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POINCARÉ-BIRKHOFF FIXED POINT THEOREM AND PERIODIC SOLUTIONS OF ASYMPTOTICALLY LINEAR PLANAR HAMILTONIAN SYSTEMS

Abstract. This work, which has a self contained expository character, is devoted to the Poincaré-Birkhoff (PB) theorem and to its applications to the search of periodic solutions of nonautonomous periodic planar Hamiltonian systems. After some historical remarks, we recall the classical proof of the PB theorem as exposed by Brown and Neumann. Then, a variant of the PB theorem is considered, which enables, together with the classical version, to obtain multiplicity results for asymptotically linear planar hamiltonian systems in terms of the gap between the Maslov indices of the linearizations at zero and at infinity.

1. The Poincaré-Birkhoff theorem in the literature

In his paper [28], Poincaré conjectured, and proved in some special cases, that an area-preserving homeomorphism from an annulus onto itself admits (at least) two fixed points when some "twist" condition is satisfied. Roughly speaking, the twist condition consists in rotating the two boundary circles in opposite angular directions. This concept will be made precise in what follows.
Subsequently, in 1913, Birkhoff [4] published a complete proof of the existence of at least one fixed point but he made a mistake in deducing the existence of a second one from a remark of Poincaré in [28]. Such a remark guarantees that the sum of the indices of fixed points is zero. In particular, it implies the existence of a second fixed point in the case that the first one has a nonzero index.
In 1925 Birkhoff not only corrected his error, but he also weakened the hypothesis about the invariance of the annulus under the homeomorphism T. In fact Birkhoff himself already searched a version of the theorem more convenient for the applications. He also generalized the area-preserving condition.

Before going on with the history of the theorem we give a precise statement of the classical version of Poincaré-Birkhoff fixed point theorem and make some remarks.
In the following we denote by $\mathcal{A}$ the annulus $\mathcal{A} := \{(x, y) \in \mathbb{R}^2 : r_1^2 \le x^2 + y^2 \le r_2^2 , \ 0 < r_1 < r_2\}$ and by C_1 and C_2 its inner and outer boundaries, respectively.

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Moreover we consider the covering space $H := \mathbb{R} \times \mathbb{R}_0^+$ of $\mathbb{R}^2 \setminus \{(0,0)\}$ and the projection associated to the polar coordinate system $\Pi : H \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$ defined by $\Pi(\vartheta, r) = (r \cos\vartheta,\ r \operatorname{sen}\vartheta)$. Given a continuous map $\varphi : D \subset \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$, a map $\widetilde{\varphi} : \Pi^{-1}(D) \longrightarrow H$ is called a lifting of φ to H if

$$\Pi \circ \widetilde{\varphi} = \varphi \circ \Pi.$$

Furthermore for each set $D \subset \mathbb{R}^2 \setminus \{(0,0)\}$ we set $\tilde{D} := \Pi^{-1}(D)$.

THEOREM 1 (POINCARÉ-BIRKHOFF THEOREM). *Let $\psi : \mathcal{A} \longrightarrow \mathcal{A}$ be an area-preserving homeomorphism such that both boundary circles of $\mathcal{A}$ are invariant under ψ (i.e. $\psi(C_1) = C_1$ and $\psi(C_2) = C_2$). Suppose that ψ admits a lifting $\widetilde{\psi}$ to the polar coordinate covering space given by*

$$\widetilde{\psi}(\vartheta, r) = (\vartheta + g(\vartheta, r),\ f(\vartheta, r)), \tag{1}$$

where g and f are $2\pi-$periodic in the first variable. If the twist condition

$$g(\vartheta, r_1)\ g(\vartheta, r_2) < 0 \qquad \forall\, \vartheta \in \mathbb{R} \quad \textit{[twist condition]} \tag{2}$$

holds, then ψ admits at least two fixed points in the interior of $\mathcal{A}$.

The proof of Theorem 1 guarantees the existence of two fixed points (called F_1 and F_2) of $\widetilde{\psi}$ such that $F_1 - F_2 \neq k(2\pi, 0)$, for any $k \in \mathbb{Z}$. This fact will be very useful in the applications of the theorem to prove the multiplicity of periodic solutions of differential equations. Of course the images of F_1, F_2 under the projection Π are two different fixed points of ψ.

We make now some remarks on the assumptions of the theorem.

REMARK 1. We point out that it is essential to assume that the homeomorphism is area-preserving. Indeed, let us consider an homeomorphism $\psi : \mathcal{A} \longrightarrow \mathcal{A}$ which admits the lifting $\widetilde{\psi}(\vartheta, r) = (\vartheta + \alpha(r), \beta(r))$, where α and β are continuous functions verifying $2\pi > \alpha(r_1) > 0 > \alpha(r_2) > -2\pi$, $\beta(r_i) = r_i$ for $i \in \{1, 2\}$, β is strictly increasing and $\beta(r) > r$ for every $r \in (r_1, r_2)$. This homeomorphism, which does not preserve the area, satisfies the twist condition, but it has no fixed points. Also its projection has no fixed points.

REMARK 2. The homeomorphism ψ preserves the standard area measure $\mathrm{d}x\mathrm{d}y$ in $\mathbb{R}^2$ and hence its lift $\widetilde{\psi}$ preserves the measure $r\mathrm{d}r\mathrm{d}\vartheta$. We remark that it is possible to consider a lift in the Poincaré-Birkhoff theorem which preserves $\mathrm{d}r\mathrm{d}\vartheta$ instead of $r\mathrm{d}r\mathrm{d}\vartheta$ and still satisfies the twist condition. In fact, let us consider the homeomorphism T of $\mathbb{R} \times [r_1, r_2]$ onto itself defined by $T(\vartheta, r) = (\vartheta, a\, r^2 + b)$, where $a = \dfrac{1}{r_1 + r_2}$ and $b = \dfrac{r_1\, r_2}{r_1 + r_2}$. The homeomorphism T preserves the twist and transforms the measure $r\mathrm{d}r\mathrm{d}\vartheta$ in a multiple of $\mathrm{d}r\mathrm{d}\vartheta$. Thus, if we define $\widetilde{\psi}^* := T \circ \widetilde{\psi} \circ T^{-1}$, we note that it preserves the measure $\mathrm{d}r\mathrm{d}\vartheta$. Furthermore, there is a bijection between fixed points F

of $\widetilde{\psi}^*$ and fixed points $T^{-1}(F)$ of $\widetilde{\psi}$. Finally, it is easy to verify that $\widetilde{\psi}^*$ is the lifting of an homeomorphism ψ^* which satisfies all the assumptions of Theorem 1. This remark implies that Theorem 1 is equivalent to Theorem 7 in Section 2.

It is interesting to observe that if slightly stronger assumptions are required in Theorem 1, then its proof is quite simple (cf. [25]). Indeed, we have the following proposition.

PROPOSITION 1. *Suppose that all the assumptions of Theorem* 1 *are satisfied and that*

(3) $\quad g(\vartheta, \cdot)$ *is strictly decreasing* (*or strictly increasing*) *for each* ϑ.

Then, ψ *admits at least two fixed points in the interior of* $\mathcal{A}$.

Proof. According to (2) and (3), it follows that for every $\vartheta \in \mathbb{R}$ there exists a unique $r(\vartheta) \in (r_1, r_2)$ such that $g(\vartheta, r(\vartheta)) = 0$. By the periodicity of g in the first variable, we have that $g(\vartheta + 2k\pi, r(\vartheta)) = g(\vartheta, r(\vartheta)) = 0$ for every $k \in \mathbb{Z}$ and $\vartheta \in \mathbb{R}$. Hence as $g(\vartheta + 2k\pi, r(\vartheta + 2k\pi)) = 0$, we deduce from the uniqueness of $r(\vartheta)$ that $r : \vartheta \longmapsto r(\vartheta)$ is a $2\pi-$periodic function. Moreover, we claim that it is continuous too. Indeed, by contradiction, let us assume that there exist $\vartheta \in \mathbb{R}$ and a sequence ϑ_n converging to ϑ which admits a subsequence ϑ_{n_k} satisfying $\lim\limits_{k\to+\infty} r(\vartheta_{n_k}) = b \neq r(\vartheta)$. Passing to the limit, from the equality $g(\vartheta_{n_k}, r(\vartheta_{n_k})) = 0$, we immediately obtain $g(\vartheta, b) = 0 = g(\vartheta, r(\vartheta))$, which contradicts $b \neq r(\vartheta)$.
By construction, $\widetilde{\psi}(\vartheta, r(\vartheta)) = (\vartheta + g(\vartheta, r(\vartheta)),\ f(\vartheta, r(\vartheta))) = (\vartheta\ ,\ f(\vartheta, r(\vartheta)))$. Hence, each point of the continuous closed curve $\Gamma \subset \mathcal{A}$ defined by

$$\Gamma = \{(x, y) \in \mathbb{R}^2 \ : \ x = r(\vartheta)\cos\vartheta \ , \ \ y = r(\vartheta)\,\mathrm{sen}\,\vartheta \ , \ \ \vartheta \in \mathbb{R}\}$$

is “radially” mapped into another one under the operator ψ. Being ψ area-preserving and recalling the invariance of the boundary circles C_1, C_2 of $\mathcal{A}$ under ψ, we can deduce that the region bounded by the curves C_1 and Γ encloses the same area as the region bounded by the curves C_1 and $\psi(\Gamma)$. Therefore, there exist at least two points of intersection between Γ and $\psi(\Gamma)$. In fact as the two regions mentioned above have the same measure, we can write

$$\int_0^{2\pi}\int_{r_1}^{r(\vartheta)} r\,\mathrm{d}r\,\mathrm{d}\vartheta \ = \int_0^{2\pi}\int_{r_1}^{f(\vartheta, r(\vartheta))} r\,\mathrm{d}r\,\mathrm{d}\vartheta\ ,$$

which implies $\displaystyle\int_0^{2\pi}\Big(r^2(\vartheta) - f^2(\vartheta, r(\vartheta))\Big)\,\mathrm{d}\vartheta = 0$. Being the integrand continuous and $2\pi-$periodic, it vanishes at least at two points which give rise to two distinct fixed points of $\widetilde{\psi}(\cdot, r(\cdot))$ in $[0, 2\pi)$. Hence, we have found two fixed points of ψ and the proposition follows.

$\square$

Morris [26] applied this version of the theorem to prove the existence of infinitely many $2\pi-$periodic solutions for

$$x'' + 2x^3 = e(t),$$

where e is continuous, $2\pi-$periodic and it satisfies

$$\int_0^{2\pi} e(t)\ \mathrm{d}t = 0.$$

If we assume monotonicity of $\vartheta + g(\vartheta, r)$ in ϑ, for each r, then also in this case the existence of at least one fixed point easily follows (cf. [25]).

PROPOSITION 2. *Assume that all the hypotheses of Theorem* 1 *hold. Moreover, suppose that*

(4) $\quad \vartheta + g(\vartheta, r)$ *is strictly increasing* (*or strictly decreasing*) *in* ϑ *for each* r.

Then, the existence of at least one fixed point follows, when ψ *is differentiable.*

Proof. Let us suppose that $\vartheta \longmapsto \vartheta + g(\vartheta, r)$ is strictly increasing for every $r \in [r_1, r_2]$. Thus, since $\dfrac{\partial\,(\vartheta + g(\vartheta, r))}{\partial\,\vartheta} > 0$ for every r, it follows that the equation $\vartheta^* = \vartheta + g(\vartheta, r)$ defines implicitly ϑ as a function of ϑ^* and r. Moreover, taking into account the $2\pi-$periodicity of g in the first variable, it turns out that $\vartheta = \vartheta(\vartheta^*, r)$ satisfies $\vartheta(\vartheta^* + 2\pi, r) = \vartheta(\vartheta^*, r) + 2\pi$ for every ϑ^* and r. We set $r^* = f(\vartheta, r)$. Combining the area-preserving condition and the invariance of the boundary circles under ψ, then the existence of a generating function $W(\vartheta^*, r)$ such that

$$\text{(5)} \qquad \begin{cases} \vartheta &= \dfrac{\partial\,W}{\partial r}(\vartheta^*, r) \\[2ex] r^* &= \dfrac{\partial\,W}{\partial \vartheta^*}(\vartheta^*, r) \end{cases}$$

is guaranteed by the Poincaré Lemma.
Now we consider the function $w(\vartheta^*, r) = W(\vartheta^*, r) - \vartheta^* r$. Since, according to (5), the following equalities hold

$$\begin{cases} \dfrac{\partial\,w}{\partial \vartheta^*} &= r^* - r \\[2ex] \dfrac{\partial\,w}{\partial r} &= \vartheta - \vartheta^*, \end{cases}$$

the critical points of w give rise to fixed points of ψ.

It is easy to verify that w has period 2π in ϑ^*. Indeed, according to the hypothesis of boundary invariance and to (5), we get

$$W(\vartheta^* + 2\pi, r_1) - W(\vartheta^*, r_1) = \int_{\vartheta^*}^{\vartheta^*+2\pi} r^*(s, r_1)\,\mathrm{d}s = \int_{\vartheta^*}^{\vartheta^*+2\pi} r_1\,\mathrm{d}s = 2\pi\, r_1.$$

Furthermore, combining (5) with the equality $\vartheta(\vartheta^* + 2\pi, r) = \vartheta(\vartheta^*, r) + 2\pi$, we deduce

$$\begin{aligned} W(\vartheta^* + 2\pi, r) - W(\vartheta^* + 2\pi, r_1) &= \int_{r_1}^{r} \vartheta(\vartheta^* + 2\pi, s)\,\mathrm{d}s \\ &= \int_{r_1}^{r} \vartheta(\vartheta^*, s)\,\mathrm{d}s + 2\pi(r - r_1) \\ &= W(\vartheta^*, r) - W(\vartheta^*, r_1) + 2\pi(r - r_1). \end{aligned}$$

Finally, we infer that

$$\begin{aligned} w(\vartheta^* + 2\pi, r) - w(\vartheta^*, r) &= W(\vartheta^* + 2\pi, r) - W(\vartheta^*, r) - 2\pi\, r \\ &= W(\vartheta^* + 2\pi, r_1) - W(\vartheta^*, r_1) - 2\pi\, r_1 = 0, \end{aligned}$$

and the periodicity of w in the first variable follows.
Consider now the external normal derivatives of w

$$\left.\frac{\partial w}{\partial n}\right|_{\widetilde{C_1}} = (r^* - r, \vartheta - \vartheta^*) \cdot (0, -1) = \vartheta^* - \vartheta, \tag{6}$$

$$\left.\frac{\partial w}{\partial n}\right|_{\widetilde{C_2}} = (r^* - r, \vartheta - \vartheta^*) \cdot (0, 1) = \vartheta - \vartheta^*. \tag{7}$$

The twist condition (2) implies that $\vartheta^* - \vartheta$ has opposite signs on the two boundary circles. Hence, by (6) and (7), the two external normal derivatives in $\widetilde{C_1}$ and $\widetilde{C_2}$ have the same sign. Being w a $2\pi-$periodic function in ϑ^*, critical point theory guarantees the existence of a maximum or a minimum of w in the interior of the covering space $\widetilde{\mathcal{A}}$. Such a point is the required critical point of w.

□

It is interesting to notice that as a consequence of the periodicity of g and f in ϑ, the existence of a second fixed point (a saddle) follows from critical point theory.

As we previously said, in order to apply the twist fixed point theorem to prove the existence of periodic solutions to planar Hamiltonian systems, Birkhoff tried to replace the invariance of the annulus by a weaker assumption. Indeed, he was able to require that only the inner boundary is invariant under T. He also generalized the area-preserving condition. More precisely, in his article [5] the homeomorphism T is defined on a region R bounded by a circle C and a closed curve Γ surrounding C. Such an homeomorphism takes values on a region R_1 bounded by C and by a closed curve Γ_1 surrounding C. Under these hypotheses, Birkhoff proved the following theorem

THEOREM 2. *Let $T : R \longrightarrow R_1$ be an homeomorphism such that $T(C) = C$ and $T(\Gamma) = \Gamma_1$, with Γ and Γ_1 star-shaped around the origin. If T satisfies the twist condition, then either*

- *there are two distinct invariant points P of R and R_1 under T*

or
• *there is a ring in* R *(or* R_1*) around* C *which is carried into part of itself by* T *(or* T^{-1}*).*

Since Birkhoff's proof was not accepted by many mathematicians, Brown and Neumann [6] decided to publish a detailed and convincing proof (based on the Birkhoff's one) of Theorem 1. In the same year, Neumann in [27] studied generalizations of such a theorem. For completeness, we will recall the proof given in [6] and also the details of a remark stated in [27] in the next section.

After Birkhoff's contribution, many authors tried to generalize the hypothesis of invariance of the annulus, in view of studying the existence of periodic solutions for problems of the form

$$x'' + f(t,x) = 0,$$

with $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ continuous and T-periodic in t.
In this sense we must emphasize the importance of the works by Jacobowitz and W-Y Ding. In his article [22] Jacobowitz (see also [23]), gave a version of the twist fixed point theorem in which the area-preserving twist homeomorphism is defined on an annulus whose internal boundary (roughly speaking) degenerates into a point, while the external one is a simple curve around it. More precisely, he first considered two simple curves $\Gamma_i = (\vartheta_i(\cdot), r_i(\cdot))$, $i = 1, 2$, defined in [0, 1], with values in the (ϑ, r) half-plane $r > 0$, such that $\vartheta_i(0) = -\pi$, $\vartheta_i(1) = \pi$, $\vartheta_i(s) \in (-\pi, \pi)$ for each $s \in (0, 1)$ and $r_i(0) = r_i(1)$. Then, he considered the corresponding 2π-periodic extensions, which he called again Γ_i. Denoting by A_i the regions bounded by the curve Γ_i (included) and the axis $r = 0$ (excluded), Jacobowitz proved the following theorem

THEOREM 3. *Let* $\psi : A_1 \longrightarrow A_2$ *be an area-preserving homeomorphism, defined by*

$$\psi(\vartheta, r) = (\vartheta + g(\vartheta, r), f(\vartheta, r)),$$

where
- g *and* f *are* 2π*−periodic in the first variable;*
- $g(\vartheta, r) < 0$ *on* Γ_1*;*
- $\liminf_{r \to 0} g(\vartheta, r) > 0$.

Then, ψ *admits at least two fixed points, which do not differ from a multiple of* $(2\pi, 0)$.

Unfortunately the proof given by Jacobowitz is not very easy to follow. Subsequently, using the result by Jacobowitz, W-Y Ding in [15] and [16] treated the case in which also the inner boundary can vary under the area-preserving homeomorphism. He considered an annular region $\mathcal{A}$ whose inner boundary C_1 and the outer one C_2 are two closed simple curves. By D_i he denoted the open region bounded by C_i, $i = 1, 2$. Using the result by Jacobowitz, he proved the following theorem

THEOREM 4. *Let* $T : \mathcal{A} \longrightarrow T(\mathcal{A}) \subset \mathbb{R}^2 \setminus \{(0, 0)\}$ *be an area-preserving home-*

omorphism. Suppose that

(a) C_1 *is star-shaped about the origin;*

(b) T *admits a lifting* $\widetilde{T}$ *onto the polar coordinate covering space, defined by*

$$\widetilde{T}(\vartheta, r) = (\vartheta + g(\vartheta, r), f(\vartheta, r)),$$

where f *and* g *are* $2\pi-$*periodic in the first variable,* $g(\vartheta, r) > 0$ *on the lifting of* C_1 *and* $g(\vartheta, r) < 0$ *on the lifting of* C_2*;*

(c) *there exists an area-preserving homeomorphism* $T_0 : \overline{D_2} \longrightarrow \mathbb{R}^2$*, which satisfies* $T_{0|\mathcal{A}} = T$ *and* $(0, 0) \in T_0(D_1)$*.*

Then, $\widetilde{T}$ *has at least two fixed points such that their images under the usual covering projection* Π *are two different fixed points of* T*.*

We point out that condition (c) cannot be removed.
Indeed, we can define $\mathcal{A} := \{(x, y) : 2^{-2} < x^2 + y^2 < 2^2\}$ and consider an homeomorphism $T : \mathcal{A} \longrightarrow \mathbb{R}^2 \setminus \{(0, 0)\}$ whose lifting is given by $\widetilde{T}(\vartheta, r) = \left(\vartheta + 1 - r, \sqrt{r^2 + 1}\right)$. It easily follows that $\widetilde{T}$ preserves the measure $r\mathrm{d}r\mathrm{d}\vartheta$ and, consequently, T preserves the measure $\mathrm{d}x\mathrm{d}y$. Moreover, the twist condition is satisfied, being $g(\vartheta, r) = 1 - r$ positive on $r = \dfrac{1}{2}$ and negative on $r = 2$. We also note that it is not possible to extend the homeomorphism into the interior of the circle of radius $1/2$ as an area-preserving homeomorphism, and hence (c) is not satisfied. Since $f(\vartheta, r) = \sqrt{r^2 + 1} > r$ for every $r \in \left(\dfrac{1}{2}, 2\right)$, we can conclude that $\widetilde{T}$ has no fixed points.

In [29], Rebelo obtained a proof for Jacobowitz and Ding versions of the Poincaré-Birkhoff theorem directly from Theorem 7.
The W-Y Ding version of the theorem seems the most useful in terms of the applications. In 1998, Franks [18] proved a quite similar result using another approach. In fact he considered an homeomorphism f from the open annulus $\mathcal{A} = \mathrm{S}^1 \times (0, 1)$ into itself. He replaced the area-preserving condition with the weaker condition that every point of $\mathcal{A}$ is non-wandering under f. We recall that a point x is non-wandering under f if for every neighbourhood U of x there is an $n > 0$ such that $f^n(U) \cap U \neq \emptyset$.
Being $\widetilde{f}$, from the covering space $\widetilde{\mathcal{A}} = \mathbb{R} \times (0, 1)$ onto itself, a lift of f, it is said that there is a positively returning disk for $\widetilde{f}$ if there is an open disk $U \subset \widetilde{A}$ such that $\widetilde{f}(U) \cap U = \emptyset$ and $\widetilde{f}^n(U) \cap (U + k) \neq \emptyset$ for some $n,\ k > 0$. A negatively returning disk is defined similarly, but with $k < 0$. We recall that by $U + k$ it is denoted the set $\{(x + k, t) : (x, t) \in U\}$. Franks generalized the twist condition on a closed annulus assuming the existence of both a positive and a negative returning disk on the open annulus, since this hypothesis holds if the twist condition is verified. Under these generalized assumptions, Franks obtained the existence of a fixed point (for the open annulus). However, he observed that reducing to the case of the closed annulus, one can conclude the existence of two fixed points.

On the lines of Birkhoff [5], some mathematicians generalized the Poincaré-Birkhoff theorem, replacing the area-preserving requirement by a more general topological condition. Among others, we quote Carter [8], who, as Birkhoff, considered an homeomorphism g defined on an annulus $\mathcal{A}$ bounded by the unit circle T and a simple, closed, star-shaped around the origin curve γ that lies in the exterior of T. She also supposed that $g(T) = T$, $g(\gamma)$ is star-shaped around the origin and lies in the exterior of T. Before stating her version of the Poincaré-Birkhoff theorem, we only remark that a simple, closed curve in $\mathcal{A}$ is called essential if it separates T from γ.

THEOREM 5. *If g is a twist homeomorphism of the annulus $\mathcal{A}$ and if g has at most one fixed point in the interior of $\mathcal{A}$, then there is an essential, simple, closed curve C in the interior of $\mathcal{A}$ which meets its image in at most one point.* (*If the curve C intersects its image, the point of intersection must be the fixed point of g in the interior of $\mathcal{A}$*).

We point out that Theorem 2 can be seen as a consequence of Theorem 5 above.

Recently, in [24], Margheri, Rebelo and Zanolin proved a modified version of the Poincaré-Birkhoff theorem generalizing the twist condition. They assumed that the points of the external boundary circle rotate in one angular direction, while only some points of the inner boundary circle move in the opposite direction. The existence of one fixed point is guaranteed. More precisely, they proved the following

THEOREM 6. *Let $\psi : \mathcal{A} \longrightarrow \mathcal{A}$ be an area-preserving homeomorphism in $\mathcal{A} = \mathbb{R} \times [0, R]$, $R > 0$ such that*

$$\psi(\vartheta, r) = (\vartheta_1, r_1),$$

with

$$\begin{cases} \vartheta_1 = \vartheta + g(\vartheta, r) \\ r_1 = f(\vartheta, r), \end{cases}$$

where f and g are $2\pi-$periodic in the first variable and satisfy the conditions

- *$f(\vartheta, 0) = 0$, $f(\vartheta, R) = R$ for every $\vartheta \in \mathbb{R}$ (boundary invariance),*
- *$g(\vartheta, R) > 0$ for every $\vartheta \in \mathbb{R}$ and there is $\overline{\vartheta}$ such that $g(\overline{\vartheta}, 0) < 0$ (modified twist condition).*

Then, ψ admits at least a fixed point in the interior of $\mathcal{A}$.

2. Proof of the classical version of the Poincaré-Birkhoff theorem

In this section we recall the proof of the classical version of the Poincaré-Birkhoff theorem given by Brown and Neumann [6] and give the details of the proof of an important remark (see Remark 3 below) made by Neumann in [27].

THEOREM 7. *Let us define* $\widetilde{\mathcal{A}} = \mathbb{R} \times [r_1, r_2]$, $0 < r_1 < r_2$. *Moreover, let* $h : \widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{A}}$ *be an area-preserving homeomorphism satisfying*

$$h(x, r_2) = (x - s_1(x), r_2)\,,$$

$$h(x, r_1) = (x + s_2(x), r_1)\,,$$

$$h(x + 2\pi, y) = h(x, y) + (2\pi, 0),$$

for some 2π*−periodic positive continuous functions* s_1, s_2. *Then,* h *has two distinct fixed points* F_1 *and* F_2 *which are not in the same periodic family, that is* $F_1 - F_2$ *is not an integer multiple of* $(2\pi, 0)$.

Note that Theorem 7 and Theorem 1 are the same. In fact, taking into account Remark 2, Theorem 7 corresponds to Theorem 1 choosing $h = \widetilde{\psi}$.

Before giving the proof of the theorem we give some useful preliminary definitions and results.

We define the direction from P to Q, setting $\mathrm{D}(P, Q) = \dfrac{Q - P}{\|Q - P\|}$, whenever P and Q are distinct points of $\mathbb{R}^2$. If we consider $X \subset \mathbb{R}^2$, $\mathcal{C}$ a curve in X and $h : X \longrightarrow \mathbb{R}^2$ an homeomorphism with no fixed points, then we will denote by $i_h(\mathcal{C})$ the index of $\mathcal{C}$ with respect to h. This index represents the total rotation that the direction $\mathrm{D}(P, h(P))$ performs as P moves along $\mathcal{C}$. In order to give a precise definition, we set $\mathcal{C} : [a, b] \longrightarrow \mathbb{R}^2$ and define the map $\overline{\mathcal{C}} : [a, b] \longrightarrow \mathrm{S}^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ by $\overline{\mathcal{C}}(t) := \mathrm{D}(\,\mathcal{C}(t)\,,\, h(\mathcal{C}(t))\,)$. If we denote by $\pi : \mathbb{R} \longrightarrow \mathrm{S}^1$ the covering map $\pi(r) = (\cos r, \operatorname{sen} r)$, then we can lift the function $\overline{\mathcal{C}}$ into $\widetilde{\mathcal{C}} : [a, b] \longrightarrow \mathbb{R}$ assuming $\overline{\mathcal{C}} = \pi \circ \widetilde{\mathcal{C}}$. Finally, we set

$$i_h(\mathcal{C}) = \frac{\widetilde{\mathcal{C}}(b) - \widetilde{\mathcal{C}}(a)}{2\pi}\,,$$

which is well defined, since it is independent of the lifting.

This index satisfies the following properties:

1. For a one parameter continuous family of curves $\mathcal{C}$ or homeomorphisms h, $i_h(\mathcal{C})$ varies continuously with the parameter. (*Homotopy lifting property*).
2. If $\mathcal{C}$ runs from a point A to a point B, then $i_h(\mathcal{C})$ is congruent modulo 1 to $\frac{1}{2\pi}$ times the angle between the directions $\mathrm{D}(A, h(A))$ and $\mathrm{D}(B, h(B))$.
3. If $\mathcal{C} = \mathcal{C}_1\mathcal{C}_2$ consists of $\mathcal{C}_1$ and $\mathcal{C}_2$ laid end to end (i.e. $\mathcal{C}_1 = \mathcal{C}|_{[a,c]}$ and $\mathcal{C}_2 = \mathcal{C}|_{[c,b]}$ with $a < c < b$), then $i_h(\mathcal{C}) = i_h(\mathcal{C}_1) + i_h(\mathcal{C}_2)$. In particular, $i_h(-\mathcal{C}) = -i_h(\mathcal{C})$.
4. $i_h(\mathcal{C}) = i_{h^{-1}}(h(\mathcal{C}))$.

As a consequence of properties 1 and 2 we have that in order to calculate the index we can make first an homotopy on $\overline{\mathcal{C}}$ so long as we hold the endpoints fixed, this will be very important in what follows.

In the following it will be useful to consider an extension of the homeomorphism $h : \widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{A}}$ to all $\mathbb{R}^2$.

To this aim, we introduce the following notations:

$$H_+ = \{(x, y) \in \mathbb{R}^2 : y \geq r_2\},$$

$$H_- = \{(x, y) \in \mathbb{R}^2 : y \leq r_1\}$$

and consider the extension of h (which we still denote by h)

$$h(x, y) := \begin{cases} (x - s_1(x), y) & y \geq r_2 \\ (x + s_2(x), y) & y \leq r_1 \\ h(x, y) & r_1 < y < r_2 . \end{cases}$$

The following lemma will be essential in order to prove the theorem.

LEMMA 1. *Suppose that all the assumptions of Theorem 7 are satisfied and that h has at most one family of fixed points of the form $(2k\pi, r^*)$ with $r^* \in (r_1, r_2)$. Then, for any curve $\mathcal{C}$ running from H_- to H_+ and not passing through any fixed point of h,*

(a) $i_h(\mathcal{C}) \equiv \dfrac{1}{2}$ (mod 1),

(b) $i_h(\mathcal{C})$ *is independent of* $\mathcal{C}$.

Proof of the lemma. From Property 2 of the index, it is easy to deduce that part (a) is verified.
Let us now consider two curves $\mathcal{C}_i$ $(i = 1, 2)$ running from $A_i \in H_-$ to $B_i \in H_+$ and not passing through any fixed point of h. Our aim consists in proving that $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same index. Let us take a curve $\mathcal{C}_3$ from B_1 to B_2 in H_+ and a curve $\mathcal{C}_4$ from A_2 to A_1 in H_-. Being $\mathrm{D}(P, h(P))$ constant in H_+ and H_-, we immediately deduce that $i_h(\mathcal{C}_3) = i_h(\mathcal{C}_4) = 0$. Now, we can calculate the index of the closed curve $\mathcal{C}' := \mathcal{C}_1\, \mathcal{C}_3\, (-\mathcal{C}_2)\, \mathcal{C}_4$. In particular, from Property 3 we get

$$i_h(\mathcal{C}') = i_h(\mathcal{C}_1) + i_h(\mathcal{C}_3) + i_h(-\mathcal{C}_2) + i_h(\mathcal{C}_4) = i_h(\mathcal{C}_1) - i_h(\mathcal{C}_2).$$

Hence, in order to prove (b), it remains to show that such an index is zero. To this purpose, we give some further definitions. We denote by $\mathrm{Fix}(h)$ the fixed point set of h and by $\pi_1(\mathbb{R}^2 \setminus \mathrm{Fix}(h), A_1)$ the fundamental group of $\mathbb{R}^2 \setminus \mathrm{Fix}(h)$ in the base-point A_1. We recall that such a fundamental group is the set of all the loops (closed curves defined on closed intervals and taking values in $\mathbb{R}^2 \setminus \mathrm{Fix}(h)$) based on A_1, i.e. whose initial and final points coincide with A_1. The fundamental group is generated by paths which start from A_1, run along a curve $\mathcal{C}_0$ to near a fixed point (if there are any), loop around this fixed point and return by $-\mathcal{C}_0$ to A_1. Hence, since $\mathcal{C}'$ belongs to $\pi_1(\mathbb{R}^2 \setminus \mathrm{Fix}(h), A_1)$, it is deformable into a composition of such paths. Thus, it is sufficient to show that i_h is zero for any path belonging to the set of generators of the fundamental group. Since h has at most one family of fixed points of the form

$(2k\pi, r^*)$ with $r^* \in (r_1, r_2)$, then a loop surrounding a single fixed point can be deformed into the loop $\mathcal{D}' := \mathcal{D}_1\, \mathcal{D}_2\, \mathcal{D}_3\, \mathcal{D}_4$, where
$\mathcal{D}_1$ covers $[-\pi, \pi] \times \{r_0\}$ with $r_0 < r_1$, moving horizontally from $\widetilde{A}_1 = (-\pi, r_0)$ to $\widetilde{A}_2 = (\pi, r_0)$;
$\mathcal{D}_2$ covers $\{\pi\} \times [r_0, r_3]$ with $r_3 > r_2$, moving vertically from $\widetilde{A}_2$ to $\widetilde{A}_3 = (\pi, r_3)$;
$\mathcal{D}_3$ covers $[-\pi, \pi] \times \{r_3\}$, moving horizontally from $\widetilde{A}_3 = (\pi, r_3)$ to $\widetilde{A}_4 = (-\pi, r_3)$;
$\mathcal{D}_4$ covers $\{-\pi\} \times [r_0, r_3]$, moving vertically from $\widetilde{A}_4$ to $\widetilde{A}_1$.
Roughly speaking, $\mathcal{D}'$ is the boundary curve of a rectangle with vertices $(\pm\pi, r_0), (\pm\pi, r_3)$.
As $\mathcal{D}_1$ and $\mathcal{D}_3$ lie in H_- and H_+ respectively, their index is zero. Moreover, being $h(x, y) - (x, 0)$ a $2\pi-$periodic function in its first variable, it follows that $i_h(\mathcal{D}_4) = -i_h(\mathcal{D}_2)$. Thus, Property 3 of the index ensures that $i_h(\mathcal{D}') = 0$. This completes the proof.

□

Proof of Theorem 7. To prove the theorem, we will argue by contradiction. Assume that h has at most one family of fixed points $F = (\vartheta^*, r^*) + k\,(2\pi, 0)$, with $k \in \mathbb{Z}$. It is not restrictive to suppose $\vartheta^* = 0$. Indeed, we can always reduce to this case with a simple change of coordinates. In order to get the contradiction, we will construct two curves, with different indices, satisfying the hypotheses of Lemma 1.
Now we define the set

$$W = \{(x, y) \in \mathbb{R}^2 \,:\, 2k\pi + \frac{\pi}{2} \le x \le 2k\pi + \frac{3}{2}\pi \;, \quad k \in \mathbb{Z}\}\,.$$

Since the fixed points of h (if there are any) are of the form $(2k\pi, r^*)$, we can conclude that h has no fixed points in this region. Moreover, there exists $\varepsilon > 0$ such that

$$\varepsilon < \|P - h(P)\| \qquad \forall\, P \in W. \tag{8}$$

Indeed, by the periodicity of $(x, 0) - h(x, y)$ in its first variable, it is sufficient to find $\varepsilon > 0$ which satisfies the above inequality only for every $P \in W_1 := \{(x, y) \,:\, \frac{\pi}{2} \le x \le \frac{3}{2}\pi\}$. If we choose $\varepsilon < \min s_i$, for $i \in \{1, 2\}$ the inequality is satisfied on the sets $W_1 \cap H_\pm$. On the region $V := \{(x, y) \,:\, \frac{\pi}{2} \le x \le \frac{3}{2}\pi \,,\, r_1 \le y \le r_2\}$, the function $\|\mathrm{Id} - h\|$ is continuous and positive, hence it has a minimum on V, which is positive too.
Define the area-preserving homeomorphism $T \,:\, \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$T(x, y) = (x, y + \frac{\varepsilon}{2}(|\cos x| - \cos x))\,.$$

We point out that it moves only points of W and $\|T(P) - P\| \le \varepsilon$ for every $P \in \mathbb{R}^2$. Combining this fact with (8), we deduce that $T \circ h$ (just like h) has no fixed points in W. Furthermore, fixed points of $T \circ h$ coincide with the ones of h in $\mathbb{R}^2 \setminus W$ and, consequently, in $\mathbb{R}^2$.
Let us introduce the following sets

$$D_0 := H_- \setminus (T \circ h)^{-1}\, H_-\,,$$

$$D_1 := (T \circ h)\, D_0 = (T \circ h)\, H_- \setminus H_-\,,$$
$$D_i := (T \circ h)^i\, D_0 \qquad \forall\, i \in \mathbb{Z}\,.$$

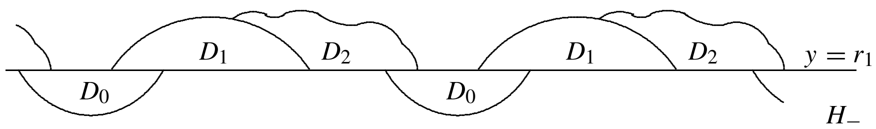

Figure 1: Some of the sets D_i

We immediately observe that $D_0 \subset H_-$, while $D_1 \subset \mathbb{R}^2 \setminus H_- = \{(x, y) : y > r_1\}$. Since $(T \circ h)(\mathbb{R}^2 \setminus H_-) \subset \mathbb{R}^2 \setminus H_-$, we can easily conclude that $D_i \subset \mathbb{R}^2 \setminus H_-$ for every $i \geq 1$. Hence, $D_i \cap D_0 = \emptyset$ for every $i \geq 1$. This implies that $D_k \cap D_j = \emptyset$ whenever $j \neq k$. Since $(T \circ h)^{-1} H_- \subset H_-$, we also get $D_i \subset H_-$ for every $i < 0$.

Furthermore, as T, h and, consequently, $(T \circ h)$ are area-preserving homeomorphisms, every D_i has the same area in the rolled-up plane $\mathbb{R}^2 / ((x, y) \equiv (x + 2\pi, y))$ and its value is 2ε. Thus, as the sets D_j are disjoint and contained in $\mathbb{R}^2 \setminus H_-$ for every $j \geq 1$, they must exhaust $\widetilde{\mathcal{A}}$ and hence intersect H_+. In particular, there exists $n > 0$ such that $D_n \cap H_+ \neq \emptyset$. Since $D_n \subset (T \circ h)^n H_-$, we also obtain that $(T \circ h)^n H_- \cap H_+ \neq \emptyset$. For such an $n > 0$, we can consider a point $P_n \in (T \circ h)^n H_- \cap H_+$ with maximal $y-$coordinate. The point P_n is not unique, but it exists since, by periodicity, it is sufficient to look at the compact region $(T \circ h)^n H_- \cap \{(x, y) : 0 \leq x \leq 2\pi\,,\ y \geq r_1\}$. Let us define

$$P_i = (x_i, y_i) := (T \circ h)^{i-n}\, P_n\,, \qquad i \in \mathbb{Z}.$$

Clearly, $P_n \in H_+$ and $P_0 = (T \circ h)^{-n}\, P_n \in H_-$. Moreover, $P_{i+1} = (T \circ h)\, P_i$ for every $i \in \mathbb{Z}$. Hence, recalling that $(T \circ h)\, H_+ \subset H_+$ and $(T \circ h)^{-1} H_- \subset H_-$, we obtain $P_{n+1} \in H_+$ and $P_{-1} \in H_-$.

Let us denote by $\mathcal{C}_0$ the straight line segment from P_{-1} to P_0 and let

$$\mathcal{C}_i = (T \circ h)^i\, \mathcal{C}_0\,, \qquad i \in \mathbb{Z}.$$

In particular, the curve $\mathcal{C}_i$ runs from P_{i-1} to P_i. Furthermore, let us define the curve $\mathcal{C} := \mathcal{C}_0\, \mathcal{C}_1 \ldots \mathcal{C}_{n-1}\, \mathcal{C}_n$. Thus, $(T \circ h)(\mathcal{C}) = \mathcal{C}_1\, \mathcal{C}_2 \ldots \mathcal{C}_n\, \mathcal{C}_{n+1}$.

We have constructed a curve $\mathcal{C}$ running from H_- to H_+. Now, we will show that it does not pass through any fixed point of h and we will calculate its index. First, we need to list and prove some properties that this curve satisfies.

1. The curve $\mathcal{C}\, \mathcal{C}_{n+1} = \mathcal{C}_0 \ldots \mathcal{C}_{n+1}$ has no double points;
2. No point of $\mathcal{C}$ has larger $y-$coordinate than P_{n+1};
3. No point of $(T \circ h)(\mathcal{C})$ has smaller $y-$coordinate than P_{-1}.

In order to prove Property 1, we first observe that as $\mathcal{C}_0$ has no double points and $T \circ h$ is a homeomorphism, each curve $\mathcal{C}_i$ has no double points. Hence, we only need to show that $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ for every $i \neq j$, exception made for the common endpoint when $|i-j| = 1$. We initially prove that this is true when i and j are both negative. We recall that the functions $f := (\mathrm{Id} + s_2) : \mathbb{R} \longrightarrow \mathbb{R}$ and f^{-1} are strictly monotone, being both continuous and bijective. From the positiveness of s_2, it immediately follows that both functions are strictly increasing. Moreover, $f^{-1}(x_0) \leq x \leq x_0$, whenever $(x, y) \in \mathcal{C}_0$. Thus, since $\mathcal{C}_0 \subset H_-$ and since f^{-1} is an increasing function, it turns out that $f^{-2}(x_0) \leq x \leq f^{-1}(x_0)$, whenever $(x, y) \in \mathcal{C}_{-1} = h^{-1}(T^{-1}(\mathcal{C}_0))$. In general, we have

$$\mathcal{C}_i \subset \{(x, y) : f^{\,i-1}(x_0) \leq x \leq f^{\,i}(x_0)\} \qquad \forall\, i < 0$$

and $\mathcal{C}_i$ intersects the boundaries of this strip only in its endpoints (because this is true for $\mathcal{C}_0$ and f^{-1} is strictly increasing). Thus, $\mathcal{C}_l$ and $\mathcal{C}_s$ intersect at most in a endpoint, if we choose l and s negative. In general, if we take $\mathcal{C}_i$ and $\mathcal{C}_j$ with $i \neq j$, then there exists $k < 0$ such that $(T \circ h)^k$ transforms such curves in two curves $\mathcal{C}_l$ and $\mathcal{C}_s$ with l and s both negative. Finally, the previous step guarantees that $\mathcal{C}_s \cap \mathcal{C}_l$ and hence $\mathcal{C}_i \cap \mathcal{C}_j$ are empty, if we exclude the intersection in the common endpoint.

Property 2 is easily proved. In fact, it is immediate to show that $\mathcal{C} \subset (T \circ h)^n\, H_-$. Thus, from the maximal choice involving the $y-$coordinate of P_n, we can conclude that for every $(x, y) \in \mathcal{C}$, we obtain $y \leq y_n$. Moreover, since $P_n \in H_+$ and $P_{n+1} = (T \circ h) P_n$, we can conclude that $y_n \leq y_{n+1}$. This completes the proof of Property 2.

With respect to Property 3, we remark that if we take $y \geq y_{-1}$ and if we define $(x', y') := (T \circ h)(x, y)$, then $y' \geq y_{-1}$. This is a consequence of the fact that $P_{-1} \in H_-$. Moreover, $y_0 \geq y_{-1}$ and hence $\mathcal{C}_0 \subset \{(x, y) : y \geq y_{-1}\}$. Thus, for every $(x, y) \in \mathcal{C}_1 = (T \circ h)\, \mathcal{C}_0$, we get $y \geq y_{-1}$. By induction, Property 3 follows.

Property 1 guarantees that $\mathcal{C}$ does not pass through any fixed point of $T \circ h$ and, consequently, of h.

We are interested in calculating the index of $\mathcal{C}$. More precisely, we will show that its value is exactly $\frac{1}{2}$. First, we will calculate $i_{(T \circ h)}(\mathcal{C})$.

The curve $\mathcal{C}$ runs from P_{-1} to P_n. Thus, recalling that $(T \circ h)\,(P_{-1}) = P_0$ and $(T \circ h)\,(P_n) = P_{n+1}$, let us consider the angle ϑ between $\mathrm{D}(P_{-1}, P_0)$ and $\mathrm{D}(P_n, P_{n+1})$. Since, by construction,

$$\begin{aligned} P_0 &= (x_0, y_0) = (x_{-1} + s_2(x_{-1}), y_{-1} + \delta_2)\,, \qquad & 0 \leq \delta_2 \leq \varepsilon, \\ P_{n+1} &= (x_{n+1}, y_{n+1}) = (x_n - s_1(x_n), y_n + \delta_1)\,, \qquad & 0 \leq \delta_1 \leq \varepsilon, \end{aligned}$$

then we can write the explicit expression of ϑ

$$\vartheta = \pi - \left(\operatorname{arctg}\left(\frac{\delta_1}{s_1(x_n)} \right) + \operatorname{arctg}\left(\frac{\delta_2}{s_2(x_{-1})} \right) \right).$$

By Property 2 of the index, we can conclude that

$$\begin{aligned} i_{(T \circ h)}(\mathcal{C}) &= \frac{\vartheta}{2\pi} \,(\mathrm{mod}\, 1) \\ &= \frac{1}{2} - \frac{1}{2\pi} \left(\operatorname{arctg}\left(\frac{\delta_1}{s_1(x_n)} \right) + \operatorname{arctg}\left(\frac{\delta_2}{s_2(x_{-1})} \right) \right) (\mathrm{mod}\, 1)\,. \end{aligned}$$

From the choice of ε, we get $0 \le \delta_i \le \varepsilon < \min s_i$ for $i \in \{1,2\}$. This implies that both $\operatorname{arctg}\left(\frac{\delta_1}{s_1(x_n)}\right)$ and $\operatorname{arctg}\left(\frac{\delta_2}{s_2(x_{-1})}\right)$ belong to the interval $[0, \frac{\pi}{4}[$. Consequently, $\frac{1}{4} < \frac{\vartheta}{2\pi} \le \frac{1}{2}$.

Our aim now consists in proving that we can cut mod 1 in the previous formula for $i_{(T\circ h)}(\mathcal{C})$. For this purpose, we will construct a suitable homotopy.

Let $P : [-1,0] \longrightarrow \mathbb{R}^2$ be a parametrization of $\mathcal{C}_0$. Setting $P(t+1) := (T\circ h)(P(t))$ for $t \in [-1,n]$, we extend the given parametrization of $\mathcal{C}_0$ into a parametrization $P : [-1,n+1] \longrightarrow \mathbb{R}^2$ of $\mathcal{C}\,\mathcal{C}_{n+1}$. Clearly, if we restrict P to the interval $[-1,n]$, we obtain a parametrization of $\mathcal{C}$. Moreover, it is immediate to see that $P(i) = P_i$ for every integer $i \in \{-1,0,\ldots,n+1\}$. In order to calculate $i_{(T\circ h)}(\mathcal{C})$, by definition, we will consider the map $\overline{P} : [-1,n] \longrightarrow \mathrm{S}^1$ given by

$$\overline{P}(t) := \mathrm{D}(\,P(t)\,,\,(T\circ h)(P(t))\,) = \mathrm{D}(P(t), P(t+1))\,.$$

Let us define now $\overline{P}_0 : [-1,2n+1] \longrightarrow \mathrm{S}^1$, setting

$$\overline{P}_0 = \begin{cases} \overline{P}(t) & -1 \le t \le n \\ \overline{P}(n) & n \le t \le 2n+1. \end{cases} \tag{9}$$

Of course, in order to evaluate the index we can use $\overline{P}_0$ instead of $\overline{P}$.

Now, we are in position to write the required homotopy. We will introduce a family of maps $\overline{P}_\lambda : [-1,2n+1] \longrightarrow \mathrm{S}^1$, with $0 \le \lambda \le n+2$. We will define this family treating separately the cases $0 \le \lambda \le n+1$ and $n+1 \le \lambda \le n+2$.

We develop the first case. The homotopy that we will exhibit will carry the initial map $\overline{P}_0$, which deals with the rotation of $\mathrm{D}\,(P, (T\circ h)(P))$ as P moves along $\mathcal{C}$, into the map $\overline{P}_{n+1}$ defined by

$$\overline{P}_{n+1}(t) = \begin{cases} \mathrm{D}\,(P(-1), P(t+1)) & -1 \le t \le n \\ \mathrm{D}\,(P(t-n-1), P(n+1)) & n \le t \le 2n+1. \end{cases} \tag{10}$$

This map corresponds to a rotation obtained if we initially move $(T\circ h)(P)$ along $(T\circ h)(\mathcal{C})$ from $(T\circ h)(P_{-1}) = P_0$ to P_{n+1}, holding P_{-1} fixed, and then we move P along $\mathcal{C}$ from P_{-1} to P_n, holding P_{n+1} fixed.

More precisely, when $0 \le \lambda \le n+1$, we set

$$\overline{P}_\lambda(t) = \begin{cases} \mathrm{D}\,(P(-1), P(t+1)) & -1 \le t \le \lambda-1 \\ \mathrm{D}\,(P(t-\lambda), P(t+1)) & \lambda-1 \le t \le n \\ \mathrm{D}\,(P(t-\lambda), P(n+1)) & n \le t \le n+\lambda \\ \mathrm{D}\,(P(n), P(n+1)) & n+\lambda \le t \le 2n+1. \end{cases}$$

Clearly, the above definition of $\overline{P}_\lambda$ in the case $\lambda = 0$ and $\lambda = n+1$ is compatible with (9) and (10), respectively. Furthermore, we note that $\overline{P}_\lambda(t)$ is always of the form

$\mathrm{D}\,(P(t_0), P(t_1))$ with $-1 \le t_0 < t_1 \le n+1$. By Property 1 of $\mathcal{C}$, we deduce that $P(t_0) \ne P(t_1)$, hence $\overline{P}_\lambda$ is well defined for every $0 \le \lambda \le n+1$.

We consider now the second case: $n+1 \le \lambda \le n+2$. The homotopy we will exhibit will carry the map $\overline{P}_{n+1}$ into the map $\overline{P}_{n+2}$ defined by

$$(11)\qquad \overline{P}_{n+2}(t) = \begin{cases} \mathrm{D}\,(P(-1), P'(t+1)) & -1 \le t \le n \\ \mathrm{D}\,(P''(t-n-1), P(n+1)) & n \le t \le 2n+1, \end{cases}$$

where by $P' : [0, n+1] \longrightarrow \mathbb{R}^2$ and $P'' : [-1, n] \longrightarrow \mathbb{R}^2$ we denote the straight line segments from $P(0)$ to $P(n+1)$ and from $P(-1)$ to $P(n)$, respectively.
The map $\overline{P}_{n+2}$ corresponds to a rotation obtained if we initially move $(T \circ h)(P)$ along the straight line segment P' from P_0 to P_{n+1}, holding P_{-1} fixed, and then if we move P along the straight line segment P'' from P_{-1} to P_n, holding P_{n+1} fixed.
More precisely, for every $0 \le \mu \le 1$, we define

$$\overline{P}_{n+1+\mu}(t) = \begin{cases} \mathrm{D}\,(P(-1)\,,\ (1-\mu)\,P(t+1) + \mu\,P'(t+1)) \\ \qquad\qquad\qquad\qquad -1 \le t \le n \\ \mathrm{D}((1-\mu)P(t-n-1) + \mu P''(t-n-1), P(n+1)) \\ \qquad\qquad\qquad\qquad n \le t \le 2n+1. \end{cases}$$

Clearly, the above definition of $\overline{P}_{n+1+\mu}$ in the case $\mu = 0$ and $\mu = 1$ is compatible with (10) and (11), respectively.
Moreover, the homotopy is well defined. To prove this, we will show first that $P(-1)$ is never equal to $Q := (1-\mu)P(t+1) + \mu P'(t+1)$ for any $t \in [-1, n]$. Indeed, by Property 3 of $\mathcal{C}$, we deduce that Q has larger $y-$coordinate than $P(-1)$, except possibly when $t = -1$ or $\mu = 0$. However, in both these cases $Q = P(t+1)$ for some $t \in [-1, n]$. Since in this interval $t+1 \ge 0 > -1$, then Property 1 of $\mathcal{C}$ guarantees that $P(t+1) \ne P(-1)$. Hence, $P(-1) \ne Q$.
Analogously, by applying Property 2 and Property 1 of $\mathcal{C}$, we can conclude that $(1-\mu)P(t-n-1) + \mu P''(t-n-1) \ne P(n+1)$. Thus, the homotopy is well defined.

In particular, $\overline{P}_{n+2}$ defined in the interval $[-1, 2n+1]$ describes an increase in the angle which corresponds exactly to ϑ, calculated above. Thus as a consequence of the homotopy property, we conclude that

$$i_{(T\circ h)}(\mathcal{C}) = \frac{\vartheta}{2\pi} = \frac{1}{2} - \frac{1}{2\pi}\left(\operatorname{arctg}\left(\frac{\delta_1}{s_1(x_n)}\right) + \operatorname{arctg}\left(\frac{\delta_2}{s_2(x_{-1})}\right)\right).$$

From the previous calculations, we get

$$\frac{1}{4} < i_{(T\circ h)}(\mathcal{C}) \le \frac{1}{2}.$$

Our aim consists now in proving that $i_h(\mathcal{C}) = \frac{1}{2}$.
To this end, we define for every $s \in [0, 1]$ the map $T_s : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, setting

$$T_s(x, y) = \left(x\,,\ y + \left(\frac{s\,\varepsilon}{2}\right)(\,|\cos x| - \cos x)\right).$$

In particular, $T_0 = \mathrm{Id}$ and $T_1 = T$. Arguing as before, we can easily see that

$$(12) \qquad i_{(T_s \circ h)}(\mathcal{C}) = \frac{1}{2} - \frac{1}{2\pi}\left(\operatorname{arctg}\left(\frac{s\,\delta_1}{s_1(x_n)}\right) + \operatorname{arctg}\left(\frac{s\,\delta_2}{s_2(x_{-1})}\right)\right) \ (\mathrm{mod}\, 1)\,.$$

Since this congruence becomes an equality in the case $s = 1$, by the continuity of the index we infer that also the congruence in (12) is an equality for every $s \in [0, 1]$. Hence, when $s = 0$ we can conclude that

$$(13) \qquad i_h(\mathcal{C}) = \frac{1}{2}.$$

In order to get the contradiction with Lemma 1, we need to construct another curve $\mathcal{C}'$ running from H_- to H_+, having index different from $\frac{1}{2}$. To this aim, we can repeat the whole argument replacing h with h^{-1}. Now everything works as before except the fact that the directions along which the two boundaries of $\widetilde{\mathcal{A}}$ move under h and under h^{-1} are opposite. In such a way, we find a curve $\widehat{\mathcal{C}}$ from H_- to H_+ with $i_{h^{-1}}(\widehat{\mathcal{C}}) = -\frac{1}{2}$. Let us define $\mathcal{C}' := h^{-1} \circ \widehat{\mathcal{C}} : H_- \longrightarrow H_+$. By Property 4 of the index, we finally infer

$$i_h\left(\mathcal{C}'\right) = i_{h^{-1}}\left(h\left(\mathcal{C}'\right)\right) = i_{h^{-1}}(\widehat{\mathcal{C}}) = -\frac{1}{2}.$$

If we compare the above equality with the equality in (13), we get the desired contradiction with Lemma 1.

□

As a consequence of the above proof, Neumann in [27] provides the following useful remark

REMARK 3. If h satisfies all the assumptions of Theorem 7 and it has a finite number of families of fixed points (finite number of fixed points in $[0, 2\pi] \times [r_1, r_2]$), then there exist fixed points with positive and negative indices.

We recall that the definition of index of a fixed point coincides with $i_h(\alpha)$ for a small circle α surrounding the fixed point when it has a positive (counter-clockwise) direction. Given a fixed point F, we will denote by $ind(F)$ its index.

Proof. Let us denote by F_i $(i = 1, 2, \ldots, k)$ the distinct fixed points in $[0, 2\pi] \times (r_1, r_2)$, belonging to different periodic families. Theorem 7 guarantees that $k \geq 2$.
It is not restrictive to assume that $F_i \in (0, 2\pi) \times (r_1, r_2)$ since we suppose that the number of families of fixed points is finite. As in the proof of Theorem 7, we extend the homeomorphism h to an homeomorphism in the whole $\mathbb{R}^2$, and we still denote it by h.
If we fix $r_0 < r_1$, arguing as in the proof of Lemma 1, we can construct a loop $\mathcal{D}' := \mathcal{D}_1\,\mathcal{D}_2\,\mathcal{D}_3\,\mathcal{D}_4 \in \pi_1(\mathbb{R}^2 \setminus \mathrm{Fix}(h), (0, r_0))$, where
$\mathcal{D}_1$ covers $[0, 2\pi] \times \{r_0\}$, moving horizontally from $(0, r_0)$ to $(2\pi, r_0)$;
$\mathcal{D}_2$ covers $\{2\pi\} \times [r_0, r_3]$ with $r_3 > r_2$, moving vertically from $(2\pi, r_0)$ to $(2\pi, r_3)$;

$\mathcal{D}_3$ covers $[0, 2\pi] \times \{r_3\}$, moving horizontally from $(2\pi, r_3)$ to $(0, r_3)$;
$\mathcal{D}_4$ covers $\{0\} \times [r_0, r_3]$, moving vertically from $(0, r_3)$ to $(0, r_0)$.
In particular, $\mathcal{D}'$ moves with a positive orientation and, by construction, the only fixed points of h it surrounds are exactly the fixed points F_i, $i \in \{1, \ldots, k\}$. We note that $i_h(\mathcal{D}_1) = i_h(\mathcal{D}_3) = 0$, since the curves $\mathcal{D}_1$ and $\mathcal{D}_3$ respectively lie in H_- and H_+ and $\mathrm{D}(P, h(P))$ is constant in these regions. Furthermore, being $h(x, y) - (x, 0)$ a $2\pi-$periodic function in its first variable, it follows that $i_h(\mathcal{D}_4) = -i_h(\mathcal{D}_2)$. Thus, Property 3 of the index guarantees that $\mathcal{D}'$ has index zero.

We recall that the fundamental group $\pi_1(\mathbb{R}^2 \setminus \mathrm{Fix}(h), (0, r_0))$ is generated by paths which start from $(0, r_0)$, run along a curve $\mathcal{C}_0$ to near a fixed point, loop around this fixed point and return by $-\mathcal{C}_0$ to $(0, r_0)$. It is possible to show that the generating paths, whose composition is deformable into the closed curve $\mathcal{D}'$, surround only the inner fixed points F_i. Consequently, the following equality holds

$$0 = i_h(\mathcal{D}') = \sum_{j=1}^{k} ind(F_j)\,. \tag{14}$$

This means that the sum of the fixed point indices is zero. We remark that such a result could have been directly obtained from the Lefschetz fixed point theorem.

Next step consists in constructing two curves with opposite indices, running from H_- to H_+ and not passing through any fixed point of h.

Since the number of fixed points in $[0, 2\pi] \times [r_1, r_2]$ is finite, it is possible to consider a non-empty vertical strip $\widehat{W} = [\alpha, \beta] \times \mathbb{R}$, for some $\alpha, \beta \in (0, 2\pi)$, which does not contain any F_i. Let us extend $2\pi-$periodically $\widehat{W}$ into the set

$$\bigcup_{m\in\mathbb{Z}} \big(\widehat{W} + (2m\pi, 0)\big) := \{(x, y) \in \mathbb{R}^2 : \ 2m\pi + \alpha \le x \le 2m\pi + \beta, \ \ m \in \mathbb{Z}\},$$

that we still denote by $\widehat{W}$.

Arguing as in the proof of Theorem 7, we can find a positive constant $\varepsilon < \min s_i$ for every $i \in \{1, 2\}$, satisfying $\varepsilon < \|P - h(P)\|$ for every $P \in \widehat{W}$. Let us now introduce the area-preserving homeomorphism $\widehat{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by setting

$$\widehat{T}_{|[0,2\pi]\times\mathbb{R}}(x, y) := \begin{cases} (x, y + \varepsilon \cos\Big(\frac{\pi}{2(\beta-\alpha)}\,(2x - \beta - \alpha)\Big) & \text{for } x \in [\alpha, \beta] \\ (x, y) & \text{otherwise}\,. \end{cases}$$

Fixed points of $\widehat{T} \circ h$ coincide with the ones of h in $\mathbb{R}^2$. If we proceed exactly as in the proof of Theorem 7, considering the homeomorphism $\widehat{T}$ instead of T and the set $\widehat{W}$ instead of W, we are able to construct a curve $\mathcal{C}$ of index $\frac{1}{2}$, which runs from $P_{-1} \in H_-$ to $P_n \in H_+$ and does not pass through any fixed point of h. Analogously, we can find another curve $\mathcal{C}'$ of index $-\frac{1}{2}$ running from $P'_{-1} \in H_-$ to $P'_n \in H_+$.

Let us consider now the closed curve $\mathcal{F} := \mathcal{C}\,\mathcal{B}\,(-\mathcal{C}')\,\mathcal{B}'$, where $\mathcal{B}$ is the straight line segment from P_n to P'_n; while $\mathcal{B}'$ is the straight line segment from P'_{-1} to P_{-1}. In particular, $\mathcal{B}$ lies in H_+ and connects the curve $\mathcal{C}$ to the curve $-\mathcal{C}'$; while $\mathcal{B}'$ lies in H_- and connects the curve $-\mathcal{C}'$ to the curve $\mathcal{C}$.

Since, by construction, $\mathcal{B}$ and $\mathcal{B}'$ have index equal to zero, we infer from Property 2 of the index that

$$i_h(\mathcal{F}) = i_h(\mathcal{C}) - i_h(\mathcal{C}') = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1\,.$$

Moreover, the loop $\mathcal{F}$ belongs to the fundamental group $\pi_1(\mathbb{R}^2 \setminus \mathrm{Fix}(h), P_{-1})$ and surrounds a finite number of fixed points. Each of them is of the form $F_i + m(2\pi, 0)$ for some $i \in \{1, 2 \ldots, k\}$ and some integer $m \in \mathbb{Z}$. Since $ind(F_i + m(2\pi, 0)) = ind(F_i)$ for every $m \in \mathbb{Z}$, we can deduce that

$$1 = i_h(\mathcal{F}) = \sum_{j=1}^{k} \nu(\mathcal{F}, F_j)\ ind(F_j)\,, \tag{15}$$

where the integer $\nu(\mathcal{F}, F_j)$ coincides with the sum of all the signs corresponding to the directions of every loop in which $\mathcal{F}$ can be deformed in a neighbourhood of every point of the form $F_j + m(2\pi, 0)$ surrounded by $\mathcal{F}$. From (15), we infer that there exists $j^* \in \{1, 2 \ldots, k\}$ such that $\nu(\mathcal{F}, F_{j^*})\, ind(F_{j^*}) > 0$ and, consequently, $ind(F_{j^*}) \neq 0$.

Hence, recalling that the sum of the fixed point indices is zero (cf. (14)), we can conclude the existence of at least a fixed point with positive index and a fixed point with negative one. This completes the proof.

□

3. Applications of the Poincaré-Birkhoff theorem

In this section we are interested in the applications of the Poincaré-Birkhoff fixed point theorem to the study of the existence and multiplicity of T-periodic solutions of Hamiltonian systems, that is systems of the form

$$\begin{cases} x' &= \frac{\partial H}{\partial y}(t, x, y) \\ \\ y' &= -\frac{\partial H}{\partial x}(t, x, y) \end{cases} \tag{16}$$

where $H : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous scalar function that we assume T−periodic in t and C^2 in $z = (x, y)$.
Under these conditions uniqueness of Cauchy problems associated to system (16) is guaranteed. Hence for each $z_0 = (x_0, y_0) \in \mathbb{R}^2$ and $t_0 \in \mathbb{R}$ there is a unique solution $(x(t), y(t))$ of system (16) such that

$$(x(t_0), y(t_0)) = (x_0, y_0) := z_0. \tag{17}$$

In the following we will denote such a solution by

$$z(t; t_0, z_0) := (x(t; t_0, z_0), y(t; t_0, z_0)) := (x(t; t_0, (x_0, y_0)), y(t; t_0, (x_0, y_0))).$$

For simplicity we set $z(t;z_0) := (x(t;z_0), y(t;z_0)) := (x(t;0,z_0), y(t;0,z_0))$. If we suppose that H satisfies further conditions which imply global existence of the solutions of Cauchy problems, then the Poincaré operator

$$\tau : z_0 = (x_0, y_0) \to (x(T;(x_0,y_0)), y(T;(x_0,y_0)))$$

is well defined in $\mathbb{R}^2$ and it is continuous. Also fixed points of the Poincaré operator are initial conditions of periodic solutions of system (16) and as a consequence of the Liouville theorem, the Poincaré operator is an area-preserving map. Hence it is natural to try to apply the Poincaré-Birkhoff fixed point theorem in order to prove the existence of periodic solutions of the Hamiltonian systems.

Before giving a version of the Poincaré-Birkhoff fixed point theorem useful for the applications, we previously introduce some notation.
Let $z : [0,T] \to \mathbb{R}^2$ be a continuous function satisfying $z(t) \neq (0,0)$ for every $t \in [0,T]$ and $(\vartheta(\cdot), r(\cdot))$ a lifting of $z(\cdot)$ to the polar coordinate system. We define the rotation number of z, and denote it by $\mathrm{Rot}(z)$ as

$$\mathrm{Rot}(z) := \frac{\vartheta(T) - \vartheta(0)}{2\pi}.$$

Note that $\mathrm{Rot}(z)$ counts the counter-clockwise turns described by the vector $\overrightarrow{0\,z(s)}$ as s moves in the interval $[0,T]$. In what follows, we will use the notation $\mathrm{Rot}(z_0)$ to indicate $\mathrm{Rot}(z(\cdot;z_0))$.

From the Poincaré-Birkhoff Theorem 4, we can obtain the following multiplicity result.

THEOREM 8 ([29]). *Let $\mathcal{A} \subset \mathbb{R}^2 \setminus \{(0,0)\}$ be an annular region surrounding $(0,0)$ and let C_1 and C_2 be its inner and outer boundaries, respectively. Assume that C_1 is strictly star-shaped with respect to $(0,0)$ and that $z(\cdot;t_0,z_0)$ is defined in $[t_0,T]$ for every $z_0 \in C_2$ and $t_0 \in [0,T]$. Suppose that*

i) $z(t;t_0,z_0) \neq (0,0) \qquad \forall\, t_0 \in [0,T[\ ,\ \forall\, z_0 \in C_1\ ,\ \forall\, t \in [t_0,T];$

ii) there exist $m_1, m_2 \in \mathbb{Z}$ with $m_1 \geq m_2$ such that

$$\mathrm{Rot}(z_0) > m_1 \qquad \forall\, z_0 \in C_1,$$

$$\mathrm{Rot}(z_0) < m_2 \qquad \forall\, z_0 \in C_2.$$

Then, for each integer l with $l \in [m_2, m_1]$, there are two fixed points of the Poincaré map which correspond to two periodic solutions of the Hamiltonian system having l as T−rotation number.

Sketch of the proof. The idea of the proof consists in applying Theorem 4 to the area-preserving Poincaré map $\tau : z_0 \longrightarrow z(T;z_0)$, considering different liftings of it. For each integer l with $m_2 \leq l \leq m_1$, it is possible to consider the liftings

$$\widetilde{\tau}_l(\vartheta, r) := (\vartheta + 2\pi(\mathrm{Rot}\,\Pi(\vartheta,r) - l)\,,\ \|\,\tau(\Pi(\vartheta,r))\,\|\,).$$

Since $z(t; z_0) \neq (0,0)$ for every $t \in [0, T]$ and for every $z_0 \in \mathcal{A}$, the liftings are well defined. We note that as a consequence of i), $(0,0)$ belongs to the image of the interior of C_1 and also that

$$\begin{aligned} &\mathrm{Rot}(z_0) - l \geq \mathrm{Rot}(z_0) - m_1 > 0 \qquad && \forall\, z_0 \in C_1, \\ &\mathrm{Rot}(z_0) - l \leq \mathrm{Rot}(z_0) - m_2 < 0 \qquad && \forall\, z_0 \in C_2. \end{aligned}$$

Hence we can easily conclude that assumptions (a) and (b) in Theorem 4 are satisfied. Moreover it is easy to show that also assumption (c) is verified. Hence, from Theorem 4 we infer the existence of at least two fixed points (ϑ_l^i, r_l^i), $i = 1, 2$, of $\widetilde{\tau}_l$ whose images z_l^i under the projection Π are two different fixed points of τ. Since (ϑ_l^i, r_l^i) are fixed points of $\widetilde{\tau}_l$, we get that $\mathrm{Rot}(z_l^i) = l$ for every $i \in \{1, 2\}$. We can finally conclude that $z(\cdot; z_l^1)$ and $z(\cdot; z_l^2)$ are the searched $T-$periodic solutions.

□

There are many examples in the literature of the application of the Poincaré-Birkhoff theorem in order to study the existence and multiplicity of $T-$periodic solutions of the equation

$$x'' + f(t, x) = 0, \tag{18}$$

with $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ continuous and T-periodic in t. Note that if we consider the system

$$\begin{cases} x'(t) = y(t) \\ y'(t) = -f(t, x(t)), \end{cases} \tag{19}$$

this system is a particular case of system (16) and its solutions give rise to solutions of equation (18). Hence we can consider equation (18) as a particular case of an Hamiltonian system and everything we mentioned above holds for the case of this equation.

Among the mathematicians who studied existence and multiplicity of periodic solutions for equation (18) via the Poincaré-Birkhoff theorem, we quote Jacobowitz [22], Hartman [20], Butler [7]. We remark that in order to reach the results, in all of these papers the authors assumed the validity of the condition $f(t, 0) \equiv 0$.

With respect to the particular case of the nonlinear Duffing's equation

$$x'' + g(x) = p(t),$$

we mention the papers [15], [11], [13], [12], [17], [32], in which the Poincaré-Birkhoff theorem was applied in order to prove the existence of periodic solutions with prescribed nodal properties. Among the applications of the Poincaré-Birkhoff theorem to the analysis of periodic solutions to nonautonomous second order scalar differential equations depending on a real parameter s, we refer to the paper [10] by Del Pino, Manásevich and Murua, which studies the following equation

$$x'' + g(x) = s(1 + h(t))$$

and also the paper [30] by Rebelo and Zanolin, which deals with the equation

$$x'' + g(x) = s + w(t, x).$$

Finally, we quote Hausrath, Manásevich [21] and Ding, Zanolin [14] for the treatment of periodically perturbed Lotka-Volterra systems of type

$$\begin{cases} x' = x\,(a(t) - b(t)\,y) \\ y' = y\,(-d(t) + c(t)\,x). \end{cases}$$

We describe now recent results obtained in [24] in which a modified version of the Poincaré-Birkhoff fixed point theorem is obtained and applied, together with the classical one, in order to obtain existence and multiplicity of periodic solutions for Hamiltonian systems. In their paper the authors study system (16) assuming that $z = 0$ is an equilibrium point, i.e. $H'_z(t, 0) \equiv 0$, and that it is an asymptotically linear Hamiltonian system. This implies that it admits linearizations at zero and infinity. More precisely, as $H'_z(t, 0) \equiv 0$, if we consider the continuous and $T-$periodic function with range in the space of symmetric matrices given by $t \to B_0(t) := H''_z(t, 0),\ t \in \mathbb{R}$, we have

$$J\,H'_z(t, z) = J\,B_0(t)\,z + o(\|z\|), \qquad \text{when } z \to 0.$$

Moreover, by definiton of asymptotically linear system, there exists a continuous, $T-$periodic function $B_\infty(\cdot)$ such that $B_\infty(t)$ is a symmetric matrix for each $t \in \mathbb{R}$, satisfying

$$J\,H'_z(t, z) = J\,B_\infty(t)\,z + o(\|z\|), \qquad \text{when } \|z\| \to \infty.$$

We remark that system (16) can be equivalently written in the following way

$$z' = J\,H'_z(t, z)\,, \qquad z = (x, y)\,, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{20}$$

Before going on with the description of the results obtained in [24], we recall some results present in the literature dealing with the study of asymptotically linear Hamiltonian systems.
In [2] and [3], Amann and Zehnder considered asymptotically linear systems in $\mathbb{R}^{2N}$ of the form of system (20) with

$$\sup_{t\in[0,T],\, z\in\mathbb{R}^{2N}} \|H''_z(t, z)\| < +\infty$$

and which admit autonomous linearizations at zero and at infinity

$$z' = J\,B_0\,z\,, \qquad z' = J\,B_\infty\,z\,,$$

respectively. In these papers an index i depending on B_0 and B_∞ was introduced and the existence of at least one nontrivial $T-$periodic solution combining nonresonance conditions at infinity with the sign assumption $i > 0$ was achieved. The authors also remarked that in the planar case $N = 1$ the condition $i > 0$ corresponds to the twist condition in the Poincaré-Birkhoff theorem.

Some years later, Conley and Zehnder studied in [9] Hamiltonian systems with bounded Hessian, considering the general case in which the linearized systems at zero and at infinity

$$z' = J B_0(t)z \qquad \text{and} \qquad z' = J B_\infty(t)z$$

can be nonautonomous. The authors assumed nonresonance conditions for the linearized systems at zero and at infinity. Hence, after defining the Maslov indices associated to the above linearizations at zero and infinity, denoted respectively by i_T^0 and i_T^∞, they proved the following result.

THEOREM 9. *If $i_T^0 \neq i_T^\infty$, then there exists a nontrivial $T-$periodic solution of $z' = J H_z'(t,z)$. If this solution is nondegenerate, then there exists another $T-$periodic solution.*

Note that in this last theorem the existence of more than two solutions is not guaranteed, even if $|i_T^0 - i_T^\infty|$ is large. This is in contrast with the fact that in the paper [9] and for the case $N = 1$ the authors mention that the Maslov index is a measure of the twist of the flow. In fact, if this is the case, a large $|i_T^0 - i_T^\infty|$ should imply large gaps between the twists of the flow at the origin and at infinity. Hence, the Poincaré-Birkhoff theorem would provide the existence of a large number of periodic solutions.

The main goal in [24] consists in clarifying the relation between i_T^0, i_T^∞ and the twist condition in the Poincaré-Birkhoff theorem, when $N = 1$, obtaining multiplicity results in the case when $|i_T^0 - i_T^\infty|$ is large.

Now we give a glint of the notion of Maslov index in the plane. We will follow [1] (see also [19]).

Let us consider the following planar Cauchy problem

$$\begin{cases} z' = J B(t)z \\ z(0) = w, \end{cases} \tag{21}$$

where $B(t)$ is a T-periodic continuous path of symmetric matrices. The matrix $\Psi(t)$ is called the fundamental matrix of the system (21) if it satisfies $\Psi(t)\, w = z(t; w)$. Clearly, $\Psi(0) = \mathrm{Id}$. Moreover, it is well known that as $B(t)$ is symmetric, the fundamental matrix $\Psi(t)$ is a *symplectic matrix* for each $t \in [0, T]$. We recall that a matrix A of order two is symplectic if it verifies

$$A^T J A = J, \tag{22}$$

where J is as in (20). Since we are working in a planar setting, condition (22) is equivalent to

$$\det A = 1\,,$$

from which it follows immediately that the symplectic 2×2 matrices form a group, usually denoted by $Sp(1)$.

We will show that, under a nonresonance condition on (21), it is possible to associate to the path $t \to \Psi(t)$ of symplectic matrices with $\Psi(0) = \mathrm{Id}$ an integer, the $T-$Maslov index $i_T(\Psi)$.
The system $z' = J\,B(t)z$ is said to be $T-$*nonresonant* if the only T-periodic solution it admits is the trivial one or, equivalently, if

$$\det(\mathrm{Id} - \Psi(T)) \neq 0,$$

where Ψ is the fundamental matrix of (21).

Before introducing the Maslov indices, we need to recall some properties of $Sp(1)$. If we take $A \in Sp(1)$, then A can be uniquely decomposed as

$$A = P \cdot O,$$

where $P \in \{\widetilde{P} \in Sp(1) \,:\, \widetilde{P} \text{ is symmetric and positive definite}\} \approx \mathbb{R}^2$ and O is symplectic and orthogonal. In particular, O belongs to the group of the rotations $SO(2) \approx S^1$. Thus we can conclude that

$$Sp(1) \approx \mathbb{R}^2 \times S^1 \approx \{z \in \mathbb{R}^2 \,:\, |z| < 1\} \times S^1 = \text{ the interior of a torus.}$$

Hence, as $[0,1) \times \mathbb{R} \times \mathbb{R}$ is a covering space of the interior of the torus, we can parametrize $Sp(1)$ with $(r, \sigma, \vartheta) \in [0,1) \times \mathbb{R} \times \mathbb{R}$. In [1] a parametrization

$$\begin{array}{rccl} \Phi \,: & [0,1) \times \mathbb{R} \times \mathbb{R} & \longrightarrow & Sp(1) \\ & (r, \sigma, \vartheta) & \to & \Phi(r, \sigma, \vartheta) = P(r, \sigma)\, R(\vartheta) \end{array}$$

is given, where ϑ is the angular coordinate on S^1 and (r, σ) are polar coordinates in $\{z \in \mathbb{R}^2 \,:\, |z| < 1\}$. In such a parametrization, for each $k \in \mathbb{Z}$ and $\sigma \in \mathbb{R}$, $\Phi(0, \sigma, 2k\pi) = \mathrm{Id}$ and $\Phi(0, \sigma, 2(k+1)\pi) = -\mathrm{Id}$ (for the details see [1]). The following sets are essential in order to define the $T-$Maslov index:

$$\begin{aligned} \Gamma^+ \,:\; &= \{A \in Sp(1) \,:\, \det(\mathrm{Id} - A) > 0\} \\ &= \Phi\{(r, \sigma, \vartheta) \,:\, r < \sin^2\vartheta \text{ and } |\vartheta| < \frac{\pi}{2} \quad \text{or} \quad |\vartheta| \geq \frac{\pi}{2}\}, \end{aligned}$$

$$\Gamma^- := \{A \in Sp(1) \,:\, \det(\mathrm{Id}-A) < 0\} = \Phi\{(r, \sigma, \vartheta) \,:\, r > \sin^2\vartheta \text{ and } |\vartheta| < \frac{\pi}{2}\},$$

$$\Gamma^0 := \{A \in Sp(1) \,:\, \det(\mathrm{Id}-A) = 0\} = \Phi\{(r, \sigma, \vartheta) \,:\, r = \sin^2\vartheta \text{ and } |\vartheta| < \frac{\pi}{2}\}.$$

The set Γ^0 is called the resonant surface and it looks like a two-horned surface with a singularity at the identity.

Now we are in position to associate to each path $t \to \Psi(t)$ defined from $[0, T]$ to $Sp(1)$, satisfying $\Psi(0) = \mathrm{Id}$ and $\Psi(T) \notin \Gamma^0$ an integer which will be called the Maslov index of Ψ. To this aim we extend such a path $t \to \Psi(t) \in Sp(1)$ in $[T, T+1]$, without intersecting Γ^0 and in such a way that

- $\Psi(T+1) = -\mathrm{Id}$, if $\Psi(T) \in \Gamma^+$,

- $\Psi(T+1)$ is a standard matrix with $\vartheta = 0$, if $\Psi(T) \in \Gamma^-$.

We define the *T-Maslov index* $i_T(\Psi)$ as the (integer) number of half turns of $\Psi(t)$ in $Sp(1)$, as t moves in $[0, T+1]$, counting each half turn ± 1 according to its orientation.

In order to compare Theorem 8 with Theorem 9 it is necessary to find a characterization of the Maslov indices in terms of the rotation numbers. To this aim, in [24] a lemma which provides a relation between the $T-$Maslov index of system

$$z' = J\,B(t)z \tag{23}$$

and the rotation numbers associated to the solutions of (23) was given.

LEMMA 2. *Let Ψ be the fundamental matrix of system* (23) *and let i_T and ψ be, respectively, its T-Maslov index and the Poincaré map defined by*

$$\psi \;:\; w \;\to\; \Psi(T)w\,.$$

Consider the $T-$rotation number $\mathrm{Rot}_w(T)$ *associated to the solution of* (23) *satisfying $z(0) = w \in S^1$. Then,*
a) $i_T = 2\ell + 1$ with $\ell \in \mathbb{Z}$ if and only if

$$\deg(\mathrm{Id} - \psi, B(1), 0) = 1 \text{ and } \ell < \min_{w\in S^1} \mathrm{Rot}_w(T) \le \max_{w\in S^1} \mathrm{Rot}_w(T) < \ell + 1;$$

b) $i_T = 2\ell$ with $\ell \in \mathbb{Z}$ if and only if

$$\deg(\mathrm{Id} - \psi, B(1), 0) = -1 \text{ and } \ell - \frac{1}{2} < \min_{w\in S^1} \mathrm{Rot}_w(T) \le \max_{w\in S^1} \mathrm{Rot}_w(T) < \ell + \frac{1}{2};$$

moreover, in this case there are $w_1, w_2 \in S^1$ such that

$$\mathrm{Rot}_{w_1}(T) < \ell < \mathrm{Rot}_{w_2}(T)\,.$$

In the statement of Lemma 2 the $T-$rotation number $\mathrm{Rot}_w(T)$ associated to the solution of (23) with $z(0) = w \in S^1$ was considered. We observe that from the linearity of system (23) it follows that $\mathrm{Rot}_w(T) = \mathrm{Rot}_{\lambda w}(T)$ for every $\lambda > 0$.
Now, we are in position to make a first comparison between Theorem 8 and Theorem 9.
Let us consider the second order scalar equation

$$x'' + q(t,x)x = 0\,, \tag{24}$$

where the continuous function $q : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is $T-$periodic in its first variable t and it satisfies

$$q(t,0) \equiv q_0 \in \mathbb{R} \quad \text{and} \quad \lim_{|x|\to+\infty} q(t,x) = q_\infty \in \mathbb{R}, \tag{25}$$

uniformly with respect to $t \in [0, T]$. Hence, the linearizations of (24) at zero and infinity are respectively $x'' + q_0 x = 0$ and $x'' + q_\infty x = 0$. We observe that equation (24) can be equivalently written in the following form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ -q(t,x)x \end{pmatrix} = J \begin{pmatrix} q(t,x)x \\ y \end{pmatrix}.$$

Analogously, the corresponding linearizations at zero and infinity are given respectively by

$$z' = J \begin{pmatrix} q_0 & 0 \\ 0 & 1 \end{pmatrix} z = \begin{pmatrix} 0 & 1 \\ -q_0 & 0 \end{pmatrix} z$$

and

$$z' = J \begin{pmatrix} q_\infty & 0 \\ 0 & 1 \end{pmatrix} z = \begin{pmatrix} 0 & 1 \\ -q_\infty & 0 \end{pmatrix} z.$$

If we choose $q_0 = -\frac{1}{2}$ and $q_\infty = 5$, we can easily deduce that there exists $r_0 > 0$ such that $\mathrm{Rot}_w(T) \in (-\frac{1}{2}, \frac{1}{2})$ for every $\|w\| = r_0$ and there exists $R_0 > r_0$ such that $\mathrm{Rot}_w(T) \in (-3, -2)$ for every $\|w\| = R_0$. Hence applying Theorem 8 we can guarantee the existence of four nontrivial T-periodic solutions to equation (24). On the other hand, since $i_T^0 = 0$ and $i_T^\infty = -5$, Theorem 9 ensures the existence of at least one periodic solution.

We recall that, even if the gap between q_0 and q_∞ is large, Theorem 9 guarantees only the existence of at least one solution (or at least two solutions if the first one is nondegenerate) while it is quite clear that the number of nontrivial periodic solutions we can find by applying Theorem 8 depends on the gap between q_0 and q_∞.

On the other hand, there are particular situations in which Theorem 9 can be applied, while Theorem 8 cannot, because the twist condition is not satisfied.

For instance, let us set $q_0 = -\frac{1}{2}$ and $q_\infty = \frac{1}{2}$. The corresponding indices are different, since $i_T^0 = 0$, as before, and $i_T^\infty = -1$. Hence, from Theorem 9, we know that there exists a periodic solution of (24). As far as the rotation numbers are concerned, one can prove the existence of $R_0 > r_0$ such that $\mathrm{Rot}_w(T) \in (-1, 0)$ for every $\|w\| = R_0$; while, from Lemma 2, there exist $w_1,\ w_2 \in \mathbb{R}^2$ with $\|w_i\| = r_0$ $(i = 1, 2)$ such that $-\frac{1}{2} < \mathrm{Rot}_{w_1}(T) < 0 < \mathrm{Rot}_{w_2}(T) < \frac{1}{2}$. Consequently, the twist condition is not verified and Theorem 8 is not applicable. In [24] the authors tried to sharpen the results obtained via the Poincaré-Birkhoff theorem in order to obtain periodic solutions in cases like this one. For this purpose, they developed a suitable version of the Poincaré-Birkhoff theorem. Before describing this result we can obtain a first result of multiplicity of T-periodic solutions for system (16) which is a consequence of Lemma 2 and of Theorem 8.

We will use the notation: for each $s \in \mathbb{R}$, we denote by $\lfloor s \rfloor$ the integer part of s, while we denote by $\lceil s \rceil$ the smallest integer larger than or equal to s.

COROLLARY 1. *Assume that $z' = J\, H_z'(t, z)$ is asymptotic at infinity and at zero to the $T-$periodic and $T-$nonresonant linear systems $z' = J\, B_\infty(t) z$ and $z' = J\, B_0(t) z$,* *respectively. Let i_T^∞ and i_T^0 be the corresponding $T-$Maslov indices. If $i_T^0 \neq i_T^\infty$ then the Hamiltonian system admits at least*

- $|i_T^\infty - i_T^0|$ *nontrivial $T-$periodic solutions if i_T^0 and i_T^∞ are odd;*
- $|i_T^\infty - i_T^0| - 2$ *nontrivial $T-$periodic solutions if i_T^0 and i_T^∞ are even;*
- $2\left\lfloor \dfrac{|i_T^\infty - i_T^0|}{2} \right\rfloor$ *nontrivial $T-$periodic solutions otherwise.*

REMARK 4. If i_T^0 and i_T^∞ are either consecutive integers or consecutive even integers, the previous corollary does not guarantee the existence of $T-$periodic solutions. Indeed in these cases the twist condition in Theorem 8 is not satisfied.
However, if i_T^0 and i_T^∞ are consecutive integers the excision property of the degree implies the existence of a $T-$periodic solution.

THEOREM 10 (MODIFIED POINCARÉ-BIRKHOFF THEOREM). *Let $\psi : \mathcal{A} \longrightarrow \mathcal{A}$ be an area-preserving homeomorphism in $\mathcal{A} = \mathbb{R} \times [0, R]$, $R > 0$ such that*

$$\psi(\vartheta, r) = (\vartheta_1, r_1),$$

with

$$\begin{cases} \vartheta_1 = \vartheta + g(\vartheta, r) \\ r_1 = f(\vartheta, r), \end{cases}$$

where f and g are $2\pi-$periodic in the first variable and satisfy the conditions

- *$f(\vartheta, 0) = 0$, $f(\vartheta, R) = R$ for every $\vartheta \in \mathbb{R}$ (boundary invariance),*
- *$g(\vartheta, R) > 0$ for every $\vartheta \in \mathbb{R}$ and there is $\overline{\vartheta}$ such that $g(\overline{\vartheta}, 0) < 0$ (modified twist condition).*

Then, ψ admits at least a fixed point in the interior of $\mathcal{A}$. If ψ admits only one fixed point in the interior of $\mathcal{A}$, then its fixed point index is nonzero.

Idea of the proof. By contradiction, it is assumed that there are no fixed points in the interior of $\mathcal{A}$. As in the proof of Theorem 7, the homeomorphism ψ is extended to an homeomorphism $\widehat{\psi} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. If the fixed point set of $\widehat{\psi}$ is not empty, it is union of vertical closed halflines in the halfplane $r \leq 0$ with origin on the line $r = 0$.
Without loss of generality, one can assume $\overline{\vartheta} \in (0, 2\pi)$. Hence, denoting by $\mathcal{N}$ the maximal strip contained in $\mathbb{R}\times] - \infty, 0]$ such that $(\overline{\vartheta}, 0) \in \mathcal{N}$ and $g(\vartheta, 0) < 0$ in $\mathcal{N}$, the following important property of $\mathcal{N}$ holds:
if $(\vartheta, r) \in \cup_{k\in\mathbb{Z}}(\mathcal{N} + (2k\pi, 0))$, then for each $n > 0$ we have that $\widehat{\psi}^{-n}(\vartheta, r)$ belongs to the connected component of $\cup_{k\in\mathbb{Z}}(\mathcal{N} + (2k\pi, 0))$ which contains (ϑ, r).
Then the proof follows steps analogous to those in Theorem 7, taking into account this property. The contradiction follows from the existence of a curve Γ with $i_{\widehat{\psi}}(\Gamma) =$

$-\frac{1}{2}$, which runs from the point $(\overline{\vartheta}, 0)$ to a point into $\{(\vartheta, r) : r \geq R\}$ and such that $i_{\widehat{\psi}^{-1}}(\widehat{\psi}(\Gamma)) = \frac{1}{2}$.

The fact that if ψ admits only one fixed point in the interior of $\mathcal{A}$, then its fixed point index is nonzero can be proved following similar steps to those in the proof of Remark 3. Now it will be important to take into account the property of $\mathcal{N}$ mentioned above.

□

At this point, the authors in [24] obtain a variant of Theorem 10 in which the invariance of the outer boundary is not assumed. The proof of this Corollary follows the same steps as the proof of Theorem 1 in [29].

Let Γ_1 be a circle with center in the origin and radius $R > 0$ and Γ_2 be a simple closed curve surrounding the origin. For each $i \in \{1, 2\}$ we denote by $\mathcal{B}_i$ the finite closed domain bounded by Γ_i. Let $\widetilde{\Gamma}_i$ be the lifting of Γ_i and $\widetilde{A}_i$ be the lifting of $\mathcal{A}_i$, where $\mathcal{A}_i := \mathcal{B}_i \setminus \{0\}$. Then, the following result holds.

COROLLARY 2. *Let $\psi : \mathcal{A}_1 \to \mathcal{A}_2$ be an area-preserving homeomorphism. Assume that ψ admits a lifting which can be extended to an homeomorphism $\widetilde{\psi} : \widetilde{\mathcal{A}}_1 \cup \{(\vartheta, r) : r = 0\} \to \widetilde{\mathcal{A}}_2 \cup \{(\vartheta, r) : r = 0\}$ given by $\widetilde{\psi}(\vartheta, r) = (\vartheta + g(\vartheta, r), f(\vartheta, r))$, where g and f are $2\pi-$periodic in the first variable. Moreover, suppose that $g(\vartheta, r) > 0$ for every $(\vartheta, r) \in \widetilde{\Gamma}_1$ and there is $\overline{\vartheta}$ such that $g(\overline{\vartheta}, 0) < 0$ (modified twist condition). Then, $\widetilde{\psi}$ admits at least a fixed point in the interior of $\widetilde{\mathcal{A}}_1$ whose image under the usual covering projection Π is a fixed point of ψ in $\mathcal{A}_1$. If ψ admits only one fixed point in the interior of $\mathcal{A}_1$, then its fixed point index is nonzero.*

We point out that the proof of Corollary 2 cannot be repeated if we modify the twist condition by supposing that there is $(\overline{\vartheta}, \overline{r}) \in \widetilde{\Gamma}_1$ such that $g(\overline{\vartheta}, \overline{r}) > 0$, while $g(\vartheta, 0) < 0$ for every $\vartheta \in \mathbb{R}$.

Now, we will show how the application of Theorem 10 to the scalar equation (24) can improve the multiplicity results achieved by applying Theorem 8 and Theorem 9.

First let us set once again $q_0 = -\frac{1}{2}$ and $q_\infty = \frac{1}{2}$ in (25). By the modified Poincaré-Birkhoff theorem, there is a fixed point P_0 of ϕ (that corresponds to a nontrivial $T-$periodic solution) and, if it is the unique fixed point, then

$$ind(P_0) \neq 0\,.$$

We recall that Simon in [31] has shown that an isolated fixed point of an area-preserving homeomorphism in $\mathbb{R}^2$ has index less than or equal to 1. In particular, the fixed point index of P_0 satisfies

$$ind(P_0) \leq 1\,.$$

As the fixed point index of ψ changes from -1 (near the origin) to $+1$ (near infinity), there is at least another fixed point P_1 of ϕ. Hence, in this case we can guarantee the existence of at least two nontrivial $T-$periodic solutions. We recall that applying Theorem 9 only the existence of one nontrivial periodic solution could be guaranteed.

Now we choose $q_0 = -\frac{1}{2}$ and $q_\infty = 3$ in (25). The Poincaré-Birkhoff theorem, according to Theorem 8, guarantees that there are at least two fixed points P_1 and P_2 of ϕ (which correspond to two $T-$periodic solutions with rotation number -1). If they are unique, then from Remark 3 we obtain that

$$ind(P_1) = +1\,, \qquad ind(P_2) = -1\,.$$

Moreover, by the modified Poincaré-Birkhoff theorem there is a fixed point P_3 of ϕ (which corresponds to a $T-$periodic solution with rotation number 0) and, if it is unique,

$$0 \neq ind(P_3) \leq 1\,.$$

As the fixed point index of ϕ changes from -1 (near the origin) to $+1$ (near infinity), there exists at least a fourth fixed point P_4 of ϕ. Summarizing, Theorem 8 combined with the modified Poincaré-Birkhoff theorem guarantees the existence of at least four nontrivial $T-$periodic solutions to (24). We recall that also in this case Theorem 9 is applicable and it ensures that there exists at least one nontrivial periodic solution.

Finally, we state the main multiplicity theorem. We point out that the multiplicity results achieved in the above examples can be also obtained by applying the following theorem.

THEOREM 11. *Assume that the conditions of Corollary* 1 *hold.*
Then if $i_T^0 \neq i_T^\infty$ *the Hamiltonian system* (16) *admits at least* $\max\left\{1, 2\left\lfloor \frac{|i_T^\infty - i_T^0|}{2} \right\rfloor\right\}$ *nontrivial* $T-$*periodic solutions.*
If i_T^0 *is even then the Hamiltonian system admits at least* $2\left\lceil \frac{|i_T^\infty - i_T^0|}{2} \right\rceil$ *nontrivial* $T-$*periodic solutions.*

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