration u of the pollutant. The term $\psi(u)\mathbf{V}(x,t)$, represents the advective flux i.e. the flux carried by the water at the velocity $\mathbf{V}(x,t)$. The fundamental balance equation for the transport of a pollutant concentration in a porous medium, is given by the advenctive–dispersion equation $u_t = div(\nabla\varphi(u) - \psi(u)\mathbf{V}(x,t))$.

2. Preliminaries

We study the problem (P) under the assumptions:

$$H_\varphi)\ \varphi \in C([0,\infty)) \cap C^1((0,\infty)),\ \varphi(0)=0,\ \varphi'(s)>0 \text{ for } s \neq 0\,;$$

$H_\psi)$ $\psi \in C([0,\infty)) \cap C^1((0,\infty))$, $\psi(0)=0$, ψ locally Lipschitz continuous;
$H_V)$ $\mathbf{V} \in \prod_{i=1}^n C(\overline{Q}_T) \cap C^1(Q_T)$, $\mathbf{V}(x,.)$ is ω–periodic, $div\mathbf{V}(x,t)=0$ in Q_T and $\mathbf{V}(x,t)\cdot\mathbf{n}<0$ on S_T;
$H_g)$ $g \in L^\infty(S_T)$, $g>0$, $g(x,.)$ is ω–periodic and admits an extension on all Q_T such that $g_x \in L^\infty(Q_T)$.

REMARK 1. The assumptions $H_\varphi)$ and $H_\psi)$ include both the case of degenerate equations i.e. $\varphi'(0)=0$ and $\psi'(0)=\pm\infty$, while the assumptions $H_g)$ allows to apply the result of [5, thm. 6.2].

DEFINITION 1. *A function $u \in C([0,T];\ L^2(\Omega)) \cap L^\infty(Q_T)$, is a periodic weak solution to (P), if $u(x,t+\omega)=u(x,t)$, $\varphi(u) \in L^2((0,T);\ H^1(\Omega))$ and*

$$\int_0^T \int_\Omega (u\zeta_t + \varphi(u)\Delta\zeta + \psi(u)\mathbf{V}(x,t)\cdot\nabla\zeta)dxdt+ \tag{1}$$

$$\int_0^T \int_{\partial\Omega} (g(x,t)\zeta - \varphi(u)\frac{\partial\zeta}{\partial\mathbf{n}})dSdt = \int_\Omega (u(x,T)\zeta(x,T) - u(x,0)\zeta(x,0))dx$$

for any ζ such that $\zeta,\ \zeta_t,\ \Delta\zeta \in L^2(Q_T)$ and $\frac{\partial\zeta}{\partial\mathbf{n}} \in L^2(S_T)$.

The existence of the positive weak periodic solutions for the problem (P) shall be obtained as the limit of approximated periodic solutions whose existence is showed by means of the Schauder fixed point theorem, applied to the Poincaré map of a nondegenerate initial–boundary value problem associated to (P). In the light of what has been said, we begin by proving the existence of the positive periodic solutions for the approximated nondegenerate problem

$$(P_\varepsilon)\left\{\begin{array}{c} u_{\varepsilon t} = div(\nabla\varphi_\varepsilon(u_\varepsilon) - \psi_\varepsilon(u_\varepsilon)\mathbf{V}(x,t)),\ \text{ in } Q_T \\ (\nabla\varphi_\varepsilon(u_\varepsilon) - \psi_\varepsilon(u_\varepsilon)\mathbf{V}(x,t))\cdot\mathbf{n} = g_\varepsilon(x,t),\ \text{ on } S_T \\ u_\varepsilon(x,t+\omega) = u_\varepsilon(x,t),\ \text{ in } Q_T \end{array}\right\}$$

where

$$H_{\varphi_\varepsilon})\ \varphi_\varepsilon \in C^1([0,\infty)),\ \varphi_\varepsilon(0)=0,\ \varphi_\varepsilon'(s)\geq\varepsilon,\ \varphi_\varepsilon(s)=\varphi(s) \text{ if } s\geq\varepsilon/2$$

and $\varphi_\varepsilon \to \varphi$ uniformly on compact sets of R_+ as $\varepsilon \to 0^+$;
$H_{\psi_\varepsilon})$ $\psi_\varepsilon \in C^1([0,\infty))$, $\psi_\varepsilon(0)=0$, with $\psi_\varepsilon(s)=\psi(s)$ if $s\geq\varepsilon/2$
and $\psi_\varepsilon \to \psi$ uniformly on compact sets of R_+ as $\varepsilon \to 0^+$;
$H_{g_\varepsilon})$ $g_\varepsilon \in C^\infty(\overline{Q}_T)$, $g_\varepsilon>0$, $g_\varepsilon(x,.)$ is ω–periodic and $g_\varepsilon \to g$

uniformly on compact sets of $\overline{Q}_T$ as $\varepsilon \to 0^+$.

The existence of the positive periodic solutions to (P_ε) derives from the Schauder fixed point theorem for the Poincaré map of the associated initial–boundary value problem

$$(P'_\varepsilon)\left\{\begin{array}{c} u_{\varepsilon t} = div(\nabla\varphi_\varepsilon(u_\varepsilon) - \psi_\varepsilon(u_\varepsilon)\mathbf{V}(x,t)),\ \text{in } Q_T \\ (\nabla\varphi_\varepsilon(u_\varepsilon) - \psi_\varepsilon(u_\varepsilon)\mathbf{V}(x,t))\cdot\mathbf{n} = g_\varepsilon(x,t),\ \text{on } S_T \\ u_\varepsilon(x,0) = u_{0\varepsilon},\ \text{in } \Omega \end{array}\right\}$$

where

$H_{0\varepsilon}$) $u_{0\varepsilon} \in C^2(\overline{\Omega})$, $u_{0\varepsilon} \geq 0$ for all $x \in \overline{\Omega}$ and satisfies the compatibility condition $(\nabla\varphi_\varepsilon(u_{0\varepsilon}(x)) - \psi_\varepsilon(u_{0\varepsilon}(x))\mathbf{V}(x,0))\cdot\mathbf{n} = g_\varepsilon(x,0)$, on $\partial\Omega$.

The uniqueness of the positive weak periodic solutions, follows from a more general result which shows the continuous dependence of the solutions with respect to the data. This result shall be established extending, to our periodic case, the method utilized in [4], [6], [8] for the study of the Cauchy or the Cauchy–Dirichlet problems. As a conclusive fact of this extention, we show a comparison principle for the periodic solutions. According to the knowledges of the author, the topic considered here has not been discussed previously, in the literature. Related papers to ours are [1] where the blow–up in finite time is studied for a problem of reaction–diffusion and [2] where the existence and uniqueness of the solution for a non periodic problem (P) is showed in a unbounded domain Ω. See also [9].

3. Existence and uniqueness for the approximating problem

The classical theory of parabolic equations asserts that the problem (P'_ε) has a unique solution $u_\varepsilon \in C^{2,1}(\overline{Q}_T)$. Moreover, problem (P'_ε) has ε as a lower–solution if we assume that

$$-\psi(\varepsilon)\mathbf{V}(x,t)\cdot\mathbf{n} \leq g_\varepsilon(x,t),\ \text{on } S_T\ .$$

If we suppose that there exists a constant $M > 0$ such that $\psi(M) > 0$ and

$$-\psi(M)\mathbf{V}(x,t)\cdot\mathbf{n} \geq g_\varepsilon(x,t),\ \text{on } S_T\ ,$$

then, M is an upper–solution for (P'_ε).

If $u_{0\varepsilon}$ verifies

$$\varepsilon \leq u_{0\varepsilon}(x) \leq M\ ,\ \text{for all } x \in \overline{\Omega}\ , \tag{1}$$

the comparison principle asserts that

$$\varepsilon \leq u_\varepsilon(x,t) \leq M\ ,\ \text{in } \overline{Q}_T\ . \tag{2}$$

For $\varphi(u_\varepsilon)$ holds this uniform estimate

PROPOSITION 1. *There exists a constant $C > 0$, independent of ε, such that*

$$\int_0^T \|\varphi(u_\varepsilon)\|^2_{H^1(\Omega)}dt \leq C\ . \tag{3}$$

Proof. Multiply the equation in (P'_ε) by $\varphi(u_\varepsilon)$ and integrate by parts using Young's inequality, we have

$$\frac{d}{dt}\int_\Omega \Phi_\varepsilon(u_\varepsilon)dx + \frac{1}{2}\int_\Omega |\nabla\varphi(u_\varepsilon)|^2 dx \leq \tag{4}$$

$$\int_{\partial\Omega} g_\varepsilon(x,t)\varphi(M)dS + \frac{1}{2}\int_\Omega \|\mathbf{V}(x,t)\|^2_{R^n}|\psi(u_\varepsilon)|^2 dx \ ,$$

where $\Phi_\varepsilon(u_\varepsilon) := \int_\varepsilon^{u_\varepsilon}\varphi(s)ds$. Integrating (4) over $(0,T)$, one has

$$\int_\Omega \Phi_\varepsilon(u_\varepsilon(x,T))dx - \int_\Omega \Phi_\varepsilon(u_{0\varepsilon}(x))dx + \frac{1}{2}\int_0^T\int_\Omega |\nabla\varphi(u_\varepsilon)|^2 dxdt \leq C_1$$

and from

$$\frac{d}{du_\varepsilon}|\varphi(u_\varepsilon)|^2 = 2\varphi(u_\varepsilon)\varphi'(u_\varepsilon) \leq 2C_2\varphi(u_\varepsilon) \ ,$$

$(C_2 = \sup\{|\varphi'(\xi)|, \varepsilon < \xi < M\})$, one obtains

$$|\varphi(u_\varepsilon)|^2 \leq 2C_2\Phi_\varepsilon(u_\varepsilon) + |\varphi(\varepsilon)|^2 \ .$$

By (1) follows that

$$\int_0^T\int_\Omega |\varphi(u_\varepsilon)|^2 xdt + \frac{1}{2}\int_0^T\int_\Omega |\nabla\varphi(u_\varepsilon)|^2 dxdt \leq C \ .$$

□

Taking into account that $u_{0\varepsilon} \in C^2(\overline{\Omega})$, we can utilize the regularity result given in [5], which establishes that the sequence of solutions u_ε is equicontinuous in $\overline{Q}_T$.

PROPOSITION 2. ([5]). *If $u_{0\varepsilon}$ is continuous on $\overline{\Omega}$, then the sequence $\{u_\varepsilon\}$ of solutions of (P'_ε) is equicontinuous in $\overline{Q}_T$ in the sense that there exists $\omega_0 : R_+ \to R_+$, $\omega_0(0) = 0$ continuous and nondecreasing such that*

$$|u_\varepsilon(x_1,t_1) - u_\varepsilon(x_2,t_2)| \leq \omega_0(|x_1 - x_2| + |t_1 - t_2|^{1/2}) \ , \tag{5}$$

for any (x_1,t_1), $(x_2,t_2) \in \overline{Q}_T$.

In order to mobilize the Schauder fixed point theorem, we introduce the closed and convex set

$$K_\varepsilon := \{w \in C(\overline{\Omega}) : \varepsilon \leq w(x) \leq M, \ \forall x \in \overline{\Omega}\}$$

and the Poincaré map associated to the problem (P'_ε), defined as follows

$$\mathcal{F}(u_{0\varepsilon}(.)) = u_\varepsilon(.,\omega)$$

where u_ε is the unique solution of (P'_ε).

From the formula (2) and the Proposition 2, we deduce that

i) $\mathcal{F}(K_\varepsilon) \subset K_\varepsilon$

ii) $\mathcal{F}(K_\varepsilon)$ is relatively compact in $C(\overline{\Omega})$.

It remains to prove that

iii) $\mathcal{F}|_{K_\varepsilon}$ is continuous.

PROPOSITION 3. *If $u^n_{0\varepsilon}$, $u_{0\varepsilon} \in K_\varepsilon$ and $u^n_{0\varepsilon} \to u_{0\varepsilon}$ uniformly on $\overline{\Omega}$ as $n \to \infty$, then, if u^n_ε and u_ε are solutions of (P'_ε) with initial data $u^n_{0\varepsilon}$ and $u_{0\varepsilon}$ respectively, we have that $u^n_\varepsilon(.,t)$ converges to $u_\varepsilon(.,t)$ uniformly as $n \to \infty$ for any $t \in [0,T]$.*

Proof. Multiplying the equation in (P'_ε) by $sgn(u^n_\varepsilon - u_\varepsilon)$ and integrating over Q_t, we get

$$\frac{d}{dx}\int_0^t \int_\Omega |u^n_\varepsilon(x,s) - u_\varepsilon(x,s)| dx ds = 0$$

i.e.

$$\int_\Omega |u^n_\varepsilon(x,t) - u_\varepsilon(x,t)| dx = \int_\Omega |u^n_{0\varepsilon}(x) - u_{0\varepsilon}(x)| dx.$$

The uniform convergence of $u^n_{0\varepsilon}(x) \to u_{0\varepsilon}(x)$ when $n \to \infty$, implies that u^n_ε strongly converges to u_ε in $L^1(\Omega)$ as n goes to infinity. Consequently, for a subsequence, $u^n_\varepsilon(x,t)$ converges to $u_\varepsilon(x,t)$ a.e. $x \in \Omega$. Since $\varepsilon \le u^n_\varepsilon(x,t) \le M$, the Lebesgue theorem allows to conclude that $u^n_\varepsilon \to u_\varepsilon$ in $L^p(\Omega)$, for any $1 \le p \le \infty$. The uniform convergence of $u^n_\varepsilon(.,t)$ to $u_\varepsilon(.,t)$ when $n \to \infty$ is due to the fact that $u^n_\varepsilon(.,t)$, $u_\varepsilon(.,t) \in C(\overline{\Omega})$ for any $t \in [0,T]$.

□

Now, we can apply the Schauder fixed point theorem and conclude that the Poincaré map has at least one point, which is a periodic solution of (P'_ε). Closing this section, we state our main result

THEOREM 1. *If the assumptions $H_\varphi) - H_g)$ hold, there exist positive weak ω–periodic solutions to the problem (P).*

Proof. When $\varepsilon \to 0^+$, the above estimates and the compactness result yield

(6) $\qquad u_\varepsilon \to u$ uniformly on $\overline{Q}_T$, by the Ascoli–Arzelà theorem

and

(7) $\qquad u_\varepsilon \to u$ strongly in $L^2(Q_T)$, because of (6) and (2) .

From (4), one has

(8) $\qquad \varphi(u_\varepsilon) \to \varphi(u)$ in $L^2((0,T); H^1(\Omega))$,

while (6) and the Lebesgue theorem imply that

(9) $\qquad \varphi(u_\varepsilon) \to \varphi(u)$ in $L^2(Q_T)$.

Finally, assumption $H_{g_\varepsilon})$ gives

$$g_\varepsilon(x,t) \to g(x,t) \text{ uniformly on } \partial\Omega \times [0,T]\,. \tag{10}$$

This easily leads to conclude that u satisfies (1).

$\square$

4. Uniqueness and comparison principle

To obtain the uniqueness of the solutions, we need some preliminary inequalities. Let u_ε and v_ε be any positive ω–periodic solutions of (P_ε) with boundary data g_ε, g_ε^* respectively, such that

$$\varepsilon \le \max\{u_\varepsilon(x,t), v_\varepsilon(x,t)\}$$

then,

$$\begin{aligned}(11)\quad &\int_0^T \int_\Omega [(u_\varepsilon - v_\varepsilon)\zeta_t + (\varphi(u_\varepsilon) - \varphi(v_\varepsilon))\Delta\zeta + (\psi(u_\varepsilon) - \psi(v_\varepsilon))\mathbf{V}(x,t)\cdot\nabla\zeta]dxdt + \\ &\int_0^T \int_{\partial\Omega} (g_\varepsilon(x,t) - g_\varepsilon^*(x,t))\zeta(x,t)dSdt - \int_0^T \int_{\partial\Omega} (\varphi(u_\varepsilon) - \varphi(v_\varepsilon))\frac{\partial\zeta}{\partial\mathbf{n}})dSdt = \\ &\int_\Omega (u_\varepsilon(x,T) - v_\varepsilon(x,T))\zeta(x,T)dx - \int_\Omega (u_\varepsilon(x,0) - v_\varepsilon(x,0))\zeta(x,0)dx,\end{aligned}$$

for any ζ such that $\zeta, \zeta_t, \Delta\zeta \in L^2(Q_T)$ and $\frac{\partial\zeta}{\partial\mathbf{n}} \in L^2(S_T)$.

Define

$$A_\varepsilon(x,t) := \frac{\varphi(u_\varepsilon) - \varphi(v_\varepsilon)}{u_\varepsilon - v_\varepsilon} = \int_0^1 \varphi'(\theta u_\varepsilon(x,t) + (1-\theta)v_\varepsilon(x,t))d\theta$$

$$B_\varepsilon(x,t) := \frac{\psi(u_\varepsilon) - \psi(v_\varepsilon)}{u_\varepsilon - v_\varepsilon} = \int_0^1 \psi'(\theta u_\varepsilon(x,t) + (1-\theta)v_\varepsilon(x,t))d\theta,$$

so that (11) becomes

$$\begin{aligned}(12)\quad &\int_0^T \int_\Omega (u_\varepsilon - v_\varepsilon)(\zeta_t + A_\varepsilon(x,t)\Delta\zeta + B_\varepsilon(x,t)\mathbf{V}(x,t)\cdot\nabla\zeta)dxdt = \\ &\int_\Omega (u_\varepsilon(x,T) - v_\varepsilon(x,T))\zeta(x,T)dx - \int_\Omega (u_\varepsilon(x,0) - v_\varepsilon(x,0))\zeta(x,0)dx + \\ &\int_0^T \int_{\partial\Omega} (g_\varepsilon^*(x,t) - g_\varepsilon(x,t))\zeta(x,t)dSdt + \int_0^T \int_{\partial\Omega} (\varphi(u_\varepsilon) - \varphi(v_\varepsilon))\frac{\partial\zeta}{\partial\mathbf{n}})dSdt.\end{aligned}$$

There exist some positive constants α and L depending only on ε and M, such that (see [6])

$$\varepsilon \le A_\varepsilon(x,t) \le \alpha := \sup\{\varphi'(s), \varepsilon \le s \le M\}\,, \quad \forall\,(x,t) \in \overline{Q}_T$$

$$|B_\varepsilon(x,t)| \le L := \sup\{|\psi'(s)|, \varepsilon \le s \le M\}\,, \quad \forall\,(x,t) \in \overline{Q}_T .$$

Let $\zeta_{\varepsilon,m}$ be the solution of the backward linear parabolic problem with smooth coefficients

$$(P_{\zeta_{\varepsilon,m}}) \left\{ \begin{array}{c} \zeta_{\varepsilon,mt} + A_{\varepsilon,m}(x,t)\Delta\zeta_{\varepsilon,m} + B_{\varepsilon,m}(x,t)\mathbf{V}(x,t)\cdot\nabla\zeta_{\varepsilon,m} = f, \text{ in } Q_T \\ \zeta_{\varepsilon,m}(x;T) = \Theta(x), \text{ in } \overline{\Omega} \\ \nabla\zeta_{\varepsilon,m}\cdot\mathbf{n} = 0, \text{ on } S_T \end{array} \right\}$$

with $A_{\varepsilon,m}$, $B_{\varepsilon,m}$, $f \in C^\infty(\overline{Q}_T)$, $A_{\varepsilon,m} \to A_\varepsilon$, $B_{\varepsilon,m} \to B_\varepsilon$ uniformly on $\overline{Q}_T$ as m goes to infinity and $\Theta \in C_0^\infty(\overline{\Omega})$, $0 \le \Theta(x) \le 1$.

Also for $A_{\varepsilon,m}$ and $B_{\varepsilon,m}$ hold

$$\varepsilon \le A_{\varepsilon,m}(x,t) \le \alpha, \ \forall\,(x,t) \in \overline{Q}_T$$

$$|B_{\varepsilon,m}(x,t)| \le L, \ \forall\,(x,t) \in \overline{Q}_T .$$

The existence, uniqueness and regularity of $\zeta_{\varepsilon,m}(x,t)$ as solution of $(P_{\zeta_{\varepsilon,m}})$ follow from the classical theory of linear parabolic equations with smooth coefficients (see [7]).

For the solution $\zeta_{\varepsilon,m}$ the following estimates hold

LEMMA 1. *Let $\zeta_{\varepsilon,m}(x,t)$ be the solution of $(P_{\zeta_{\varepsilon,m}})$, then*

(13) $$\max_{\overline{Q}_T} |\zeta_{\varepsilon,m}(x,t)| \le k_1 = k_1(\|f\|_{\infty,\overline{Q}_T}) .$$

(14) $$\int_0^T \int_\Omega |\nabla\zeta_{\varepsilon,m}(x,t)|^2 dxdt \le k$$

(15) $$\int_0^T \int_\Omega |\Delta\zeta_{\varepsilon,m}(x,t)|^2 dxdt \le k$$

where $k := k(\varepsilon, L, \|f\|_{2,Q_T})$. Moreover, if $f \le 0$ one has

(16) $$0 \le \zeta_{\varepsilon,m}(x,t)\,, \quad \forall\,(x,t) \in \overline{Q}_T .$$

Proof. Inequalities (13) and (16) are a straightforward consequence of the maximum principle. To prove (14), multiply the equation in $(P_{\zeta_{\varepsilon,m}})$ by $\Delta\zeta_{\varepsilon,m}$ and integrate by parts over $\overline{\Omega} \times [\tau, T]$. This yields

$$-\int_\tau^T \int_\Omega \nabla\zeta_{\varepsilon,m}(x,t)\nabla\zeta_{\varepsilon,mt}(x,t)dxdt + \int_\tau^T \int_\Omega A_{\varepsilon,m}(x,t)|\Delta\zeta_{\varepsilon,m}(x,t)|^2 dxdt +$$

$$\int_\tau^T \int_\Omega B_{\varepsilon,m}(x,t)\Delta\zeta_{\varepsilon,m}(x,t)\mathbf{V}(x,t)\cdot\nabla\zeta_{\varepsilon,m}(x,t)dxdt = \int_\tau^T \int_\Omega f(x,t)\Delta\zeta_{\varepsilon,m}(x,t)dxdt .$$

Applying Young's inequality, we get

$$-\frac{1}{2}\int_{\Omega}|\nabla\Theta(x)|^2dx+\frac{1}{2}\int_{\Omega}|\nabla\zeta_{\varepsilon,m}(x,\tau)|^2dx+ \tag{17}$$

$$\frac{\varepsilon}{4}\int_{\tau}^{T}\int_{\Omega}|\nabla\zeta_{\varepsilon,m}(x,t)|^2dxdt\le$$

$$\frac{C_3L^2}{\varepsilon}\int_{\tau}^{T}\int_{\Omega}|\nabla\zeta_{\varepsilon,m}(x,\tau)|^2dxdt+\frac{1}{2\varepsilon}\int_{\tau}^{T}\int_{\Omega}|f(x,t)|^2dxdt\ ,$$

($C_3=\sup\{\|\mathbf{V}(x,t)\|^2_{R^n},\ (x,t)\in\overline{Q}_T\}$) from which

$$\frac{1}{2}\int_{\Omega}|\nabla\zeta_{\varepsilon,m}(x,\tau)|^2dx\le\frac{1}{2}\int_{\Omega}|\nabla\Theta(x)|^2dx+$$

$$\frac{1}{2\varepsilon}\int_{\tau}^{T}\int_{\Omega}|f(x,t)|^2dxdt+\frac{C_3L^2}{\varepsilon}\int_{\tau}^{T}\int_{\Omega}|\nabla\zeta_{\varepsilon,m}(x,\tau)|^2dxdt\ .$$

Gronwall's inequality and integration with respect to τ gives (14) and therewith by substitution in (17)), the (15).

□

PROPOSITION 4. *For any $f\in C^{\infty}(\overline{Q}_T)$ and any $\Theta\in C_0^{\infty}(\overline{\Omega})$, $0\le\Theta(x)\le 1$ we have*

$$\int_{\Omega}(u_\varepsilon(x,T)-v_\varepsilon(x,T))\Theta(x)dx-\int_0^T\int_{\Omega}(u_\varepsilon(x,t)-v_\varepsilon(x,t))f(x,t)dxdt\le \tag{18}$$

$$k_1\left(\int_{\Omega}|u_\varepsilon(x,0)-v_\varepsilon(x,0)|dx+\int_0^T\int_{\partial\Omega}|g_\varepsilon(x,t)-g^*_\varepsilon(x,t)|dSdt\right)\ .$$

If $f\le 0$, then

$$\int_{\Omega}(u_\varepsilon(x,T)-v_\varepsilon(x,T))\Theta(x)dx-\int_0^T\int_{\Omega}(u_\varepsilon(x,t)-v_\varepsilon(x,t))f(x,t)dxdt\le \tag{19}$$

$$k_1\left(\int_{\Omega}(u_\varepsilon(x,0)-v_\varepsilon(x,0))^+dx+\int_0^T\int_{\partial\Omega}(g_\varepsilon(x,t)-g^*_\varepsilon(x,t))^+dSdt\right)\ .$$

Proof. Substituting $\zeta_{\varepsilon,m}$ in (12) we obtain

$$\int_0^T\int_{\Omega}(u_\varepsilon-v_\varepsilon)[f(x,t)+(A_\varepsilon(x,t)-A_{\varepsilon,m}(x,t))\Delta\zeta_{\varepsilon,m}(x,t)+ \tag{20}$$

$$(B_\varepsilon(x,t)-B_{\varepsilon,m}(x,t))\mathbf{V}(x,t)\cdot\nabla\zeta_{\varepsilon,m}]dxdt=$$

$$\int_{\Omega}(u_\varepsilon(x,T)-v_\varepsilon(x,T))\Theta(x)dx-\int_{\Omega}(u_\varepsilon(x,0)-v_\varepsilon(x,0))\zeta_{\varepsilon,m}(x,0)dx+$$

$$\int_0^T \int_{\partial\Omega} (g_\varepsilon^*(x,t) - g_\varepsilon(x,t))\zeta_{\varepsilon,m}(x,t)dSdt\ .$$

By Lemma 1, one concludes that

$$\begin{aligned}
(21) \qquad & \int_\Omega (u_\varepsilon(x,T) - v_\varepsilon(x,T))\Theta(x)dx - \\
& \int_0^T \int_\Omega (u_\varepsilon(x,t) - v_\varepsilon(x,t)) f(x,t)dxdt \le k_1 \left(\int_\omega |u_\varepsilon(x,0) - v_\varepsilon(x,0)|dx + \right. \\
& \left. \int_0^T \int_{\partial\Omega} |g_\varepsilon(x,t) - g_\varepsilon^*(x,t)|dSdt \right) + \\
& \left(\max_{\overline{Q}_T} |u_\varepsilon(x,t) - v_\varepsilon(x,t)| \right) \left\{ \max_{\overline{Q}_T} |A_\varepsilon(x,t) - A_{\varepsilon,m}(x,t)|(kT|\Omega|)^{1/2} + \right. \\
& \left. \max_{\overline{Q}_T} |B_\varepsilon(x,t) - B_{\varepsilon,m}(x,t)|\sqrt{k} \left(\int_0^T \int_\Omega \|\mathbf{V}(x,t)\|^2_{R^n} dxdt \right)^{1/2} \right\}.
\end{aligned}$$

Passing to the limit in (19) as $m \to \infty$, one obtains the desired result.

□

COROLLARY 1. *Let u and v be any periodic solutions to (P) with boundary data g, g^*, respectively. Then one has*

$$\begin{aligned}
(22) \qquad & \int_\Omega (u(x,T) - v(x,T))^+ dx \le k_1 \left(\int_0^T \int_{\partial\Omega} (g(x,t) - g^*(x,t))^+ dSdt + \right. \\
& \left. \int_\Omega (u(x,0) - v(x,0))^+ dx \right)
\end{aligned}$$

and the continuous dependence result

$$\begin{aligned}
(23) \qquad & \int_0^T \int_\Omega |u(x,t) - v(x,t)|^2 dxdt \le k_1 \left(\int_\Omega |u(x,0) - v(x,0)|dx + \right. \\
& \left. \int_0^T \int_{\partial\Omega} |g(x,t) - g^*(x,t)|dSdt \right).
\end{aligned}$$

Proof. Choosing $f(x,t) \equiv 0$ and $\Theta = \Theta_j \in C_0^\infty(\overline{\Omega})$, with $\Theta_j \to sgn_+(u_\varepsilon(x,T) - v_\varepsilon(x,T))$ in $L^1(\Omega)$ as $j \to \infty$, inequality (17) gives

$$\begin{aligned}
(24) \qquad & \int_\Omega (u_\varepsilon(x,T) - v_\varepsilon(x,T))^+ dx \le k_1 \left(\int_\Omega (u_\varepsilon(x,0) - v_\varepsilon(x,0))^+ dx + \right. \\
& \left. \int_0^T \int_{\partial\Omega} (g_\varepsilon(x,t) - g_\varepsilon^*(x,t))^+ dSdt \right).
\end{aligned}$$

Taking the limit as $\varepsilon \to 0^+$ in (24), we have (22). The (23) is deduced by $\Theta \equiv 0$, $f = f_j \in C^\infty(\overline{Q}_T)$, $f_j \to -(u_\varepsilon - v_\varepsilon)$ in $L^2(Q_T)$ as $j \to \infty$ and letting $\varepsilon \to 0^+$.

□

Our main result of this section is

THEOREM 2. *Problem (P) has a unique positive ω–periodic weak solution.*

Proof. Choosing $T = n\omega$ in (23), the periodicity of u and v gives us

$$n \int_0^\omega \int_\Omega |u(x,t) - v(x,t)|^2 dxdt \leq k_1 \Bigg(\int_\Omega |u(x,0) - v(x,0)| dx \leq k_2$$

for any $n \in N$, hence $u = v$.

□

Finally, we can to show a comparison result for the positive ω–periodic weak solutions

COROLLARY 2. *Let u and v be positive ω–periodic weak solutions to (P) with boundary data g, g^*. If $g \leq g^*$, then $u \leq v$ in Q_ω.*

Proof. Under the above assumptions, formula (17), with $\Theta \equiv 0$, $f_j \in L^\infty(Q_T)$, $f_j \to sgn_+(u(x,t) - v(x,t))$ in $L^\infty(Q_T)$, says in the limit $\varepsilon \to 0^+$

$$\int_0^T \int_\Omega (u(x,t) - v(x,t))^+ dxdt \leq k_1 \Bigg(\int_\Omega (u(x,0) - v(x,0))^+ dx +$$

$$\int_0^T \int_{\partial\Omega} (g(x,t) - g^*(x,t))^+ dSdt \Bigg) .$$

Taking $T = n\omega$, one has

$$n \int_0^\omega \int_\Omega (u(x,t) - v(x,t))^+ dxdt \leq 2k_1 N |\Omega| ,$$

where $N \geq \max\{\|u\|_{\infty,Q_T}, \|v\|_{\infty,Q_T}\}$, which implies $u(x,t) \leq v(x,t)$ in Q_ω.

□

Acknowledgment. The author is very grateful to the referee.

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MSC Classification: 35K65, 35D05, 35B10, 76S05.

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Lavoro pervenuto in redazione il 20.01.2003 e, in forma definitiva, il 26.05.2003.

Rend. Sem. Mat. Univ. Pol. Torino
Vol. 61, 1 (2003)

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LOCAL SOLVABILITY FOR SEMILINEAR PARTIAL DIFFERENTIAL EQUATIONS OF CONSTANT STRENGTH

Abstract. The main goal of the present paper is to study the local solvability of semilnear partial differential operators of the form

$$F(u) = P(D)u + f(x, Q_1(D)u, ..., Q_M(D)u),$$

where $P(D), Q_1(D), ..., Q_M(D)$ are linear partial differntial operators of constant coefficients and $f(x, v)$ is a C^∞ function with respect to x and an entire function with respect to v.
Under suitable assumptions on the nonlinear function f and on $P, Q_1, ..., Q_M$, we will solve locally near every point $x^0 \in \mathbb{R}^n$ the next equation

$$F(u) = g, \;\; g \in B_{p,k},$$

where $B_{p,k}$ is a wieghted Sobolev space as in Hörmander [13].

1. Introduction

During the last years the attention in the literature has been mainly addressed to the semilinear case:

$$P(x, D)u + f(x, D^\alpha u)_{|\alpha|\leq m-1} = g(x) \tag{1}$$

where the nonlinear function $f(x, v)$, $x \in \mathbb{R}^n$, $v \in \mathbb{C}^M$, is in $C^\infty(\mathbb{R}^n, \mathcal{H}(\mathbb{C}^M))$ with $\mathcal{H}(\mathbb{C}^M)$ the set of the holomorphic functions in $\mathbb{C}^M$ and where the local solvability of the linear term $P(x, D)$ is assumed to be already known.
See Gramchev-Popivanov[10] and Dehman[4] where, exploiting the fact that the nonlinear part of the equation (1) involves derivatives of order $\leq m-1$, one is reduced to applications of the classical contraction principle and Brower's fixed point Theorem, provided the linear part is invertible in some sense. The general case of $P(x, D)$ satisfying the $(\mathcal{P})$ condition of Nirenberg and Trèves [21] has been settled in Hounie-Santiago[12], by combining the contraction principle with compactness arguments.
Corcerning the case of linear part with multiple characteristics, we mention the recent results of Gramchev-Rodino[11], Garello[6], Garello[5], Garello-Gramchev-Popivanov-Rodino[7], Garello-Rodino[8], Garello-Rodino[9], De Donno-Oliaro[3], Marcolongo[17], Marcolongo-Oliaro[18], Oliaro[22].