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PARAMETRIC SURFACES WITH PRESCRIBED MEAN CURVATURE

Abstract. This article contains an overview on some old and new problems concerning two-dimensional parametric surfaces in $\mathbb{R}^3$ with prescribed mean curvature. Part of this exposition has constituted the subject of a series of lectures held by the first author at the Department of Mathematics of the University of Torino, during the Third Turin Fortnight on Nonlinear Analysis (September 23-28, 2001).

1. Introduction

The main focus of this article is the following problem: given a smooth, real function H in $\mathbb{R}^3$, find surfaces M having exactly mean curvature $H(p)$ at any point p belonging to M.

In order to get some intuition in the geometric and analytical aspects of this question, we believe that it might be of interest to consider first its two dimensional analog, where most concepts become rather elementary. Therefore, in this introductory part we will first discuss the following questions:

(Q_0) Given a smooth, real function κ on the plane $\mathbb{R}^2$, *find a closed curve* $\mathcal{C}$, such that for any point p in $\mathcal{C}$ the curvature of the curve at this point is exactly $\kappa(p)$ (we may possibly impose furthermore that $\mathcal{C}$ has no self intersection: $\mathcal{C}$ is then topologically a circle).

(Q_1) [*Planar Plateau problem*] Given two points a and b in the plane, and a smooth, real function κ on $\mathbb{R}^2$, *find a curve* $\mathcal{C}$ with $\partial\mathcal{C} = \{a, b\}$, such that for any point p in $\mathcal{C}$ the curvature of the curve at p is exactly $\kappa(p)$.

1.1. Parametrization

In order to provide an analytical formulation of these problems, the most natural approach is to introduce a parametrization of the curve $\mathcal{C}$, i.e., a map $u\colon I \to \mathbb{R}^2$, such that $|\dot{u}| = 1$, $u(I) = \mathcal{C}$, where I represents some compact interval of $\mathbb{R}$, and the notation $\dot{u} = \frac{du}{ds}$ is used. Notice that, nevertheless there are possible alternative approaches to parametrization: we will discuss this for surfaces in the next sections.

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Then, questions (Q_0) and (Q_1) can be formulated in terms of ordinary differential equations. More precisely, the fact that $\mathcal{C}$ has curvature $\kappa(u(s))$ at every point $u(s)$ belonging to $\mathcal{C}$ reads

$$\ddot{u} = i\kappa(u)\dot{u} \quad \text{on } I, \tag{1}$$

where i denotes the rotation by $\frac{\pi}{2}$. Note that the sign of the term of the r.h.s. depends on a choice of orientation, and the curvature might therefore take negative values.

The constraint $|\dot{u}| = 1$ might raise difficulties in order to find solutions to (Q_0) and (Q_1). It implies in particular that $|I| =$ length of $\mathcal{C}$, and this quantity is not know a priori. This difficulty can be removed if we consider instead of (1) the following equivalent formulation

$$\frac{|I|^{\frac{1}{2}}}{(\int_I |\dot{u}|^2\, ds)^{\frac{1}{2}}}\ddot{u} = i\kappa(u)\dot{u} \quad \text{on } I. \tag{2}$$

To see that (2) is an equivalent formulation of (1), note first that any solution u to (2) verifies

$$\frac{1}{2}\frac{d}{ds}(|\dot{u}|^2) = \ddot{u}\cdot\dot{u} = \frac{(\int_I |\dot{u}|^2\, ds)^{\frac{1}{2}}}{|I|^{\frac{1}{2}}}\kappa(u)i\dot{u}\cdot\dot{u} = 0,$$

so that $|\dot{u}| = C_0 =$ *const*. and, introducing the new parametrization $v(s) = u(s/C_0)$, we see that $|\dot{v}| = 1$, and v solves (1).

Hence, an important advantage of formulation (2) is that we do not have to impose any auxiliary condition on the parametrization since equation (2) is independent of the interval I. Thus, we may choose $I = [0, 1]$ and (2) reduces to

$$\ddot{u} = iL(u)\kappa(u)\dot{u} \quad \text{on } [0, 1]\,, \tag{3}$$

where

$$L(u) := \left(\int_I |\dot{u}|^2\, ds\right)^{\frac{1}{2}}.$$

Each of the questions (Q_0) and (Q_1) has then to be supplemented with appropriate boundary conditions:

$$u(0) = u(1),\ \dot{u}(0) = \dot{u}(1) \quad \text{for } (Q_0)$$

(or alternatively, to consider $\mathbb{R}/\mathbb{Z}$ instead of [0, 1]), and

$$u(0) = a,\ u(1) = b, \quad \text{for } (Q_1). \tag{4}$$

1.2. The case of constant curvature

We begin the discussion of these two questions with the simplest case, namely when the function κ is a constant $\kappa_0 > 0$. It is then easily seen that the *only* solutions to equations (1) (or (3)) are portions of circles of radius $R_0 = \frac{1}{\kappa_0}$. Therefore, for (Q_0) we obtain the simple answer: the solutions are circles of radius $1/\kappa_0$.

For question (Q_1) a short discussion is necessary: we have to compare the distance $l_0 := |a - b|$ with the diameter $D_0 = 2R_0$. Three different possibilities may occur:

(i) $l_0 > D_0$, i.e., $\frac{1}{2}l_0\kappa_0 > 1$. In this case there is no circle of diameter D_0 containing simultaneously a and b, and therefore problem (Q_1) has no solution.

(ii) $l_0 = D_0$, i.e., $\frac{1}{2}l_0\kappa_0 = 1$. There is exactly *one* circle of diameter D_0 containing simultaneously a and b. Therefore (Q_1) has exactly *two* solutions, each of the half-circles joining a to b.

(iii) $l_0 < D_0$, i.e., $\frac{1}{2}l_0\kappa_0 < 1$. There are exactly *two* circles of diameter D_0 containing simultaneously a and b. These circles are actually symmetric with respect to the axis ab. Therefore (Q_1) has exactly *four* solutions: two *small* solutions, symmetric with respect to the axis ab, which are arcs of circles of angle strictly smaller than π, and two *large* solutions, symmetric with respect to the axis ab, which are arcs of circles of angle strictly larger than π. Notice that the length of the small solutions is $2\arccos(\frac{1}{2}l_0\kappa_0))\kappa_0^{-1}$, whereas the length of the large solutions is $2(\pi - \arccos(\frac{1}{2}l_0\kappa_0))\kappa_0^{-1}$, so that the sum is the length of the circle of radius R_0.

As the above discussion shows, the problem can be settled using very elementary arguments of geometric nature.

We end this subsection with a few remarks concerning the parametric formulation, and its analytical background: these remarks will be useful when we will turn to the general case.

Firstly, we observe that equation (3) in the case $\kappa \equiv \kappa_0$ is variational: its solutions are critical points of the functional

$$F_{\kappa_0}(v) = L(v) - \kappa_0 S(v)$$

where $L(v)$ has been defined above and

$$S(v) := \frac{1}{2}\int_0^1 iv \cdot \dot{v}\, ds\ .$$

The functional space for (Q_0) is the Hilbert space

$$H_{\mathrm{per}} := \{v \in H^1([0,1], \mathbb{R}^2) \mid v(0) = v(1)\}\ ,$$

whereas the functional space for (Q_1) is the affine space

$$H_{a,b} := \{v \in H^1([0,1], \mathbb{R}^2) \mid v(0) = a,\ v(1) = b\}\ .$$

The functional $S(v)$ have a nice geometric interpretation. Indeed, for v belonging to the space H_{per}, $S(v)$ represents the (signed) *area* of the (inner) domain bounded by the curve $\mathcal{C}(v) = v([0,1])$. Whereas, for v in H_{per} or $H_{a,b}$, the quantity $L(v)$ is less or equal to the length of $\mathcal{C}(v)$ and equality holds if and only $|\dot{v}|$ is constant. In particular, for v in H_{per}, we have the inequality

$$4\pi|S(v)| \leq L^2(v),$$

which is the analytical form of the isoperimetric inequality in dimension two. Therefore solutions of (Q_0), with $\kappa \equiv \kappa_0$ are also solutions to the isoperimetric problem

$$\sup\{S(v) \mid v \in H_{\text{per}}\,,\ L(v) = 2\pi\kappa_0^{-1}\}\,.$$

This, of course, is a well known fact.

Finally, we notice that the small solutions to (Q_0), in case (iii) are local minimizers of F. More precisely, it can be proved that they minimize F on the set $\{v \in H_{a,b} \mid \|v\|_\infty \leq \kappa_0^{-1}\}$ (in this definition, the origin is taken as the middle point of ab). In this context, the large solution can then also be analyzed (and obtained) variationally, as a mountain pass solution. We will not go into details, since the arguments will be developed in the frame of H-surfaces (here however they are somewhat simpler, since we have less troubles with the Palais-Smale condition).

1.3. The general case of variable curvature

In the general case when the prescribed curvature $\kappa(p)$ depends on the point p, there are presumably no elementary geometric arguments which could lead directly to the solution of (Q_0) and (Q_1). In that situation, the parametric formulation offers a natural approach to the problems.

In this subsection we will leave aside (Q_0), since it is probably more involved and we will concentrate on question (Q_1). We will see in particular, that we are able to extend (at least partially) some of the results of the previous subsection to the case considered here using analytical tools.

We begin with the important remark that (3) is variational, even in the nonconstant case: solutions of (3) and (4) are critical points on $H_{a,b}$ of the functional

$$F_\kappa(v) = L(v) - S_\kappa(v)\,,$$

where

$$S_\kappa(v) = \int_0^1 i\,Q(v)\cdot\dot{v}\,ds,$$

for any vector field $Q\colon \mathbb{R}^2 \to \mathbb{R}^2$ verifying the relation $\operatorname{div} Q(w) = \kappa(w)$ for $w = (w_1, w_2) \in \mathbb{R}^2$. A possible choice for such as a vector field is

$$Q(w_1, w_2) = \frac{1}{2}\Big(\int_0^{w_1} \kappa(s, w_2)\,ds, \int_0^{w_2} \kappa(w_1, s)\,ds\Big)\,.$$

Notice that in the case $\kappa \equiv \kappa_0$ is constant, the previous choice of Q yields $Q(w) = \frac{1}{2}w$, and we recover the functional F_{κ_0}, as written in the previous subsection.

The existence of "small" solutions to (Q_1) can be established as follows.

PROPOSITION 1. *Assume that $l_0 > 0$ and $\kappa \in C^1(\mathbb{R}^2)$ verify the condition*

$$\frac{1}{2} l_0 \|\kappa\|_\infty < 1.$$

Then equation (3) *possesses a solution $\underline{u}$, which minimizes F_κ on the set*

$$M_0 := \{v \in H_{a,b} \mid \|v\|_\infty \le \|\kappa\|_\infty^{-1}\} .$$

In the context of H-surfaces, this type of result has been established first by S. Hildebrandt [30], and we will explain in details his proof in section 4. The proof of Proposition 1 is essentially the same and therefore we will omit it. Note that, in view of the corresponding results for the constant case, i.e., case (iii) of the discussion in the previous subsection, Proposition 1 seems rather optimal.

We next turn to the existence problem for "large" solutions. It is presumably more difficult to obtain a general existence result, in the same spirit as in the previous proposition (i.e., involving only some norms of the function κ). We leave to the reader to figure out some possible counterexamples. We believe that the best one should be able to prove is a perturbative result, i.e., to prove existence of the large solution for functions κ that are close, in some norm, to a constant. In this direction, we may prove the following result.

PROPOSITION 2. *Let $l_0, \kappa_0 > 0$, and assume that*

$$\frac{1}{2} l_0 \kappa_0 < 1 .$$

Then, there exists $\varepsilon > 0$ (depending only on the number $l_0\kappa_0$), such that, for every function $\kappa \in C^1(\mathbb{R}^2)$ verifying

$$\|\kappa - \kappa_0\|_{C^1} < \varepsilon ,$$

equation (3) *has four different solutions $\underline{u}_1$, $\underline{u}_2$, $\overline{u}_1$ and $\overline{u}_2$, where one of the small solutions $\underline{u}_1$ and $\underline{u}_2$ corresponds to the minimal solution given by proposition 1.*

The new solutions $\overline{u}_1$ and $\overline{u}_2$ provided by proposition 2 correspond to the large solutions of the problem: one can actually prove that they converge, as $\|\kappa - \kappa_0\|_{C^1}$ goes to zero, to the large portion of the two circles of radius κ_0^{-1}, joining a to b, given in case (iii) of the previous subsection.

Proof. A simple proof of Proposition 2 can be provided using the implicit function theorem. Indeed, consider the affine space

$$C^2_{a,b} := \{v \in C^2([0,1], \mathbb{R}^2) \mid v(0) = a,\ v(1) = b\} ,$$

and the map $\Phi\colon C^2_{a,b} \times \mathbb{R} \to C^0 := C^0([0,1], \mathbb{R}^2)$ defined by

$$\Phi(v,t) = -\ddot{v} + i(\kappa_0 + t(\kappa(v) - \kappa_0))L(v)\dot{v} .$$

Clearly Φ is of class C^1 and for $w \in C^2_{0,0}$ one has:

$$\begin{aligned}\partial_v \Phi(v,t)(w) &= -\ddot{w} + iL(v)((\kappa_0 + t(\kappa(v) - \kappa_0))\dot{w} + t\kappa'(v)\cdot \dot{v}\, w) \\ &\quad + i(\kappa_0 + t(\kappa(v)-\kappa_0))L(v)^{-1}\dot{v}\int_0^1 \dot{v}\cdot\dot{w}\, ds\ .\end{aligned}$$

Let u_0 be one of the four solutions for κ_0. Notice that for an appropriate choice of orthonormal coordinates in the plane, u_0 is given by the explicit formula

$$u_0(s) = \kappa_0^{-1}\exp(iL_0\kappa_0 s),$$

where $L_0 = L(u_0)$ (recall that $L_0 = 2\kappa_0^{-1}\arccos(\frac{1}{2}l_0\kappa_0))$ for small solutions, or $L_0 = 2\kappa_0^{-1}(\pi - \arccos(\frac{1}{2}l_0\kappa_0))$ for large solutions). We compute the derivative at the point $(u_0, 0)$:

$$\partial_v\Phi(u_0,0)(w) = -\ddot{w} + iL_0\kappa_0\dot{w} + i\kappa_0 L_0^{-1}\dot{u}_0\int_0^1 \dot{u}_0\cdot\dot{w}\, ds\ ,$$

It remains merely to prove that $\partial_v\Phi(u_0,0)$ is invertible, i.e., by Fredholm theory, that $\ker\partial_v\Phi(u_0,0) = \{0\}$. If $w \in \ker\partial_v\Phi(u_0,0)$, then

$$\ddot{w} = iL_0\kappa_0\dot{w} - \alpha(w)L_0\kappa_0\exp(iL_0\kappa_0 s), \tag{5}$$

where $\alpha(w) = L_0^{-1}\int_0^1 \dot{u}_0\cdot\dot{w}\, ds$. Taking α as a parameter, equation (5) can be solved explicitly and its solution is given by:

$$w(x) = C_1 + C_2\exp(iL_0\kappa_0 s) + i\alpha s\exp(iL_0\kappa_0 s)$$

where C_1 and C_2 are some (complex-valued) constants. The boundary conditions $w(0) = w(1) = 0$ determine C_1 and C_2 as functions of α. In view of the definition of α, one deduces an equation for α. After computations, since $\frac{1}{2}l_0\kappa_0 < 1$, it turns out that the only solution is $\alpha = 0$, and then $w = 0$. Thus the result follows by an application of the implicit function theorem. □

The result stated in proposition 2 can be improved if one uses instead a variational approach based on the mountain pass theorem. More precisely, one may replace the C^1 norm there, by the L^∞ norm, i.e., prove that if, for some small $\varepsilon > 0$, depending only on the value $l_0\kappa_0$ one has

$$\|\kappa - \kappa_0\|_\infty < \varepsilon,$$

then a large solution exists, for the problem (Q_1) corresponding to the curvature function κ. The analog of this result for surfaces will be discussed in Section 6, and it is one of the important aspects of the question we want to stress.

At this point, we will leave the planar problem for curves, and we turn to its version for surfaces in the three dimensional space $\mathbb{R}^3$. It is of course only for one dimensional objects that the curvature could be expressed by a simple scalar function. For higher dimensional submanifolds, one needs to make use of a tensor (in the context of surfaces, the second fundamental form). However, some "curvature" functions, deduced from this tensor are of great geometric interest. For surfaces in $\mathbb{R}^3$ the Gaussian curvature and the mean curvature in particular are involved in many questions.

2. Some geometric aspects of the mean curvature

In this section, we will introduce the main definitions and some natural problems involving the notion of *curvature*. Although this notion is important in arbitrary dimension and arbitrary codimension, we will mainly restrict ourselves to *two-dimensional surfaces embedded in* $\mathbb{R}^3$. More precisely, our main goal is to introduce some problems of *prescribed mean curvature*, and their links to *isoperimetric problems*.

We remark that mean curvature concerns problems in *extrinsic* geometry, since it deals with the way objects are embedded in the ambient space. In contrast, problems in *intrinsic* geometry do not depend on the embedding and for this kind of problems one considers the Gaussian curvature.

Let us start by recalling some geometric background.

2.1. Basic definitions

Let M be a two-dimensional regular surface in $\mathbb{R}^3$. Fixed $p_0 \in M$, let us consider near p_0 a parametrization of M, that is a map $u : \mathcal{O} \to M$ with $\mathcal{O}$ open neighborhood of 0 in $\mathbb{R}^2$, $u(0) = p_0$, and u diffeomorphism of $\mathcal{O}$ onto an open neighborhood of p_0 in M. Note that, denoting by $\wedge$ the exterior product in $\mathbb{R}^3$, one has $u_x \wedge u_y \neq 0$ on $\mathcal{O}$, and

$$\overrightarrow{n} = \frac{u_x \wedge u_y}{|u_x \wedge u_y|} \tag{6}$$

(evaluated at $(x, y) \in \mathcal{O}$) defines a unit normal vector at $u(x, y)$.

The metric on N is given by the **first fundamental form**

$$g_{ij}\, du^i du^j = E\,(dx)^2 + 2F\,dx\,dy + G\,(dy)^2$$

where

$$E = |u_x|^2, \quad F = u_x \cdot u_y, \quad G = |u_y|^2.$$

The notion of curvature can be expressed in terms of the second fundamental form. More precisely, let $\gamma : (-1, 1) \to M$ be a parametric curve on M of the form $\gamma(t) = u(x(t), y(t))$, with $x(0) = y(0) = 0$. Thus $\gamma(0) = p_0$.

Since $\frac{d\gamma}{dt}$ and $\overrightarrow{n}$ are orthogonal, one has

$$\frac{d^2\gamma}{dt^2} \cdot \overrightarrow{n} = u_{xx} \cdot \overrightarrow{n} \left(\frac{dx}{dt}\right)^2 + 2\,u_{xy} \cdot \overrightarrow{n}\,\frac{dx}{dt}\frac{dy}{dt} + u_{yy} \cdot \overrightarrow{n} \left(\frac{dy}{dt}\right)^2. \tag{7}$$

Setting

$$L = u_{xx} \cdot \overrightarrow{n}, \quad M = u_{xy} \cdot \overrightarrow{n}, \quad N = u_{yy} \cdot \overrightarrow{n},$$

the right hand side of (7), evaluated at $(x, y) = (0, 0)$,

$$L\,(dx)^2 + 2M\,dx\,dy + N\,(dy)^2$$

defines the **second fundamental form**. By standard linear algebra, there is a basis (e_1, e_2) in $\mathbb{R}^2$ (depending on p_0) such that the quadratic forms

$$A = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad Q = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

can be simultaneously diagonalized; in particular $du(e_1)$ and $du(e_2)$ are orthogonal. The unit vectors

$$\nu_1 = \frac{du(e_1)}{|du(e_1)|}, \quad \nu_2 = \frac{du(e_2)}{|du(e_2)|}$$

are called **principal directions** at p_0, while the **principal curvatures** at p_0 are the values

$$\kappa_1 = \left\langle \frac{d^2\gamma_1}{dt^2}, \vec{n} \right\rangle, \quad \kappa_2 = \left\langle \frac{d^2\gamma_2}{dt^2}, \vec{n} \right\rangle$$

for curves $\gamma_i \colon (-1, 1) \to M$ such that $\gamma_i(0) = p_0$ and $\gamma_i'(0) = \nu_i$ $(i = 1, 2)$.

The **mean curvature** at p_0 is defined by

$$\boxed{H = \tfrac{1}{2}(\kappa_1 + \kappa_2)}$$

(homogeneous to the inverse of a length), whereas the **Gaussian curvature** is

$$K = \kappa_1 \kappa_2.$$

Notice that H and K *do not depend* on the choice of the parametrization.

In terms of the first and second fundamental forms, we have

$$\boxed{2H = \tfrac{1}{EG-F^2}\,(GL - 2FM + EN) = \operatorname{tr}\left(A^{-1}Q\right).} \tag{8}$$

REMARK 1. Suppose that M can be represented as a *graph*, i.e. M has a parametrization of the form

$$u(x, y) = (x, y, f(x, y))$$

with $f \in C^1(\mathcal{O}, \mathbb{R})$. Using the formula (8) for H, a computation shows that

$$2H = \operatorname{div}\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right), \tag{9}$$

whereas the Gaussian curvature is

$$K = \frac{f_{xx} f_{yy} - f_{xy}^2}{1 + |\nabla f|^2}.$$

Let us note that every regular surface admits locally a parametrization as a graph. Moreover, if $p_0 = (x_0, y_0, f(x_0, y_0))$, by a suitable choice of orthonormal coordinates one may also impose that $\nabla f(x_0, y_0) = 0$.

2.2. Conformal parametrizations and the H-system

In problems concerning mean curvature, it is convenient to use conformal parametrizations, since this leads to an equation for the mean curvature that can be handle with powerful tools in functional analysis.

DEFINITION 1. Let M be a two-dimensional regular surface in $\mathbb{R}^3$ and let $u\colon \mathcal{O} \to M$ be a (local) parametrization, $\mathcal{O}$ being a connected open set in $\mathbb{R}^2$. The parametrization u is said to be **conformal** if and only if for every $z \in \mathcal{O}$ the linear map $du(z)\colon \mathbb{R}^2 \to T_{u(z)}M$ preserves angles (and consequently multiplies lengths by a constant factor), that is there exists $\lambda(z) > 0$ such that

$$\langle du(z)h, du(z)k\rangle_{\mathbb{R}^3} = \lambda(z)\langle h, k\rangle_{\mathbb{R}^2} \quad \text{for every } h, k \in \mathbb{R}^2\,. \tag{10}$$

In other words, u is conformal if and only if for every $z \in \mathcal{O}$ $du(z)$ is the product of an isometry and a homothety from $\mathbb{R}^2$ into $\mathbb{R}^3$. Note also that the condition of conformality (10) can be equivalently written as:

$$|u_x|^2 - |u_y|^2 = 0 = u_x \cdot u_y \tag{11}$$

at every point $z \in \mathcal{O}$. In what will follow, an important role is played by the *Hopf differential*, which is the complex-valued function:

$$\omega = \Big(|u_x|^2 - |u_y|^2\Big) - 2i u_x \cdot u_y.$$

In particular, u is conformal if and only if $\omega = 0$.

REMARK 2. If the target space of a conformal map u has dimension two, then u is analytical. This follows by the fact that, given a domain $\mathcal{O}$ in $\mathbb{R}^2$, a mapping $u \in C^1(\mathcal{O}, \mathbb{R}^2)$ is conformal if and only if u is holomorphic or anti-holomorphic (we identify $\mathbb{R}^2$ with the complex field $\mathbb{C}$). However for conformal maps $u\colon \mathcal{O} \to \mathbb{R}^k$ with $k \geq 3$ there is no such as regularity result.

We turn now to the expression of H for conformal parametrizations. If u is conformal, then

$$\begin{cases} E = |u_x|^2 = \left|u_y\right|^2 = G \\ F = u_x \cdot u_y = 0, \end{cases}$$

so that

$$2H(u) = \frac{\Delta u \cdot \overrightarrow{n}}{|u_x|^2} \quad \text{on } \mathcal{O}\,. \tag{12}$$

On the other hand, deriving conformality conditions (11) with respect to x and y, we can deduce that Δu is orthogonal both to u_x and to u_y. Hence, recalling the expression (6) of the normal vector $\overrightarrow{n}$, we infer that Δu and $\overrightarrow{n}$ are parallel. Moreover, by (11), $|u_x \wedge u_y| = |u_x|^2 = |u_y|^2$, and then, from (12) it follows that

$$\boxed{\Delta u = 2H(u) u_x \wedge u_y \quad \text{on } \mathcal{O}\,.} \tag{13}$$

Let us emphasize that (13) is a system of equations, often called **H-system**, or also **H-equation**, and for this system the scalar coefficient $H(u)$ has the geometric meaning of mean curvature for the surface M parametrized by u at the point $u(z)$ provided that u is conformal and $u(z)$ is a regular point, i.e., $u_x(z) \wedge u_y(z) \neq 0$.

2.3. Some geometric problems involving the H-equation

Equation (13) is the main focus of this article. In order to justify its importance let us list some related geometric problems.

It is useful to recall that the area of a two-dimensional regular surface M parametrized by some mapping $u \colon \mathcal{O} \to \mathbb{R}^3$ is given by the integral

$$A(u) = \int_{\mathcal{O}} |u_x \wedge u_y| \, .$$

In particular, if u is conformal, the area functional equals the Dirichlet integral:

$$E_0(u) = \frac{1}{2} \int_{\mathcal{O}} |\nabla u|^2 \tag{14}$$

One of the most famous geometric problems is that of **minimal surfaces**.

DEFINITION 2. A two-dimensional regular surface in $\mathbb{R}^3$ is said to be minimal if and only if it admits a parametrization u which is a critical point for the area functional, that is, $\frac{dA}{ds}(u + s\varphi)\big|_{s=0} = 0$ for every $\varphi \in C_c^\infty(\mathcal{O}, \mathbb{R}^3)$.

An important fact about minimal surfaces is given by the following statement.

PROPOSITION 3. *A two-dimensional regular surface M in $\mathbb{R}^3$ is minimal if and only if $H \equiv 0$ on M.*

Proof. Fixing a point p_0 in the interior of M, without loss of generality, we may assume that a neighborhood M_0 of p_0 in M is parametrized as a graph, namely there exist a neighborhood $\mathcal{O}$ of 0 in $\mathbb{R}^2$ and a function $f \in C^1(\mathcal{O}, \mathbb{R})$ such that M_0 is the image of $u(x, y) = (x, y, f(x, y))$ as $(x, y) \in \mathcal{O}$. In terms of f, the area functional (restricted to M_0) is given by

$$A_0(f) = \int_{\mathcal{O}} \sqrt{1 + |\nabla f|^2}$$

and then

$$\frac{dA_0}{ds}(f + s\psi)\bigg|_{s=0} = -\int_{\mathcal{O}} \operatorname{div}\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right)\psi$$

for every $\psi \in C_c^\infty(\mathcal{O}, \mathbb{R})$. Hence, keeping into account of (9), the thesis follows. □

Another famous geometric problem is given by the so-called **isoperimetric problem** that we state in the following form. Given any two-dimensional regular *compact* surface M *without boundary*, let $V(M)$ be the volume enclosed by M. The general principle says that:

> *Surfaces which are critical for the area, among surfaces enclosing a prescribed volume, (i.e., solutions of isoperimetric problems) verify* $H \equiv$ *const.*

REMARK 3. Consider for instance the standard isoperimetric problem:

> *Fixing* $\lambda > 0$*, minimize the area of* M *among compact surfaces* M *without boundary such that* $V(M) = \lambda$.

It is well known that this problem admits a unique solution, corresponding to the sphere of radius $\sqrt[3]{\frac{3\lambda}{4\pi}}$. This result agrees with the previous general principle since the sphere has constant mean curvature. Nevertheless, there are many variants for the isoperimetric problem, in which one may add some constrains (on the topological type of the surfaces, or boundary conditions, etc.).

In general, the isoperimetric problem can be phrased in analytical language as follows: consider any surface M admitting a conformal parametrization $u\colon \mathcal{O} \to \mathbb{R}^3$, where $\mathcal{O}$ is a standard reference surface, determined by the topological type of M (for instance the sphere $\mathbb{S}^2$, the torus $\mathbb{T}^2$, etc.). For the sake of simplicity, suppose that M is parametrized by the sphere $\mathbb{S}^2$ that can be identified with the (compactified) plane $\mathbb{R}^2$ through stereographic projection. Hence, if $u\colon \mathbb{R}^2 \to \mathbb{R}^3$ is a conformal parametrization of M, the area of M is given by (14), whereas the (algebraic) volume of M is given by

$$V(u) = \frac{1}{3} \int_{\mathbb{R}^2} u \cdot u_x \wedge u_y \ .$$

In this way, the above isoperimetric problem can be written as follows:

> *Fixing* $\lambda > 0$*, minimize* $\int_{\mathbb{R}^2} |\nabla u|^2$ *with respect to the class of conformal mappings* $u\colon \mathbb{R}^2 \to \mathbb{R}^3$ *such that* $\int_{\mathbb{R}^2} u \cdot u_x \wedge u_y = 3\lambda$.

One can recognize that if u solves this minimization problem, or also if u is a critical point for the Dirichlet integral satisfying the volume constraint, then, by the Lagrange multipliers Theorem, u solves an H-equation with H constant.

As a last remarkable example, let us consider the **prescribed mean curvature problem**: given a mapping $H\colon \mathbb{R}^3 \to \mathbb{R}$ study existence and possibly multiplicity of two-dimensional surfaces M such that for all $p \in M$ the mean curvature of p at M equals $H(p)$. Usually the surface M is asked to satisfy also some geometric or topological side conditions.

This kind of problem is a generalization of the previous ones and it appears in various physical and geometric contexts. For instance, it is known that in some evolution problems, interfaces surfaces move according to mean curvature law. Again, nonconstant mean curvature arises in capillarity theory.

3. The Plateau problem: the method of Douglas-Radó

In this section we consider the classical *Plateau problem* for minimal surfaces. Let γ be a Jordan curve in $\mathbb{R}^3$, that is γ is the support of a smooth mapping $g : \mathbb{S}^1 \to \mathbb{R}^3$ with no self-intersection. The question is:

$$\boxed{\begin{array}{c}\textit{Is there any surface } M \textit{ minimizing (or critical for)} \\ \textit{the area, among all surfaces with boundary } \gamma \textit{ ?}\end{array}}$$

In view of our previous discussions, the Plateau problem becomes:

$$(P_0) \qquad \boxed{\begin{array}{c}\textit{Find a surface } M \textit{ such that } \partial M = \gamma \textit{ and having} \\ \textit{zero mean curvature at all points.}\end{array}}$$

Note that in general, this problem may admit more than one solution.

We will discuss this problem by following the method of Douglas-Radó, but we point out that many methods have been successfully proposed to solve the Plateau problem. Here is a nonexhaustive list of some of them.

1. When γ is a *graph*, try to find M as a graph. More precisely suppose γ to be close to a plane curve γ_0. Note that for γ_0 the obvious solution is the planar region bounded by γ_0 itself. Let $g : \mathbb{S}^1 \to \mathbb{R}$ be such that $\gamma = \widetilde{g}(\mathbb{S}^1)$ where $\widetilde{g}(z) = (z, g(z))$ as $z = (x, y) \in \mathbb{S}^1$. If g is "small", we may use *perturbation techniques* (Schauder method) to solve the *nonlinear* problem

$$\begin{cases} \operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = 0 & \text{in } D^2 \\ f = g & \text{on } \partial D^2 = \mathbb{S}^1 \end{cases}$$

   where D^2 is the open unit disc in $\mathbb{R}^2$ (compare with (9), being now $H = 0$).

2. Given a Jordan curve γ, find a surface M spanning γ, with M parametrized in conformal coordinates. This is the Douglas-Radó method that we will develop in more details. Here we just note that, differently from the previous case, now the conformal parametrization u of M solves the *linear* equation $\Delta u = 0$.

3. Use the tools from *geometric measure theory* ([21], [39], [40]), especially designed for that purpose. The advantage of this method is that it is free from conformality equations, and it is very good for minimization problems, but it needs a lot of work to recover regularity of the solutions. Actually, this method is not very useful to handle with saddle critical points.

4. Use *singular limit problems*:

$$E_\varepsilon(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{\varepsilon^2}\int \left(1 - |u|^2\right)^2 .$$

   As the previous one, this method does not use any parametrization.

Let us turn now to the Douglas-Radó method. Looking for a conformal parametrization of M, by (13), the Plateau problem is reduced to the following form:

$$(P_0) \qquad \boxed{\begin{array}{l} \textit{Find } u \in C^0\left(\overline{D^2}, \mathbb{R}^3\right) \cap C^2(D^2, \mathbb{R}^3) \textit{ such that} \\ \begin{cases} \Delta u = 0 & \textit{in } D^2 \\ |u_x|^2 - |u_y|^2 = 0 = u_x \cdot u_y & \textit{in } D^2 \\ u\big|_{\partial D^2} \textit{ monotone parametrization of } \gamma \ . \end{cases} \end{array}}$$

On one hand the Laplace equation is completely standard. On the other hand, the boundary condition is less usual than the Dirichlet one and, besides, one has to deal with the conformality conditions.

The first step in the Douglas-Radó approach consists in translating problem (P_0) into a minimization problem. To this aim let us introduce the Sobolev space $H^1 = H^1(D^2, \mathbb{R}^3)$ and the set

$$(15) \qquad W = \{v \in H^1 \ : \ v\big|_{\partial D^2} \text{ continuous, monotone, parametrization of } \gamma\}$$

and for every $v \in W$ let us denote by $E_0(v)$ the Dirichlet integral of v on D^2, as in (14). Recall that if v is conformal then $E_0(v)$ gives the area of the surface parametrized by v.

LEMMA 1. *If $u \in W$ minimizes E_0 on W, then u is a solution of the Plateau problem (P_0).*

The most surprising result in this statement is that the conformality conditions come out as part of the Euler-Lagrange equation.

Proof. Since u minimizes the Dirichlet integral for all H^1 maps with the same boundary value, u is a weak solution to $\Delta u = 0$ in D^2. In fact, from regularity theory, $u \in C^\infty$. Now, let

$$\omega = |u_x|^2 - |u_y|^2 - 2i u_x \cdot u_y$$

be the Hopf differential associated to u. Since u solves the Laplace equation, it is easy to verify that $\frac{\partial \omega}{\partial \bar{z}} = 0$, and then ω is constant. In order to prove that $\omega \equiv 0$, i.e., u is conformal, the idea is to use *variations of the domain*. More precisely, let $\overrightarrow{X}$ be an arbitrary vector field on D^2 such that $\overrightarrow{X} \cdot \overrightarrow{n} = 0$ on ∂D^2, and let $\phi(t, z)$ be the flow generated by $\overrightarrow{X}$, i.e.

$$\begin{cases} \dfrac{\partial \phi}{\partial t} = \overrightarrow{X}(\phi) \\ \\ \phi(0, z) = z \ . \end{cases}$$

Then $\phi(t, z) = z + t\overrightarrow{X}(z) + o\left(t^2\right)$ and $\phi_t := \phi(t, \cdot)\colon D^2 \to D^2$ is a diffeomorphism for every $t \geq 0$. If we set $u_t = u \circ \phi_t$ then $u \in W$ implies $u_t \in W$ for every $t \geq 0$ and therefore, by the minimality of u,

$$\frac{d}{dt}E_0(u_t) = 0,$$

i.e.

$$\frac{d}{dt}\int_{D^2}\left|\nabla u\left(z + t\overrightarrow{X}(z)\right)\right|^2 = 0. \tag{16}$$

After few computations, (16) can be rewritten as

$$\int_{D^2}\omega\cdot\frac{\partial\overrightarrow{X}}{\partial\overline{z}} = 0$$

which holds true for every $\overrightarrow{X} \in C^\infty\left(D^2, \mathbb{R}^2\right)$ such that $\overrightarrow{X}\cdot\overrightarrow{n} = 0$ on ∂D^2. This implies $\omega \equiv 0$, that is the thesis.

□

Thanks to Lemma 1, a solution to problem (P_0) can be found by solving the following minimization problem:

$$(Q_0) \qquad \boxed{\textit{Find } u \in W \textit{ such that } E_0(u) = \inf_{v\in W} E_0(v)}$$

where $E_0(v)$ is the Dirichlet integral of v and W is defined in (15).

Conformal invariance

The greatest difficulty in the study of problem (Q_0) is that minimizing sequences are not necessarily compact in W, because of the *conformal invariance* of the problem. Let us consider the group G of all conformal diffeomorphisms of D^2:

$$\begin{aligned} G = \{\phi \in C^1(D^2, D^2) \quad : \quad & \phi \text{ one to one and orientation preserving,} \\ & |\phi_x|^2 - |\phi_y|^2 = 0 = \phi_x\cdot\phi_y\}. \end{aligned}$$

It is easy to verify that, given any $v \in W$ and $\phi \in G$ one has $|\nabla(v\circ\phi)| = \lambda|(\nabla v)\circ\phi|$ where $\lambda = |\phi_x| = |\phi_y|$. Since $\lambda^2 = |\mathrm{Jac}\,\phi|$, one obtains

$$\int_{D^2}|\nabla(v\circ\phi)|^2 = \int_{D^2}|\nabla v|^2$$

that is

$$E_0(v\circ\phi) = E_0(v),$$

the energy is invariant under a conformal change on D^2. Note also that

$$u \in W,\ \phi \in G \ \Rightarrow\ u\circ\phi \in W$$

because if $\phi \in G$ then $\phi\big|_{\partial D^2} : \partial D^2 \to \partial D^2$ is monotone.

As a consequence of conformal invariance, we are going to see that W *is not sequentially weakly closed* in H^1. In order to do that, let us first describe G. As already mentioned in remark 2, conformal maps $\phi \in G$ are holomorphic or antiholomorphic; by a choice of orientation, we can restrict ourselves to holomorphic diffeomorphisms. It is then a (not so easy) exercise in complex analysis to prove that

$$G = \Big\{\phi \in C^1(D^2, \mathbb{C}) : \exists a \in \mathbb{C},\ |a| < 1,\ \exists \theta \in [0, 2\pi)\ s.t.\ \phi = \phi_{\theta,a}\Big\}$$

where

$$\phi_{\theta,a}(z) = \frac{z+a}{1-\overline{a}z}e^{i\theta} \qquad \Big(z \in D^2\Big).$$

Hence G is parametrized by $D^2 \times \mathbb{S}^1$, a noncompact three-dimensional manifold with boundary.

Note now that, given $v \in W \cap C(\overline{D^2}, \mathbb{R}^3)$ and $(a_n) \subset D^2$, if $a_n \to a \in \partial D^2$ then $v \circ \phi_{0,a_n} \to v(a)$ pointwise and weakly in H^1 (but not strongly), and the weak limit in general does not belong to W which does not contain any constant.

The three points condition

In order to remove conformal invariance, we have to "fix a gauge", choosing for every $v \in W$ a special element in the orbit $\{v \circ \phi\}_{\phi \in G}$. For this purpose, let us fix a monotone parametrization $g \in C(\mathbb{S}^1, \gamma)$ of γ and then, let us introduce the class

$$W^* = \Big\{v \in W\ :\ v\big(e^{\frac{2ik\pi}{3}}\big) = g\big(e^{\frac{2ik\pi}{3}}\big),\ k = 1, 2, 3\Big\}.$$

Since $W^* \subset W$ and for every $v \in W$ there exists $\varphi \in G$ such that $v \circ \varphi \in W^*$, one has that:

LEMMA 2. $\inf_{v \in W^*} E_0(v) = \inf_{v \in W} E_0(v)$.

Hence, in order to find a solution to the Plateau problem (P_0), it is sufficient to solve the minimization problem defined by $\inf_{v \in W^*} E_0(v)$. This can be accomplished by using the following result.

LEMMA 3 (COURANT-LEBESGUE). *W^* is sequentially weakly closed in H^1.*

Proof. We limit ourselves to sketch the proof. To every $v \in W^*$, one associates (in a unique way) a continuous mapping $\varphi\colon [0, 2\pi] \to [0, 2\pi]$ such that

$$v\big(e^{i\theta}\big) = g\big(e^{i\varphi(\theta)}\big),\ \ \varphi(0) = 0. \tag{17}$$

The function φ turns out to be increasing and satisfying

$$\varphi\left(\frac{2k\pi}{3}\right) = \frac{2k\pi}{3} \text{ for } k = 0, \dots, 3. \tag{18}$$

Take a sequence $(v_n) \subset W^*$ converging to some v weakly in H^1. Let $(\varphi_n) \in C([0, 2\pi])$ be the corresponding sequence, defined according to (17). Since every φ_n

is increasing and satisfies (18), for a subsequence, $\varphi_n \to \varphi$ almost everywhere, being φ an increasing function on $[0, 2\pi]$ satisfying (18). One can show that φ is continuous on $[0, 2\pi]$, this is the hard step in the proof. Then, from monotonicity, $\varphi_n \to \varphi$ uniformly on $[0, 2\pi]$. By continuity of g, from (17) it follows that $u\big|_{\partial D^2}$ is a continuous monotone parametrization of γ and then u belongs to W^*.

$\square$

Hence, apart from regularity at the boundary, we proved that the Plateau problem (P_0) admits at least a solution, characterized as a minimum.

4. The Plateau problem for H-surfaces (the small solution)

A natural extension of the previous Plateau problem (P_0) is to look for surfaces with prescribed mean curvature bounding a given Jordan curve γ, that is

(P_H) *Find a surface M such that $\partial M = \gamma$ and the mean curvature of M at p equals $H(p)$, for all $p \in M$.*

where $H : \mathbb{R}^3 \to \mathbb{R}$ is a given function (take for instance a constant).

Some restrictions on the function H or on γ are rather natural. This can be seen even for the equivalent version of problem (P_H) in lower dimension. Indeed, a curve in the plane with constant curvature $K_0 > 0$ is a portion of a circle with radius $R_0 = 1/K_0$. Therefore, fixing the end points $a, b \in \mathbb{R}^2$, such as a curve joining a and b exists provided that $|a - b| \leq 2R_0$. Choosing the origin in the middle of the segment $\overline{ab}$, this condition becomes $\sup\{|a|, |b|\}K_0 \leq 1$.

The necessity of some smallness condition on H or on γ is confirmed by the following *nonexistence result* proved by E. Heinz in 1969 [26]:

THEOREM 1. *Let γ be a circle in $\mathbb{R}^3$ of radius R. If $H_0 > 1/R$ then there exists no surface of constant mean curvature H_0 bounding γ.*

Hence we are led to assume a condition like $\|H\|_\infty \|\gamma\|_\infty \leq 1$. Under this condition, in 1969 S. Hildebrandt [30] proved the next *existence result*:

THEOREM 2. *Let γ be a Jordan curve in $\mathbb{R}^3$ and let $H : \mathbb{R}^3 \to \mathbb{R}$ be such that*

$$\|H\|_\infty \|\gamma\|_\infty \leq 1.$$

Then there exists a surface of prescribed mean curvature H, bounding γ.

We will give some ideas of the proof of the Hildebrandt theorem. Firstly, by virtue of what discussed in section 2, problem (P_H) can be expressed analytically as follows:

$$(P_H) \qquad \boxed{\begin{array}{c}\textit{Find a (regular) } u\colon \overline{D^2} \to \mathbb{R}^3 \textit{ such that}\\ \begin{cases}\Delta u = 2H(u)u_x \wedge u_y & \textit{in } D^2\\ |u_x|^2 - |u_y|^2 = 0 = u_x \cdot u_y & \textit{in } D^2\\ u\big|_{\partial D^2} \textit{ monotone parametrization of } \gamma\ .\end{cases}\end{array}}$$

The partial differential equation for u is now *nonlinear* and this is of course the main difference with the Plateau problem (P_0) for minimal surfaces. The solution of (P_H) found by Hildebrandt is characterized as a minimum, and it is often called **small solution**. In fact, under suitable assumptions, one can find also a second solution to (P_H) which does not correspond to a minimum point but to a saddle critical point, the so-called **large solution** (see section 6).

The conformality condition can be handled as in the Douglas-Radó approach (three-point condition). In doing that, we are led to consider the more standard Dirichlet problem

$$(D_H) \qquad \begin{cases}\Delta u = 2H(u)u_x \wedge u_y & \text{in } D^2\\ u = g & \text{on } \partial D^2\ ,\end{cases}$$

where g is a fixed continuous, monotone parametrization of γ.

The main point in Hildebrandt's proof is the existence of solutions to the problem (D_H), that is:

THEOREM 3. *Let $g \in H^{1/2}(\mathbb{S}^1, \mathbb{R}^3) \cap C^0$ and let $H : \mathbb{R}^3 \to \mathbb{R}$ be such that*

$$\|g\|_\infty \|H\|_\infty \le 1.$$

Then problem (D_H) admits a solution.

Proof. Let us show this result in case the strict inequality $\|g\|_\infty \|H\|_\infty < 1$ holds. We will split the proof in some steps.

Step 1: Variational formulation of problem (D_H).

Problem (D_H) is variational, that is, solutions to (D_H) can be detected as critical points of a suitable energy functional, defined as follows. Let $Q_H : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field such that

$$\text{div } Q_H(u) = H(u) \text{ for all } u \in \mathbb{R}^3.$$

For instance, take

$$Q_H(u) = \frac{1}{3}\left(\int_0^{u_1} H(t, u_2, u_3)\, dt, \int_0^{u_2} H(u_1, t, u_3)\, dt, \int_0^{u_3} H(u_1, u_2, t)\, dt\right).$$

Then, denote

$$H^1_g = \{u \in H^1(D^2, \mathbb{R}^3) : u\big|_{\partial D^2} = g\}$$

and

$$E_H(u) = \frac{1}{2}\int_{D^2} |\nabla u|^2 + 2\int_{D^2} Q_H(u)\cdot u_x \wedge u_y \ .$$

Note that in case of constant mean curvature $H(u) \equiv H_0$ one can take $Q_{H_0}(u) = \frac{1}{3}H_0 u$ and E_H turns out to be the sum of the Dirichlet integral with the volume integral.

One can check that critical points of E_H on H^1_g correspond to (weak) solutions to problem (D_H). Actually, as far as concerns the regularity of E_H on the space H^1_g some assumptions on H are needed. For instance, E_H is of class C^1 if $H \in C^0(\mathbb{R}^3)$ and $H(u)$ is constant for $|u|$ large. A reduction to this case will be done in the next step.

Step 2: Truncation on H and study of a minimization problem.

By scaling, we may assume $h = \|H\|_\infty < 1$ and $\|g\|_\infty \le 1$. Then, let $h' \in (h, 1)$ and $\widetilde{H} : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function such that

$$\widetilde{H}(u) = \begin{cases} H(u) & \text{as } |u| \le 1 \ , \\ 0 & \text{as } |u| \ge \frac{1}{h'} \ , \end{cases}$$

and with $\|\widetilde{H}\|_\infty < h'$. Let us denote by $Q_{\widetilde{H}}$ and $E_{\widetilde{H}}$ the functions corresponding to $\widetilde{H}$. Since $|Q_{\widetilde{H}}(u)| \le 1$ for all $u \in \mathbb{R}^3$, we obtain

$$\frac{1}{3}E_0(u) \le E_{\widetilde{H}}(u) \le \frac{5}{3}E_0(u) \text{ for all } u \in H^1_g \ .$$

Moreover, $E_{\widetilde{H}}$ turns out to be weakly lower semicontinuous on H^1_g. Therefore

$$\inf_{v\in H^1_g} F_{\widetilde{H}}(v)$$

is achieved by some function $u \in H^1_g$. By standard arguments, u is a critical point of $E_{\widetilde{H}}$ and thus, a (weak) solution of

$$(D_{\widetilde{H}}) \qquad \begin{cases} \Delta u = 2\widetilde{H}(u) u_x \wedge u_y & \text{in } D^2 \\ u = g & \text{on } \partial D^2 \ . \end{cases}$$

Step 3: Application of the maximum principle.

In order to prove that u is solution to the original problem (D_H), one shows that $\|u\|_\infty \le 1$. One has that (in a weak sense)

$$-\Delta |u|^2 = -2\left(|\nabla u|^2 + u\cdot \Delta u\right) \le -2|\nabla u|^2\big(1 - |u|\big|\widetilde{H}(u)\big|\big) \le 0.$$

Hence $|u|^2$ is subharmonic and the maximum principle yields

$$\|u\|_{L^\infty(D^2)} \le \|u\|_{L^\infty(\partial D^2)} = \|g\|_\infty \le 1.$$

Since $\widetilde{H}(u) = H(u)$ as $|u| \le 1$, u turns out to solve (D_H).

□

REMARK 4. 1.The implementation of the Douglas-Radó method passing from the Dirichlet problem (D_H) to the Plateau problem (P_H) is made possible by the fact that the functional E_H is conformally invariant. Actually, note that the volume functional

$$V_H(u) = \int_{D^2} Q_H(u) \cdot u_x \wedge u_y$$

is invariant with respect to the (larger) group of the orientation preserving diffeomorphisms of D^2 into itself.
2. When H is constant (e.g. $H \equiv 1$) and $u \in H^1_g$ is regular, the functional $V_H(u)$ has a natural geometric interpretation as a (signed) volume of the region bounded by the surface parametrized by u and a fixed surface given by the portion of cone with vertex at the origin and spanning g. When H is nonconstant a similar interpretation holds, considering $\mathbb{R}^3$ endowed with an H-weighted metric (see Steffen [39]).
3. Although the condition $\|\gamma\|_\infty \|H\|_\infty \leq 1$ is natural and sufficient for existence of solutions to problem (P_H), it is not necessary. Think for instance of long and narrow "strips". In this direction there are some existence results (by Heinz [25], Wente [47], and K. Steffen [40]) both for the Dirichlet problem (D_H) and for the Plateau problem (P_H) where a solution characterized as a minimum is found assuming that

$$\|H\|_\infty \sqrt{A_\gamma} \leq C_0$$

where A_γ denotes the minimal area bounding γ and C_0 is some explicit positive constant.
4. In case of constant mean curvature $H(u) \equiv H_0 > 0$, if γ is a curve lying on a sphere of radius $R_0 = 1/H_0$, the solution given by the above Hildebrandt theorem corresponds to the smaller part of the sphere spanning γ (*small solution*). In this special case, the larger part of the sphere is also a solution to (P_H), the *large solution*. We will see below that this kind of multiplicity result holds true for more general γ and H, but it does not happen, in general, for minimal surfaces.
5. There are also conformal solutions of the H-equation which define compact surfaces (this is impossible for minimal surfaces). A typical example is the sphere $\mathbb{S}^2$. More surprisingly, Wente in [49] constructed also immersed tori of constant mean curvature.

5. Analytical aspects of the H-equation

In this section we will study properties of solutions of the H-equation (13). More precisely, we will study:

(i) the *regularity theory* as well as some aspects of the energy functional E_H (Wente's result [47] and its extensions by Heinz [27], [28], Bethuel and Ghidaglia [8], [9], Bethuel [7]),

(ii) *a priori bounds* of solutions of problem (P_H) (or also (D_H)),

(iii) *isoperimetric inequalities*.

Clearly, questions (i) and (ii) are elementary for the minimal surface equation $\Delta u = 0$. For the H-equation (13), they are rather involved, because the nonlinearity is "critical".

5.1. Regularity theory

Here we consider *weak solutions* of the equation

$$\Delta u = 2H(u)u_x \wedge u_y \text{ on } \mathcal{O} \tag{19}$$

where $\mathcal{O}$ is any domain in $\mathbb{R}^2$. Owing to the nonlinearity $2H(u)u_x \wedge u_y$ as well as to the variational formulation discussed in the previous section, it is natural to consider solutions of (19) which are in the space $H^1(\mathcal{O}, \mathbb{R}^3)$.

The first regularity result for (19) was given by H. Wente [47], for H constant.

THEOREM 4. *If H is constant, then any $u \in H^1(\mathcal{O}, \mathbb{R}^3)$ solution of* (19) *is smooth, i.e., $u \in C^\infty(\mathcal{O})$.*

Nowadays, this result is a special case of a more general theorem (see Theorem 5 below) that will be discussed in the sequel. In any case, we point out that the proof of Theorem 4 relies on the special structure of the nonlinearity:

$$u_x \wedge u_y = \begin{pmatrix} u_x^2 u_y^3 - u_x^3 u_y^2 \\ u_x^3 u_y^1 - u_x^1 u_y^3 \\ u_x^1 u_y^2 - u_x^2 u_y^1 \end{pmatrix} = \begin{pmatrix} \{u^2, u^3\} \\ \{u^3, u^1\} \\ \{u^1, u^2\} \end{pmatrix}.$$

Here we have made use of the notation

$$\{f, g\} = f_x g_y - f_y g_x$$

which represents the Jacobian of the map $(x, y) \mapsto (f(x, y), g(x, y))$. Thus, considering the equation (19) with H constant, we are led to study the more general *linear* equation

$$\Delta\phi = \{f, g\} \text{ in } \mathcal{O}$$

where f, g satisfy $\int_{\mathcal{O}} |\nabla f|^2 < +\infty$ and $\int_{\mathcal{O}} |\nabla g|^2 < +\infty$. Obviously $\{f, g\} \in L^1(\mathcal{O})$ but, in dimension two, $\Delta\phi \in L^1(\mathcal{O})$ implies $\phi \in W^{1,p}_{loc}(\mathcal{O})$ only for $p < 2$, while the embedding $W^{1,p} \hookrightarrow L^\infty$ holds true only as $p > 2$. However, $\{f, g\}$ has a special structure of divergence form, and precisely

$$\{f, g\} = \frac{\partial}{\partial x}\left(f g_y\right) - \frac{\partial}{\partial y}\left(f g_x\right),$$

and this can be employed to prove what stated in the following lemmata, which have been used in various forms since the pioneering work by Wente [47].

LEMMA 4. *Let $\phi \in W^{1,1}_{loc}(\mathbb{R}^2)$ be the solution of*

$$\begin{cases} -\Delta\phi = \{f, g\} & \text{on } \mathbb{R}^2 \\ \phi(z) \to 0 & \text{as } |z| \to +\infty\,. \end{cases}$$

Then

$$\|\varphi\|_{L^\infty} + \|\nabla\varphi\|_{L^2} \leq C\|\nabla f\|_{L^2}\|\nabla g\|_{L^2}.$$

Proof. Let $-\frac{1}{2\pi}\ln|z|$ be the fundamental solution of $-\Delta$. Since the problem is invariant under translations, it suffices to estimate $\phi(0)$. We have

$$\phi(0) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\ln|z|\,\{f, g\}\;dz.$$

In polar coordinates, one has

$$\{f, g\} = \frac{1}{r}\frac{\partial}{\partial\theta}(fg_r) - \frac{\partial}{\partial r}(fg_\theta).$$

Hence, integrating by parts, we obtain

$$\begin{aligned}\phi(0) &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{1}{r}fg_\theta\;dz\\ &= \frac{1}{2\pi}\int_0^{+\infty}\frac{dr}{r}\left(\int_{|z|=r} fg_\theta\;d\theta\right).\end{aligned}$$

Setting $\overline{f} = \frac{1}{2\pi r}\int_{|z|=r} f\;d\theta$, then, using Cauchy-Schwartz and Poincaré inequality,

$$\begin{aligned}\left|\int_{|z|=r} fg_\theta\;d\theta\right| &= \left|\int_{|z|=r}\left(f-\overline{f}\right)g_\theta\;d\theta\right|\\ &\leq \left(\int_{|z|=r}\left|f-\overline{f}\right|^2\;d\theta\right)^{\frac{1}{2}}\left(\int_{|z|=r}|g_\theta|^2\,d\theta\right)^{\frac{1}{2}}\\ &\leq C\left(\int_{|z|=r}|f_\theta|^2\;d\theta\right)^{\frac{1}{2}}\left(\int_{|z|=r}|g_\theta|^2\;d\theta\right)^{\frac{1}{2}}\\ &\leq Cr^2\left(\int_{|z|=r}|\nabla f|^2\,d\theta\right)^{\frac{1}{2}}\left(\int_{|z|=r}|\nabla g|^2\,d\theta\right)^{\frac{1}{2}}.\end{aligned}$$

Going back to $\phi(0)$, using again Cauchy-Schwartz inequality, we have

$$\begin{aligned}|\phi(0)| &\leq C\int_0^{+\infty}\left(r\int_{|z|=r}|\nabla f|^2\,d\theta\right)^{\frac{1}{2}}\left(r\int_{|z|=r}|\nabla g|^2\,d\theta\right)^{\frac{1}{2}}dr\\ &\leq C\left(\int_0^{+\infty}\int_{|z|=r}|\nabla f|^2\,d\theta\;r\;dr\right)^{\frac{1}{2}}\left(\int_0^{+\infty}\int_{|z|=r}|\nabla g|^2\,d\theta\;r\;dr\right)^{\frac{1}{2}}\\ &= C\|\nabla f\|_{L^2}\|\nabla g\|_{L^2}.\end{aligned}$$

Hence

$$\|\phi\|_{L^\infty} \leq C\|\nabla f\|_{L^2}\|\nabla g\|_{L^2}.$$

Finally, multiplying the equation $-\Delta\phi = \{f, g\}$ by ϕ and integrating over $\mathbb{R}^2$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla\phi|^2 &\leq \|\{f,g\}\|_{L^1}\|\phi\|_{L^\infty} \\ &\leq 2\|\phi\|_{L^\infty}\|\nabla f\|_{L^2}\|\nabla g\|_{L^2} \\ &\leq C\|\nabla f\|^2_{L^2}\|\nabla g\|^2_{L^2}\,. \end{aligned}$$

□

Using the maximum principle, it is possible to derive similarly (as obtained by H. Brezis and J.M. Coron [13]) the following analogous result:

LEMMA 5. *Assume* $f, g \in H^1(D^2, \mathbb{R})$ *and let* $\phi \in W_0^{1,1}(D^2, \mathbb{R})$ *be the solution of*

$$\begin{cases} -\Delta\phi = \{f, g\} & on\ D^2 \\ \phi = 0 & on\ \partial D^2. \end{cases}$$

Then

$$\|\phi\|_{L^\infty} + \|\nabla\phi\|_{L^2} \leq C\|\nabla f\|_{L^2}\|\nabla g\|_{L^2}.$$

Another proof of the above lemmas can be obtained by using tools of harmonic analysis. It has been proved (Coifman-Lions-Meyer-Semmes [19]) that if $f, g \in H^1(\mathbb{R}^2)$ then $\{f, g\}$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$, a strict subspace of $L^1(\mathbb{R}^2)$, defined as follows:

$$\mathcal{H}^1(\mathbb{R}^2) = \{u \in L^1(\mathbb{R}^2) : \mathrm{K}_j u \in L^1 \text{ for } j = 1, 2\},$$

where $\mathrm{K}_j = \partial/\partial x_j(-\Delta)^{1/2}$. As a consequence, since any Riesz transform $\mathrm{R}_j = \partial/\partial x_j(-\Delta)^{-1/2}$ maps $\mathcal{H}^1(\mathbb{R}^2)$ into itself, one has that if $-\Delta\phi = \{f, g\}$ on $\mathbb{R}^2$ then

$$-\frac{\partial^2\phi}{\partial x_i \partial x_j} = \mathrm{R}_i\mathrm{R}_j(-\Delta\phi) \in \mathcal{H}^1(\mathbb{R}^2) \quad for\ i, j = 1, 2$$

and hence $\phi \in W^{2,1}(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$. This argument holds similarly true in the situation of lemma 5 and can be pushed further to obtain the desired estimate, exploiting the fact that the fundamental solution (on $\mathbb{R}^2$) to the Laplace equation belongs to BMO($\mathbb{R}^2$), the dual of $\mathcal{H}^1(\mathbb{R}^2)$.

We now turn to the case of variable H. Regularity of (weak) H^1-solutions has been established under various assumptions on the function H. For instance, $H \in C^\infty(\mathbb{R}^3)$

and

$$\sup_{y\in\mathbb{R}^3} |H(y)|(1+|y|) \le \alpha < 1 \qquad \text{(Heinz, [27])}$$

$$\|H\|_\infty < +\infty,\ H(y) = H(y_1, y_2) \qquad \text{(Bethuel-Ghidaglia, [8])}$$

$$\|H\|_\infty < +\infty,\ \sup_{y\in\mathbb{R}^3} |\nabla H(y)|(1+|y|) < +\infty \qquad \text{(Heinz, [28])}$$

$$\|H\|_{L^\infty} < +\infty,\ \sup_{y\in\mathbb{R}^3} \frac{\partial H}{\partial y_3}(y)(1+|y_3|) \le C \qquad \text{(Bethuel-Ghidaglia, [9]).}$$

However we will describe another regularity theorem, due to F. Bethuel [7].

THEOREM 5. *If* $H \in C^\infty(\mathbb{R}^3)$ *satisfies*

$$\|H\|_{L^\infty} + \|\nabla H\|_{L^\infty} < +\infty \tag{20}$$

then any solution $u \in H^1(D^2, \mathbb{R}^3)$ *to* $\Delta u = 2H(u)u_x \wedge u_y$ *on* D^2 *is smooth, i.e.,* $u \in C^\infty(D^2)$.

The proof of this theorem involves the use of Lorentz spaces, which are borderline for Sobolev injections, and relies on some preliminary results. Thus we are going to recall some background on the subject, noting that the interest for Lorentz spaces, in our context, was pointed out by F. Hélein [29], who used them before for harmonic maps.

If Ω is a domain in $\mathbb{R}^N$ and μ denotes the Lebesgue measure, we define $L^{2,\infty}(\Omega)$ as the set of all measurable functions $f : \Omega \to \mathbb{R}$ such that the *weak* $L^{2,\infty}$*-norm*

$$\|f\|_{L^{2,\infty}} = \sup_{t>0}\{t^{\frac{1}{2}}\mu(\{x \in \Omega : f(x) > t\})\}$$

is finite. If $L^{2,1}(\Omega)$ denotes the dual space of $L^{2,\infty}(\Omega)$, one has $L^{2,1}(\Omega) \subset L^2(\Omega) \subset L^{2,\infty}(\Omega)$, the last inclusion being strict since, for instance, $1/r \in L^{2,\infty}(D^2)$ but $1/r \notin L^2(D^2)$. Moreover, if Ω is bounded, then $L^{2,\infty}(\Omega) \subset L^p(\Omega)$ for every $p < 2$. See [50] for thorough details.

Denoting by $B_r = B_r(z_0)$ the disc of radius $r > 0$ and center $z_0 \in \mathbb{R}^2$, let now $\phi \in W_0^{1,1}(B_r)$ be the solution of

$$\begin{cases} -\Delta\phi = \{f, g\} & \text{in } B_r \\ \phi = 0 & \text{on } \partial B_r \end{cases}$$

where $f, g \in H^1(B_r)$; recalling lemma 5, one has

$$\|\phi\|_{L^\infty} + \|\nabla\phi\|_{L^2} + \|\nabla\phi\|_{L^{2,1}(B_{r/2})} \le C\|\nabla f\|_{L^2}\|\nabla g\|_{L^2}. \tag{21}$$

The estimate of $L^{2,1}$-norm of the gradient was obtained by L. Tartar [45] using interpolation methods, but can also be recovered as a consequence of the embedding $W^{1,1} \hookrightarrow L^{2,1}$ due to H. Brezis (since, as we have already mentioned, the fact that

$\{f, g\}$ belongs to the Hardy space $\mathcal{H}^1$ implies that $\phi \in W^{2,1}$). Moreover, if g is constant on ∂B_r, then it can be proved (see [7]) that

$$\|\nabla\phi\|_{L^2} \leq C\|\nabla f\|_{L^2}\|\nabla g\|_{L^{2,\infty}}. \tag{22}$$

Finally, we recall the following classical result: if $h \in L^1(B_r)$, then the solution $\phi \in W_0^{1,1}(B_r)$ to

$$\begin{cases} -\Delta\phi = h & \text{in } B_r \\ \phi = 0 & \text{on } \partial B_r \end{cases}$$

verifies

$$\|\nabla\phi\|_{L^{2,\infty}(B_{r/2})} \leq C\|h\|_{L^1}. \tag{23}$$

Proof of Theorem 5. At first we note that the hypothesis (20) grants that $|\nabla H(u)| \leq C|\nabla u|$ and $H(u) \in H^1$. The proof is then divided in some steps.

Step 1: Rewriting equation (19).

Let $B_{2r}(z_0) \subset D^2$ and $\{H(u), u\} = (\{H(u), u^1\}, \{H(u), u^2\}, \{H(u), u^2\})$. The idea is to introduce a (Hodge) decomposition of $2H(u)\nabla u$ in B_{2r}:

$$2H(u)\nabla u = \nabla A + \nabla^\perp\beta \qquad \textit{where } \nabla^\perp = \left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right).$$

Since

$$\frac{\partial}{\partial x}(2H(u)u_y) + \frac{\partial}{\partial y}(-2H(u)u_x) = 2\{H(u), u\},$$

the solution $\beta \in W_0^{1,1}(B_{2r}, \mathbb{R}^3)$ to

$$\begin{cases} -\Delta\beta = \{H(u), u\} & \text{in } B_{2r} \\ \beta = 0 & \text{on } \partial B_{2r} \end{cases}$$

belongs, by lemma 5, to $H^1(B_{2r}, \mathbb{R}^3)$ and satisfies

$$\frac{\partial}{\partial x}(2H(u)u_y + \beta_x) + \frac{\partial}{\partial y}(-2H(u)u_x + \beta_y) = 0.$$

Hence, there exists $A \in H^1(B_{2r}, \mathbb{R}^3)$ such that

$$A_x = 2H(u)u_x - \beta_y, \quad A_y = 2H(u)u_y + \beta_x \tag{24}$$

and equation (19), on B_{2r}, rewrites:

$$\Delta u = A_x \wedge u_y + \beta_y \wedge u_y. \tag{25}$$

Step 2: "Morrey type" inequality for the $L^{2,\infty}$ norm.

Since regularity is a local property, as we may reduce the radius r we can assume without loss of generality that $\|\nabla u\|_{L^2(B_r)} < \varepsilon < 1$. We are now going to show that there exists $\theta \in (0,1)$ such that

$$\|\nabla u\|_{L^{2,\infty}(B_{\theta r})} \leq \frac{1}{2}\|\nabla u\|_{L^{2,\infty}(B_r)}. \tag{26}$$

This is the main step of the proof. Let us consider $B_{r_0} \subset B_{r/2}$ and let $\widetilde{u}$ be the harmonic extension to B_{r_0} of $u_{|\partial B_{r_0}}$. Note that the radius r_0 can be chosen such that $\|\nabla\widetilde{u}\|_{L^2(B_{r_0})} \leq C\|\nabla u\|_{L^{2,\infty}(B_r)}$ (see [7] for details). In B_{r_0}, using (25), we may write

$$u = \widetilde{u} + \psi_1 + \psi_2 + \psi_3$$

where the functions ψ_1, ψ_2, ψ_3 are defined by

$$\begin{gathered}\Delta\psi_1 = A_x \wedge (u-\widetilde{u})_y,\ \Delta\psi_2 = A_x \wedge \widetilde{u}_y,\ \Delta\psi_3 = \beta_y \wedge u_y \quad in\ \ B_{r_0}\\ \psi_1 = \psi_2 = \psi_3 = 0 \quad on\ \partial B_{r_0}.\end{gathered}$$

Note that, using (24), (21), (20) and the fact that $\varepsilon < 1$, computations give

$$\|\nabla A\|_{L^2(B_r)} \leq C\|\nabla u\|_{L^2(B_r)}.$$

By (22), we have

$$\begin{aligned}\|\nabla\psi_1\|_{L^2(B_{r_0})} &\leq C\|\nabla A\|_{L^2(B_r)}\|\nabla(u-\widetilde{u})\|_{L^{2,\infty}(B_{r_0})}\\ &\leq C\|\nabla u\|_{L^2(B_r)}\|\nabla u\|_{L^{2,\infty}(B_r)} \leq C\varepsilon\|\nabla u\|_{L^{2,\infty}(B_r)}\end{aligned} \tag{27}$$

and, using (21), we obtain

$$\begin{aligned}\|\nabla\psi_2\|_{L^2(B_{r_0})} &\leq C\|\nabla A\|_{L^2(B_{r_0})}\|\nabla\widetilde{u}\|_{L^2(B_{r_0})}\\ &\leq C\|\nabla u\|_{L^2(B_r)}\|\nabla\widetilde{u}\|_{L^2(B_{r_0})} \leq C\varepsilon\|\nabla u\|_{L^{2,\infty}(B_r)}.\end{aligned} \tag{28}$$

Using the duality of $L^{2,1}$ and $L^{2,\infty}$, (23) and (21) yield

$$\begin{aligned}\|\nabla\psi_3\|_{L^{2,\infty}(B_{r_0/2})} &\leq C\|\nabla\beta\|_{L^{2,1}(B_{r/2})}\|\nabla u\|_{L^{2,\infty}(B_{r/2})}\\ &\leq C\varepsilon^2\|\nabla u\|_{L^{2,\infty}(B_r)} \leq C\varepsilon\|\nabla u\|_{L^{2,\infty}(B_r)}.\end{aligned} \tag{29}$$

By the properties of harmonic functions, one has that

$$\forall\alpha \in (0,1) \qquad \|\nabla\widetilde{u}\|_{L^2(B_{\alpha r_0})} \leq C\alpha\|\nabla\widetilde{u}\|_{L^2(B_{r_0})} \leq C\alpha\|\nabla u\|_{L^{2,\infty}(B_r)}. \tag{30}$$

Combining (27)–(30) and recalling the decomposition of u in B_{r_0}, we finally deduce that

$$\forall\alpha \in (0,1) \qquad \|\nabla u\|_{L^{2,\infty}(B_{\alpha r_0})} \leq C(\varepsilon+\alpha)\|\nabla u\|_{L^{2,\infty}(B_r)}$$

and, by a suitable choice of ε and α, (26) follows.

Step 3: Hölder continuity.

From the last result, by iteration, we deduce that there exists $\mu \in (0,1)$ such that $\|\nabla u\|_{L^{2,\infty}(B_r(z_0))} \leq C r^{\mu}$ for every disc $B_{2r}(z_0) \subset D^2$ and, thanks to a theorem of C. Morrey (see [22] for example), this yields that $u \in C^{0,\alpha}$ for every $\alpha \in (0,\mu)$. Higher regularity can be derived by standard arguments.

□

As a first consequence of regularity, we will now prove a result which shows that, for solutions not supposed to be a priori conformal, however the defect of conformality can be "controlled".

THEOREM 6. *If $u \in H^2(\mathcal{O}, \mathbb{R}^3)$ is a solution to* (19)*, then its Hopf differential $\omega = (|u_x|^2 - |u_y|^2) - 2i u_x \cdot u_y$ satisfies (in the weak sense) $\frac{\partial \omega}{\partial \bar{z}} = 0$ in $\mathcal{O}$.*

Proof. Let $X \in C_c^{\infty}(\mathcal{O}, \mathbb{R}^2)$ be a vector field on $\mathcal{O}$ and let $\varphi = X_1 u_x + X_2 u_y$. Since we have assumed that $u \in H^2$, we deduce that $\varphi \in H_0^1$ and therefore we may take φ as a test function for (19). Being $H(u) u_x \wedge u_y \cdot \varphi = 0$, one has

$$0 = \Delta u \cdot \varphi = X_1(u_{xx} \cdot u_x + u_{yy} \cdot u_x) + X_2(u_{xx} \cdot u_y + u_{yy} \cdot u_y)$$

which yields directly the result.

□

REMARK 5. Note that the argument would fail for H^1-solutions, but it holds still true for smooth solutions and, moreover, ω turns out to be holomorphic.

5.2. L^{∞}-bounds for the H-equation

The a priori bounds on solutions to the H-equation we are going to describe are basic in the context of the analytical approach to the following geometric problem. Let us consider a Jordan curve γ in $\mathbb{R}^3$ and a surface $M \subset \mathbb{R}^3$ of mean curvature H and such that $\partial M = \gamma$. The question is:

Is it possible to bound $\sup_{p \in M} |H(p)|$
by a function of $\|\gamma\|_{L^{\infty}}$ *and the area of* M*?*

Although a direct approach to this problem is probably possible, the analytical one (based on ideas of M. Grüter [23] and rephrased by F. Bethuel and O. Rey [11]) relies on the following estimates, which play a central role also in the variational setting of the H-problem.

THEOREM 7. *Let u be a smooth solution to problem (D_H). Assume u conformal and H bounded. Then*

$$\|u\|_{L^{\infty}} \leq C \left(\|g\|_{L^{\infty}} + \|H\|_{L^{\infty}} \int_{D^2} |\nabla u|^2 + \left(\int_{D^2} |\nabla u|^2 \right)^{\frac{1}{2}} \right). \tag{31}$$

Proof. The proof is based on the introduction, for $z_0 \in D^2$ and $r > 0$ such that $\mathrm{dist}(u(z_0), \gamma) > r$, of the following sets and functions:

$$W(r) = u^{-1}(B_r(u(z_0))), \quad V(r) = \partial W(r)$$

$$\phi(r) = \int_{W(r)} |\nabla u|^2, \quad \psi(r) = \int_{V(r)} \left| \frac{\partial |u|}{\partial \nu} \right|$$

where ν is the outward normal to $V(r)$. Obviously, $B_r(u(z_0)) \cap \gamma = \emptyset$. We limit ourselves to describe briefly the steps which lead to the conclusion.

Step 1. Using the conformality condition, we have

$$\frac{d}{dr}\phi(r) \geq 2\psi(r). \tag{32}$$

In fact, assuming (without loss of generality) $u(z_0) = 0$ and noting that

$$|\nabla u|^2 = 2\left|\frac{\partial u}{\partial \nu}\right|^2 \geq 2\left|\frac{\partial |u|}{\partial \nu}\right|^2,$$

we obtain

$$\frac{d}{dr}\phi(r) \geq 2\frac{d}{dr}\int_{W(r)} \left|\frac{\partial |u|}{\partial \nu}\right|^2 = 2\frac{d}{dr}\int_{W(r)} |\nabla |u|\,|^2 = 2\psi(r),$$

where the last equality can be deduced from the coarea formula of Federer [21].

Step 2. Again by conformality, it is possible to prove that

$$\limsup_{r\to 0}\frac{\phi(r)}{r^2} \geq 2\pi \ , \quad \text{assuming } |\nabla u(z_0)| \neq 0 \,. \tag{33}$$

The idea is the following. As $r \to 0$, the image of u becomes locally flat, so that the area A_r of the image of u in $B_r(u(z_0))$ is close to πr^2. On the other hand, $\phi(r) = 2A_r$.

Step 3. Using the H-equation and (32), we have

$$2\phi(r) - r\frac{d}{dr}\phi(r) \leq 2H_0 r\phi(r). \tag{34}$$

In fact, integrating by parts, we obtain

$$\begin{aligned}
\phi(r) &= \int_{W(r)} |\nabla u|^2 = \int_{W(r)} -\Delta u \cdot u + \int_{V(r)} u \cdot \frac{\partial u}{\partial \nu} \\
&\leq H_0 \int_{W(r)} |u|\,|\nabla u|^2 + r\int_{V(r)} \left|\frac{\partial |u|}{\partial \nu}\right| \\
&\leq H_0 \int_{W(r)} |u|\,|\nabla u|^2 + r\psi(r) \\
&\leq H_0 r\phi(r) + \frac{1}{2}r\frac{d}{dr}\phi(r).
\end{aligned}$$

Step 4. Combining (32), (33) and (34), it is possible to prove that

$$\phi(r) \geq \frac{2\pi}{e} r^2 \tag{35}$$

for every $0 < r \leq \frac{1}{2H_0}$.

Step 5. Combining the estimate (35) with a covering argument, the proof of the theorem can be completed.

□

A relevant fact is that the conformality assumption of theorem 7 can be removed. More precisely, we have:

THEOREM 8. *Let u be a smooth solution to the problem (D_H). If H is smooth and bounded, then*

$$\|u\|_{L^\infty} \leq C\left(\|g\|_{L^\infty} + \|H\|_{L^\infty}\left(1 + \int_{D^2} |\nabla u|^2\right)\right). \tag{36}$$

Proof. Let us note that, if u were conformal, for the theorem 7 it would satisfy the inequality (31), which would directly yield (36). When u is not conformal, an adaptation of an argument of R. Shoen [38] allows a reduction to the conformal case. This procedure is based on the following construction. It is possible to determine a function $\psi : D^2 \to \mathbb{C}$ such that

$$\frac{\partial \psi}{\partial z} = -\frac{1}{4}\omega \quad \text{and} \quad \frac{\partial \psi}{\partial \overline{z}} = 0 \tag{37}$$

where $\omega = |u_x|^2 - |u_y|^2 - 2i u_x \cdot u_y$ is holomorphic (see remark 5). Then, defining

$$v = v_1 + i v_2 = \overline{z} + \psi + \alpha \tag{38}$$

where the constant $\alpha \in \mathbb{C}$ is to be chosen later, we have

$$\Delta v = 0 \tag{39}$$

and

$$-\frac{1}{4}\omega = \left\langle \frac{\partial v}{\partial z}, \frac{\partial v}{\partial \overline{z}} \right\rangle_{\mathbb{C}} = \frac{1}{4}\left(|v_x|^2 - |v_y|^2 - 2i\Re e\langle v_x, v_y\rangle_{\mathbb{C}}\right). \tag{40}$$

If we set

$$U = (u, v_1, v_2) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R},$$

then, by (37) and (40), we have $|U_x|^2 - |U_y|^2 - 2iU_x \cdot U_y = 0$ and, by (39) and the H-equation, we obtain $|\Delta U| \leq H_0 |\nabla U|^2$. Now, one may apply to U a generalized version of theorem 7, the proof being essentially the same. See [11] for thorough details.

□

Turning back to the geometric problem mentioned at the beginning of this subsection and as an application of the previous estimates, we quote the following result, again from [11].

THEOREM 9. *Let M be a compact surface in $\mathbb{R}^3$, diffeomorphic to $\mathbb{S}^2$ and of mean curvature H. Then*

$$\max_{p\in M} |H(p)| \geq C \frac{\operatorname{diam}(M)}{\operatorname{area}(M)}$$

where $\operatorname{diam}(M) = \max_{p,q\in M} |p-q|$.

5.3. Isoperimetric inequalities

We conclude this section recalling some central results of the work of Wente [48]. Considering the Plateau problem for H-surfaces in the case of constant H under a variational point of view, he observed that the volume functional

$$V(u) = \frac{1}{3}\int_{D^2} u \cdot u_x \wedge u_y,$$

whose existence needs u bounded, could instead be well defined by continuous extension for any $u \in H^1$ with bounded trace $u|_{\partial D^2}$. To define this extension, he used the decomposition $u = h + \phi$ where $\phi \in H_0^1$ and h is the bounded harmonic part of u (i.e., the minimizer for Dirichlet integral on $u + H_0^1$). Then, the classical isoperimetric inequality can be applied to ϕ provided that it is regular enough and, since the area functional $A(\phi)$ does not exceed the Dirichlet integral $E_0(\phi) = \frac{1}{2}\int_{D^2} |\nabla u|^2$, one has that $|V(\phi)| \leq (1/\sqrt{36\pi})A(\phi)^{3/2} \leq (1/\sqrt{36\pi})E_0(\phi)^{3/2}$ (see Bononcini [12]). From the fact $V(\phi)$ is a cubic form in ϕ, Wente deduced that V can be continuously extended on H_0^1 with the same inequality:

THEOREM 10. *Let $u \in H_0^1(D^2, \mathbb{R}^3)$. Then*

$$\left| \int_{D^2} u \cdot u_x \wedge u_y \right| \leq \frac{1}{\sqrt{32\pi}} \left(\int_{D^2} |\nabla u|^2 \right)^{3/2}.$$

Moving from this result and in order to achieve the extension to whole H^1, Wente also obtained that, for any $u \in H^1$ with bounded trace, the integral

$$\int_{D^2} \varphi \cdot u_x \wedge u_y$$

defines a continuous functional of $\varphi \in H_0^1$. This fact is of great importance in the variational setting of the H-problem, for constant H.

As far as the case of variable H is concerned, we just note that K. Steffen in [39] pointed out the intimate connection between isoperimetric inequalities and the Plateau problem with prescribed mean curvature. In particular, using the theory of integer currents, he proved the following version of isoperimetric inequality for the generalized volume functional

$$V_H(u) = \int_{D^2} Q_H(u) \cdot u_x \wedge u_y ,$$

where $Q_H : \mathbb{R}^3 \to \mathbb{R}^3$ is such that div $Q_H = H$.

THEOREM 11. *If $H \in L^\infty(\mathbb{R}^3)$, then there exists a constant C_H (depending only on $\|H\|_\infty$) such that*

$$|V_H(u)| \le C_H \left(\int_{D^2} |\nabla u|^2 \right)^{3/2} \quad \text{for every } u \in H_0^1 \cap L^\infty .$$

Moreover the functional V_H admits a unique continuous extension on H_0^1, and it satisfies the above inequality for every $u \in H_0^1$.

6. The large solution to the H-problem (Rellich's conjecture)

As we noticed in section 4, remark 4, if $H_0 > 0$ and γ is a perfect circle lying on a sphere of radius $R_0 = 1/H_0$, the solution given by the Hildebrandt's theorem 2 corresponds to the smaller part of the sphere spanning γ, the small solution. However also the larger part of the sphere is a solution to the same Plateau problem, the so-called large solution.

This example has lead to conjecture that in case of constant mean curvature $H_0 \neq 0$, if γ is Jordan curve such that $\|\gamma\|_\infty |H_0| < 1$, then there exists a pair of parametric surfaces spanning γ (*Rellich's conjecture*).

In 1984 H. Brezis and J.-M. Coron [13] proved this conjecture. Independently, also M. Struwe [42] obtained essentially the same result.

Technically, the main difficulty in showing the Rellich's conjecture is to prove that the Dirichlet problem

$$(D_{H_0}) \qquad \begin{cases} \Delta u = 2H_0 u_x \wedge u_y & \text{in } D^2 \\ u = g & \text{on } \partial D^2 \end{cases}$$

admits two different solutions. Here $g : \mathbb{S}^1 \to \gamma$ is a regular, monotone parametrization of γ. In this section we will discuss the following multiplicity result, proved by Brezis and Coron in [13].

THEOREM 12. *Let $g \in H^{1/2} \cap C^0(\partial D^2, \mathbb{R}^3)$ and let $H_0 \neq 0$ be such that*

$$\|g\|_{L^\infty} |H_0| < 1.$$

If g is nonconstant, then the problem (D_{H_0}) admits at least two solutions.

The existence of a first solution $\underline{u}$ (the small solution) is assured by theorem 2. Brezis and Coron proved the existence of a second solution $\overline{u} \neq \underline{u}$. As a consequence, even the corresponding Plateau problem has a second solution; we will not discuss this matter, we just limit ourselves to say that the proof can be deduced from the Dirichlet problem, using the usual tools (e.g. the three points condition) discussed in section 3.

We prefer to focus the discussion on the proof of a second solution to (D_{H_0}), in which the main difficulty is the behavior of the Palais-Smale sequences of the functional involved in its variational formulation. It is a typical example of a variational problem with *lack of compactness*, the overcoming of which moved on from the breakthrough analysis of Sacks and Uhlenbeck [37], and Aubin [5]. Let us notice that this kind of matters appears in many conformally invariant problems, such as harmonic maps (in dimension 2), Yamabe problem and prescribed scalar curvature problem, elliptic problems with critical exponent, Yang-Mills equations.

In the next subsections 6.1, 6.2 and 6.4 we will give an outline of the proof of theorem 12. We always assume all the hypotheses given in the statement of the theorem. Moreover, we will denote by $\underline{u}$ the small solution to (D_{H_0}) given by theorem 2.

6.1. The mountain-pass structure

Let us recall that the problem (D_{H_0}) has a variational structure (see the proof of theorem 2), i.e. its (weak) solutions are critical points of the functional

$$E_{H_0}(u) = \frac{1}{2}\int_{D^2} |\nabla u|^2 + \frac{2H_0}{3}\int_{D^2} u \cdot u_x \wedge u_y \tag{41}$$

on

$$H^1_g = \{u \in H^1(D^2, \mathbb{R}^3) : u\big|_{\partial D^2} = g\}.$$

Now, we are going to point out that the functional E_{H_0} has, essentially, a mountain pass geometry. Let us first recall the classical mountain pass lemma, stated by A. Ambrosetti and P. Rabinowitz in 1973 [4].

THEOREM 13 (MOUNTAIN PASS LEMMA). *Let X be a real Banach space and let $F \colon X \to \mathbb{R}$ be a functional of class C^1. Assume that*

(mp$_1$) *there exists $\rho > 0$ such that $\inf_{\|x\|=\rho} F(x) > F(0)$,*

(mp$_2$) *there exists $x_1 \in X$ such that $\|x_1\| > \rho$ and $F(x_1) \le F(0)$.*

Then, setting $\mathcal{P} = \{p \in C^0([0,1], X) : p(0) = 0,\ p(1) = x_1\}$, the value

$$c = \inf_{p\in\mathcal{P}} \max_{s\in[0,1]} F(p(s)) \tag{42}$$

is a generalized critical value, i.e., there exists a sequence (x_n) in X such that $F(x_n) \to c$ and $dF(x_n) \to 0$ in X'.

REMARK 6. 1. In the situation of the theorem 13, since $\|x_1\| > \rho$, by the hypothesis **(mp$_1$)**, it is clearly $\max_{s\in[0,1]} F(p(s)) \ge \alpha$ for all $p \in \mathcal{P}$, being $\alpha = \inf_{\|x\|=\rho} F(x)$. Hence, $c \ge \alpha > F(0)$.

2. A sequence $(x_n) \subset X$ satisfying $F(x_n) \to c$ and $dF(x_n) \to 0$ in X' is known as a Palais-Smale sequence for the functional F at level c.
3. Recall that a functional $F \in C^1(X, \mathbb{R})$ is said to satisfy the Palais-Smale condition if any Palais-Smale sequence for F is relatively compact, i.e., it admits a strongly convergent subsequence. Hence, if in the above theorem, the functional F satisfies the Palais-Smale condition (at level c) then it admits a critical point at level c, i.e, c is a critical value.

Coming back to our functional E_{H_0}, the possibility to apply the mountain-pass lemma is granted by the following properties.

LEMMA 6. *The functional E_{H_0} is of class C^2 on H^1_g and for all $u \in H^1_g$ one has*

$$dE_{H_0}(u) = -\Delta u + 2H_0 u_x \wedge u_y. \tag{43}$$

Here the fact that $u_x \wedge u_y \in H^{-1}$, which is implied by Wente's result given in theorem 10, is of fundamental importance, since it clearly yields $dE_{H_0}(u) \in H^{-1}$ for any $u \in H^1_g$ and hence that E_{H_0} is differentiable. We also remark that for variable H it is no longer clear and rather presumably false that $H(u)u_x \wedge u_y \in H^{-1}$ for every $u \in H^1_g$.

LEMMA 7. *The second derivative of E_{H_0} at $\underline{u}$ is coercive, i.e., there exists $\delta > 0$ such that*

$$d^2E_{H_0}(\underline{u})(\varphi,\varphi) = \int_{D^2}\Big(|\nabla\varphi|^2 + 4H_0\underline{u}\cdot\varphi_x\wedge\varphi_y\Big) \geq \delta\int_{D^2}|\nabla\varphi|^2$$

for all $\varphi \in H^1_0(D^2, \mathbb{R}^3)$.

A proof of this lemma is given in [13].

Finally, since the volume term $V_{H_0}(u) = \frac{2H_0}{3}\int_{D^2} u \cdot u_x \wedge u_y$ is cubic, whereas the Dirichlet integral is quadratic, the next result immediately follows.

LEMMA 8. $\inf_{u\in H^1_g} E_{H_0}(u) = -\infty$.

Proof. Let $v \in H^1_0$ be such that $V_{H_0}(v) \neq 0$. Taking $-v$ instead of v, if necessary, we may assume $V_{H_0}(v) < 0$. The thesis follows by noting that

$$E_{H_0}(tv + \underline{u}) = 2t^3V_{H_0}(v) + O(t^2)$$

as $t \to +\infty$.

□

Now we apply the mountain pass lemma to the functional $F\colon H^1_0 \to \mathbb{R}$ defined by

$$F(v) = E_{H_0}(v + \underline{u}) - E_{H_0}(\underline{u}) \,. \tag{44}$$

The regularity of F is assured by lemma 6, since $\underline{u} \in H^1_g = \underline{u} + H^1_0$ and

$$dF(v) = dE_{H_0}(v + \underline{u}) \,. \tag{45}$$

The condition ($\mathbf{mp_1}$) is granted by lemma 7. The condition ($\mathbf{mp_2}$) follows immediately from lemma 8. Hence, by theorem 13, the functional F admits a Palais-Smale sequence $(v_n) \subset H^1_0$ at a level $c > 0$. By (44) and (45), setting $u_n = v_n + \underline{u}$, we obtain a Palais-Smale sequence in H^1_g for the functional E_{H_0} at level $c + E_{H_0}(\underline{u})$.

Owing to the conformal invariance of the problem, the functional E_{H_0} is not expected to verify the Palais-Smale condition, and a deeper analysis of the Palais-Smale sequences for E_{H_0} is needed.

6.2. Palais-Smale sequences for E_{H_0}

Recalling remark 6, by (41) and (43), a Palais-Smale sequence for the functional E_{H_0} is a sequence $(u^n) \subset H^1_g$ such that

$$E_{H_0}(u_n) \to \bar{c} \tag{46}$$

$$\Delta u^n = 2H_0 u^n_x \wedge u^n_y + f_n \text{ in } D^2, \text{ with } f_n \to 0 \text{ in } H^{-1} \tag{47}$$

for some $\bar{c} \in \mathbb{R}$.

As a first fact, we have the following result.

LEMMA 9. *Any Palais-Smale sequence $(u_n) \subset H^1_g$ for E_{H_0} is bounded in H^1.*

Proof. Since $(u_n) \subset H^1_g$ it is enough to prove that $\sup \|\nabla u_n\|_2 < +\infty$. Setting $\varphi_n = u_n - \underline{u}$, and keeping into account that $dE_{H_0}(\underline{u}) = 0$, one has

$$\begin{aligned} E_{H_0}(u_n) &= E_{H_0}(\underline{u}) + \frac{1}{2} d^2 E_{H_0}(\underline{u})(\varphi_n, \varphi_n) + 2V_{H_0}(\varphi_n) \\ dE_{H_0}(u_n)\varphi_n &= d^2 E_{H_0}(\underline{u})(\varphi_n, \varphi_n) + 6V_{H_0}(\varphi_n). \end{aligned}$$

Hence, subtracting, one obtains

$$3E_{H_0}(u_n) = E_{H_0}(\underline{u}) + \frac{1}{2} d^2 E_{H_0}(\underline{u})(\varphi_n, \varphi_n) + dE_{H_0}(u_n)\varphi_n .$$

Using Lemma 7, one gets

$$\begin{aligned} \delta \|\nabla \varphi_n\|_2^2 &\leq d^2 E_{H_0}(\underline{u})(\varphi_n, \varphi_n) \\ &= 6(E_{H_0}(u_n) - E_{H_0}(\underline{u})) - 2dE_{H_0}(u_n)\varphi_n \\ &\leq C + \|dE_{H_0}(u_n)\| \, \|\nabla \varphi_n\|_2 . \end{aligned}$$

By (46) and (47) one infers that (φ_n) is bounded in H^1_0 and then the thesis follows. □

In the case of variable H, it is not clear whether the lemma holds or not. A method to overcome this kind of difficulty can be found in Struwe [42].

From the previous lemma we can deduce that all Palais-Smale sequences for E_{H_0} are relatively weakly compact. The next result states that the weak limit is a solution to (D_{H_0}).

LEMMA 10. *Let* $(u_n) \subset H^1_g$ *be a Palais-Smale sequence for* E_{H_0} *converging weakly in* H^1 *to some* $\bar{u} \in H^1_g$. *Then* $dE_{H_0}(\bar{u}) = 0$, *i.e.,* $\bar{u}$ *is a (weak) solution to* (D_{H_0}).

Proof. Fix an arbitrary $\varphi \in C^\infty_c(D^2, \mathbb{R}^3)$. By (47), one has

$$\int_{D^2} \nabla u_n \cdot \nabla \varphi + 2H_0 L(u_n, \varphi) \to 0$$

where we set

$$L(u, \varphi) = \int_{D^2} \varphi \cdot u_x \wedge u_y .$$

By weak convergence $\int_{D^2} \nabla u_n \cdot \nabla \varphi \to \int_{D^2} \nabla \bar{u} \cdot \nabla \varphi$. Moreover, using the divergence expression $2u_x \wedge u_y = (u \wedge u_y)_x + (u_x \wedge u)_y$, one has that

$$2L(u, \varphi) = -\int_{D^2} (\varphi_x \cdot u \wedge u_y + \varphi_y \cdot u_x \wedge u) .$$

Hence $L(u_n, \varphi) \to L(\bar{u}, \varphi)$, since $u_n \to \bar{u}$ strongly in L^2 and weakly in H^1. In conclusion, one gets

$$\int_{D^2} \nabla \bar{u} \cdot \nabla \varphi + 2H_0 \int_{D^2} \varphi \cdot \bar{u}_x \wedge \bar{u}_y = 0$$

that is the thesis.

□

However, the Palais-Smale sequences for E_{H_0} are not necessarily relatively strongly compact in H^1. In the spirit of Aubin [5] and Sacks-Uhlenbeck [37], and inspired by the concentration-compactness principle by P.-L. Lions [35], Brezis and Coron in [14] have precisely analyzed the possible defect of strong convergence, as the following theorem states.

THEOREM 14. *Suppose that* $(u_n) \in H^1_g$ *is a Palais-Smale sequence for* E_{H_0}. *Then there exist*

(i) $\overline{u} \in H^1_g$ *solving* $\Delta \overline{u} = 2H_0 \overline{u}_x \wedge \overline{u}_y$ *in* D^2,

(ii) *a finite number* $p \in \mathbb{N} \cup \{0\}$ *of nonconstant solutions* $v^1, \ldots, v^p$ *to* $\Delta u = 2H_0 u_x \wedge u_y$ *on* $\mathbb{R}^2$,

(iii) p *sequences* $(a^1_n), \ldots, (a^p_n)$ *in* D^2

(iv) *p sequences* $(\varepsilon_n^1), \ldots, (\varepsilon_n^p)$ *in* $\mathbb{R}_+$ *with* $\lim_{n\to+\infty} \varepsilon_i^n = 0$ *for any* $i = 1, \ldots, p$

such that, up to a subsequence, we have

$$\left\| u_n - \overline{u} - \sum_{i=1}^{p} v^i \left(\frac{\cdot - a_n^i}{\varepsilon_n^i} \right) \right\|_{H^1} \to 0$$

$$\int_{D^2} |\nabla u_n|^2 = \int_{D^2} |\nabla \overline{u}|^2 + \sum_{i=1}^{p} \int_{\mathbb{R}^2} |\nabla v^i|^2 + o(1)$$

$$E_{H_0}(u_n) = E_{H_0}(\overline{u}) + \sum_{i=1}^{p} \bar{E}_{H_0}(v^i) + o(1) \ ,$$

where in general $\bar{E}_{H_0}(v) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 + \frac{2H_0}{3} \int_{\mathbb{R}^2} v \cdot v_x \wedge v_y$*. In case* $p = 0$ *any sum* $\sum_{i=1}^{p}$ *is zero and* $u_n \to \bar{u}$ *strongly in* H^1.

REMARK 7. The conformal invariance is reflected in the concentrated maps $v^i \left(\frac{\cdot - a_n^i}{\varepsilon_n^i} \right)$. This theorem also emphasizes the role of solutions of the H_0-equation on whole $\mathbb{R}^2$, which are completely known (see below).

6.3. Characterization of solutions on $\mathbb{R}^2$

The solutions to the H_0-equation on the whole plane $\mathbb{R}^2$ are completely classified in the next theorem. It basically asserts that all solutions of the problem

$$\begin{cases} \Delta u = 2H_0 u_x \wedge u_y \quad \text{on } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} |\nabla u|^2 < +\infty \end{cases} \tag{48}$$

are conformal parametrizations of the sphere of radius $R_0 = 1/|H_0|$.

Note first that, if u is a solution to (48), defining $\omega = |u_x|^2 - |u_y|^2 - 2i u_x \cdot u_y$ the usual defect of conformality for u, it holds that $\frac{\partial \omega}{\partial \bar{z}} = 0$ (by the equation), and $\int_{\mathbb{R}^2} |\omega| < +\infty$ (by the summability condition on ∇u). Hence $\omega \equiv 0$, that is, u is conformal.

Pushing a little further the analysis, Brezis and Coron obtained the following result (see [14]).

THEOREM 15. *Let* $u \in L^1_{loc}(\mathbb{R}^2, \mathbb{R}^3)$ *be a solution to* (48) *with* $H_0 \neq 0$*. Then* u *has the form*

$$u(z) = \frac{1}{H_0} \Pi \left(\frac{P(z)}{Q(z)} \right) + C,$$

where C *is a constant vector in* $\mathbb{R}^3$*,* P *and* Q *are (irreducible) polynomials (in the complex variable* $z = (x, y) = x + iy$*) and* $\Pi \colon \mathbb{C} \to \mathbb{S}^2$ *is the stereographic projection.*

Moreover

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla u|^2 &= \frac{8\pi k}{H_0^2},\\ \bar{E}_{H_0}(u) &= \frac{4\pi k}{3H_0^2},\end{aligned}$$

where $k = \max\{\deg P, \deg Q\}$ *is the number of coverings of the sphere* $\mathbb{S}^2$ *by the parametrization* u.

We point out that problem (48) is invariant with respect to the conformal group. For instance, if u is a solution to (48), then $u_\lambda(z) = u(\lambda z)$ is also a solution. Note that $u_\lambda \to const$ as $\lambda \to +\infty$, or as $\lambda \to 0$.

6.4. Existence of the large solution

In this subsection, taking advantage from the results stated in the previous subsections, we will sketch the conclusion of the proof of theorem 12.

Let us recall that the functional F defined by (44) admits a mountain pass level $c > 0$. In view of the result on the Palais-Smale sequences stated in Theorem 14, it is useful also an upper bound for c, and precisely:

LEMMA 11. $c < \frac{4\pi}{3H_0^2}$.

This estimate is obtained by evaluating the functional E_{H_0} along an explicit mountain pass path which, roughly speaking, is constructed by attaching in a suitable way a sphere to the small solution.

Let now $(u_n) \subset H^1_g$ be the Palais-Smale sequence for E_{H_0} introduced at the end of the subsection 6.1. We have already seen that, up to a subsequence, (u_n) converges weakly to a solution $\bar{u}$ to (D_{H_0}). If $u_n \to \bar{u}$ strongly in H^1 then

$$E_{H_0}(\overline{u}) = E_{H_0}(\underline{u}) + c > E_{H_0}(\underline{u}) \tag{49}$$

because $c > 0$.

On the contrary, if no subsequence of (u_n) converges strongly in H^1, then we use theorem 14 on the characterization of Palais smale sequences. In particular, with the same notation of theorem 14, we have $p \geq 1$ and, denoting by $\mathcal{S}$ the set of all nonconstant solutions to (48),

$$\begin{aligned} E_{H_0}(\overline{u}) &= E_{H_0}(\underline{u}) + c - \sum_{i=1}^{p} \bar{E}_{H_0}(v_i) \\ &\leq E_{H_0}(\underline{u}) + c - p \inf_{v \in \mathcal{S}} \bar{E}_{H_0}(v) \\ &\leq E_{H_0}(\underline{u}) + c - \inf_{\omega \in \mathcal{S}} \bar{E}_{H_0}(\omega) \\ &\leq E_{H_0}(\underline{u}) + c - \frac{4\pi}{3H_0^2} \\ &< E_{H_0}(\underline{u}) \end{aligned} \tag{50}$$

according to (46), theorem 15 and lemma 11.

Thus, either from (49) or from (50), it follows that $\overline{u} \neq \underline{u}$ and the conclusion of theorem 12 is achieved.

6.5. The second solution for variable H

In the previous sections, we have seen how Brezis and Coron proved the existence of a second solution (different from the small one) to the problem (D_H), for constant H. Unfortunately, in the attempt of extending their proof to the case of variable H, lot of the main arguments fail. In view to overcome such obstacle, Struwe introduced in [44] a perturbed functional, which brings some compactness into the problem, and he succeeded to prove existence of a large solution for a class of curvature functions H, which is a dense subset in a small neighborhood of a nonzero constant, for some strong norm involving, in particular, a weighted C^1 norm. His results were then slightly improved by Wang in [46].

Here we present a result by Bethuel and Rey [11] (see also [10]), more general than the above mentioned results by Struwe and Wang, which extends theorem 12 for variable H, in a perturbative setting. A similar result is contained in [33] (see also [34]).

THEOREM 16. *Let $g \in H^{1/2} \cap C^0(\partial D^2, \mathbb{R}^3)$ be nonconstant and let $H_0 \neq 0$ be such that $\|g\|_{L^\infty}|H_0| < 1$. Then there exists $\alpha > 0$ such that for any $H \in C^1(\mathbb{R}^3)$ satisfying*

$$\|H - H_0\|_{L^\infty} < \alpha$$

the problem (D_H) admits at least two solutions.

The proof is developed by a direct variational approach (see [11]). Fundamental tools in the proof are: a careful analysis of the Palais-Smale sequences (which is more delicate than in the case of constant H); the *a priori* bound on solutions given in theorem 7, which permits the truncation on H outside a suitable ball. Indeed, replacing the original H by a function $\widetilde{H}$ such that $\widetilde{H}(u) = H(u)$ as $|u| \leq R$, $\widetilde{H}(u) = H_0$ as $|u| \geq 2R$, and solving the problem with $\widetilde{H}$, the *a priori* bound yields that the solution found to the truncated problem is also a solution to the original problem.

7. H-bubbles

In this section we deal with $\mathbb{S}^2$-type parametric surfaces in $\mathbb{R}^3$ with prescribed mean curvature H, briefly *H-bubbles*. On this subject, which might have some applications to physical problems (e.g., capillarity phenomena, see [24]), we discuss here some very recent results obtained in a series of papers by P. Caldiroli and R. Musina (see [15]–[18]).

Let us make some preliminary remarks, useful in the sequel. First, we observe that the "H-bubble problem":

> *Given a (smooth) function $H\colon \mathbb{R}^3 \to \mathbb{R}$, find an $\mathbb{S}^2$-type surface M such that the mean curvature of M at p equals $H(p)$, for all $p \in M$,*

after the identification of $\mathbb{S}^2$ with the compactified plane $\mathbb{R}^2 \cup \{\infty\}$, via stereographic projection, and using conformal coordinates, admits the following analytical formulation:

> *Find a nonconstant, conformal function $u\colon \mathbb{R}^2 \to \mathbb{R}^3$, smooth as a map on $\mathbb{S}^2$, satisfying*

$$(B)_H \qquad \begin{cases} \Delta u = 2H(u)u_x \wedge u_y \quad \text{on } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} |\nabla u|^2 < +\infty. \end{cases}$$

In principle, the two formulations of the H-bubble problem are not exactly equivalent, since in the analytical version one cannot exclude a priori the presence of branch points (i.e., self-intersection points, or points $p = u(z)$ where $\nabla u(z) = 0$). We do not enter in this aspect of geometric regularity and, from now on, we just study the analytical version $(B)_H$ of the H-bubble problem.

Observe that if $H \equiv 0$, clearly the only solutions of $(B)_H$ are the constants. Moreover, as we saw in the previous section, when the prescribed mean curvature is a nonzero constant $H(u) \equiv H_0$, Brezis and Coron in [14] completely characterized the set of solutions of (B_H) (see Theorem 15).

REMARK 8. 1. We point out that it is enough to look for weak solutions of $(B)_H$. Indeed, by regularity theory for H-systems (see Section 5), if H is smooth, then also any solution of $(B)_H$ is so. In particular, if $H \in C^1$, then any solution of $(B)_H$ turns out to be of class $C^{3,\alpha}$.

2. If u solves $(B)_H$, then u is conformal for free. Indeed, by Theorem 6, its Hopf differential is constant on $\mathbb{R}^2$, and actually, by the summability condition $\int_{\mathbb{R}^2} |\nabla u|^2 < +\infty$, it is zero, namely u is conformal. The deep reason of this rests on the fact that problem $(B)_H$ contains no boundary condition and it is invariant under the action of the conformal group of $\mathbb{S}^2 \approx \mathbb{R}^2 \cup \{\infty\}$. This invariance means that in fact we deal with a problem on the image of the unknown u, rather than on the mapping u itself.

Problem $(B)_H$ can be tackled by using variational methods. In particular, one can detect solutions of (B_H) as critical points of the energy functional

$$E_H(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + 2\int_{\mathbb{R}^2} Q_H(u)\cdot u_x \wedge u_y \ ,$$

where $Q_H\colon \mathbb{R}^3 \to \mathbb{R}^3$ is any vector field such that div $Q_H = H$. We can write $E_H(u) = E_0(u) + 2V_H(u)$, where $E_0(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2$ is the Dirichlet integral, and

$$V_H(u) = \int_{\mathbb{R}^2} Q_H(u)\cdot u_x \wedge u_y$$

is the so-called H-volume functional.

REMARK 9. This name for the functional V_H is motivated by the fact that if u is a regular parametrization of some $\mathbb{S}^2$-type surface M, then $V_H(u)$ equals the H-weighted algebraic volume of the bounded region enclosed by M. As a remarkable example, consider the mapping $\omega\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$\omega(z) = \begin{pmatrix} \mu x \\ \mu y \\ 1-\mu \end{pmatrix} \ , \ \ \mu = \mu(z) = \frac{2}{1+|z|^2} \ , \tag{51}$$

where, as usual, $z = (x, y) \in \mathbb{R}^2$. Notice that ω is a (1-degree) conformal parametrization of the unit sphere $\mathbb{S}^2$ centered at the origin. Indeed ω solves $(B)_H$ with $H \equiv 1$. One has that $E_0(\omega) = 4\pi$ = area of the unit sphere $\mathbb{S}^2$, and, by the Gauss-Green theorem,

$$V_H(\omega) = -\int_{B_1} H(q)\, dq \ , \tag{52}$$

where B_1 denotes the unit ball in $\mathbb{R}^3$. Notice also that for every $n \in \mathbb{Z}\setminus\{0\}$ the mapping $\omega^n(z) = \omega(z^n)$ (in complex notation) is a n-degree parametrization of $\mathbb{S}^2$ and $V_H(\omega^n) = nV_H(\omega)$.

Keeping into account of the shape of the functional E_H, the natural functional space to be considered as a domain of E_H seems to be the Sobolev space

$$H^1 := \{v\circ\omega \mid v \in H^1(\mathbb{S}^2, \mathbb{R}^3)\}$$

where $\omega\colon \mathbb{R}^2 \to \mathbb{S}^2$, defined in (51), is the inverse of the stereographic projection. Clearly, H^1 is a Hilbert space, endowed with the norm

$$\|u\|^2_{H^1} = \int_{\mathbb{R}^2} (|\nabla u|^2 + \mu^2|u|^2) \ ,$$

it is isomorphic to $H^1(\mathbb{S}^2, \mathbb{R}^3)$, and it can also be defined as the completion of $C^\infty_c(\mathbb{R}^2, \mathbb{R}^3)$ with respect to the Dirichlet norm.

REMARK 10. 1. Since in general Q_H is not bounded (e.g., if $H \equiv 1$, then $Q_H(u) = \frac{1}{3}u$), the H-volume functional V_H as well as the energy E_H turn out to be well defined only for $u \in H^1 \cap L^\infty$. But we can take advantage from the generalized isoperimetric inequality, due to Steffen [39] and stated in Theorem 11 for functions in $H^1_0(D^2, \mathbb{R}^3)$. In fact, using the conformal invariance, the same inequality holds true also for functions in H^1 and, in this more general version, it guarantees that V_H and E_H can be extended on the whole space H^1 in a continuous way.
2. The functionals V_H and E_H are of class C^1 on H^1 only in some special cases, like, for instance, when H is constant far out. For an arbitrary function H (smooth and bounded), we can just consider the derivatives along directions in a (dense) subspace of H^1: for every $u \in H^1$ and for every $\varphi \in H^1 \cap L^\infty$ there exists

$$\partial_\varphi E_H(u) = \int_{\mathbb{R}^2} \nabla u \cdot \nabla \varphi + 2 \int_{\mathbb{R}^2} H(u)\varphi \cdot u_x \wedge u_y \ . \tag{53}$$

In particular, from (53) one can recognize that if $u \in H^1$ is a critical point of E_H, namely $\partial_\varphi E_H(u) = 0$ for all $\varphi \in H^1 \cap L^\infty$, then u is a weak solution of $(B)_H$. In addition, by (53) one can see that the H-volume functional does not depend on the choice of the vector field Q_H.

REMARK 11. The functional E_H inherits all the invariances of problem $(B)_H$, and in particular $E_H(u \circ g) = E_H(u)$ for every conformal diffeomorphism of $\mathbb{S}^2 \approx \mathbb{R}^2 \cup \{\infty\}$. Since the conformal group of $\mathbb{S}^2$ is noncompact, this reflects into a lack of compactness in the variational problem associated to $(B)_H$, similarly to what we saw for the Plateau problem.

For several reasons, it is often meaningful to investigate the existence of H-bubbles having further properties concerning their energy or their location. Here is a list of some problems that will be discussed in the next subsections.

(i) Calling $\mathcal{B}_H$ the set of H-bubbles and assuming that $\mathcal{B}_H$ is nonempty (as it happens, for instance if H is constant, with a nonzero value, far away), is it true that $\inf_{u \in \mathcal{B}_H} E_H(u) > -\infty$?

(ii) Assuming $\mathcal{B}_H$ nonempty and $\mu_H := \inf_{u \in \mathcal{B}_H} E_H(u) > -\infty$, is μ_H attained in $\mathcal{B}_H$?

(iii) Find conditions on H ensuring the existence of an H-bubble u, possibly with minimal energy, that is, with $E_H(u) = \mu_H$.

(iv) Study the H-bubble problem in some perturbative setting, like for instance, $H(u) = H_0 + \varepsilon H_1(u)$, with $H_0 \in \mathbb{R} \setminus \{0\}$, H_1 smooth real function on $\mathbb{R}^3$, and $|\varepsilon|$ small.

7.1. On the minimal energy level for H-bubbles

Here we take $H \in C^1(\mathbb{R}^3) \cap L^\infty$ and, denoting by $\mathcal{B}_H$ the set of H-bubbles and assuming $\mathcal{B}_H \neq \emptyset$, we set

$$\mu_H = \inf_{u \in \mathcal{B}_H} E_H(u) \, . \tag{54}$$

In this subsection we will make some considerations about the minimal energy level μ_H and about the corresponding minimization problem (54). The results presented here are contained in [16].

To begin, we notice that if H is constant and nonzero, i.e., $H(u) \equiv H_0 \in \mathbb{R} \setminus \{0\}$, then by Theorem 15, $\omega^0 := \frac{1}{H_0}\omega$ belongs to $\mathcal{B}_{H_0}$ and $E_{H_0}(\omega^0) = \frac{4\pi}{3H_0^2} = \mu_{H_0}$.

REMARK 12. In case of a variable H, it is easy to see that in general it can be $\mathcal{B}_H \neq \emptyset$ and $\mu_H = -\infty$. Indeed, if there exists $u \in \mathcal{B}_H$ with $E_H(u) < 0$ then, setting $u^n(z) = u(z^n)$, for any $n \in \mathbb{N}$ the function u^n solves $(B)_H$, namely $u^n \in \mathcal{B}_H$, and $E_H(u^n) = nE_H(u)$. Consequently $\mu_H = -\infty$. One can easily construct examples of functions $H \in C^1(\mathbb{R}^3) \cap L^\infty$ for which there exist H-bubbles with negative energy. For instance, suppose that $H(u) = 1$ as $|u| = 1$, so that the mapping ω defined in (51) is an H-bubble. By (52), $E_H(\omega) = 4\pi - \int_{B_1} H(q)\, dq$. Hence, for a suitable definition of H in the unit ball B_1, one gets $E_H(\omega) < 0$.

The previous remark shows that in order that μ_H is finite, no H-bubbles with negative energy must exist. In particular, one needs some condition which prevents H to have too large variations. To this extent, in the definition of the vector field Q_H such that $\operatorname{div} Q_H = H$, it seems convenient to choose

$$Q_H(u) = m_H(u)u \, , \quad m_H(u) = \int_0^1 H(su)s^2\, ds \, .$$

Taking any H-bubble u, since $\partial_u E_H(u) = 0$, and using the identity $3m_H(u) + \nabla m_H(u) \cdot u = H(u)$, one has

$$\begin{aligned} E_H(u) &= E_H(u) - \frac{1}{3}\partial_u E_H(u) \\ &= \frac{1}{6}\int_{\mathbb{R}^2} |\nabla u|^2 - \frac{2}{3}\int_{\mathbb{R}^2} \nabla m_H(u) \cdot u \; u \cdot u_x \wedge u_y \\ &\geq \left(\frac{1}{6} - \frac{\bar{M}_H}{3}\right)\int_{\mathbb{R}^2} |\nabla u|^2 \end{aligned} \tag{55}$$

where

$$\bar{M}_H := \sup_{u \in \mathbb{R}^3} |\nabla m_H(u) \cdot u \; u| \, .$$

Hence, if $\bar{M}_H \leq \frac{1}{2}$, then $\mu_H \geq 0$.

Now, let us focus on the simplest case in which H is assumed to be constant far out. This hypothesis immediately implies that $\mathcal{B}_H$ is nonempty and the minimization problem defined by (54) reduces to investigate the semicontinuity of the energy functional E_H along a sequence of H-bubbles. As shown by Wente in [47], in general E_H is not globally semicontinuous with respect to weak convergence, even if H is constant. However, as we will see in the next result, under the condition $\bar{M}_H < \frac{1}{2}$, semicontinuity holds true at least along a sequence of solutions of $(B)_H$.

THEOREM 17. *Let $H \in C^1(\mathbb{R}^3)$ satisfy*

($\mathbf{h_1}$) $H(u) = H_\infty \in \mathbb{R} \setminus \{0\}$ *as* $|u| \geq R$*, for some* $R > 0$*,*

($\mathbf{h_2}$) $\bar{M}_H < \frac{1}{2}$.

Then there exists $\omega \in \mathcal{B}_H$ such that $E_H(\omega) = \mu_H$. Moreover $\mu_H \leq \frac{4\pi}{3H_\infty^2}$.

Proof. First, we observe that by ($\mathbf{h_1}$), $\mathcal{B}_H \neq \emptyset$, since the spheres of radius $|H_\infty|^{-1}$ placed in the region $|u| \geq R$ are H-bubbles. In particular, this implies that $\mu_H \leq \frac{4\pi}{3H_\infty^2}$. Now, take a sequence $(u^n) \subset \mathcal{B}_H$ with $E_H(u^n) \to \mu_H$. Since the problem $(B)_H$ is invariant with respect to the conformal group, we may assume that $\|\nabla u^n\|_\infty = |\nabla u^n(0)| = 1$ (normalization conditions).

Step 1 (Uniform global estimates): we may assume

$$\sup \|\nabla u^n\|_2 < +\infty \text{ and } \sup \|u^n\|_\infty < +\infty .$$

The first bound follows by (55), by ($\mathbf{h_2}$), and by the fact that (u^n) is a minimizing sequence for the energy in $\mathcal{B}_H$. As regards the second estimate, first we observe that using Theorem 7 one can prove that

$$\sup_n \operatorname{diam} u^n =: \rho < +\infty ,$$

where, in general, $\operatorname{diam} u = \sup_{z,z' \in \mathbb{R}^2} |u(z) - u(z')|$. If $\|u^n\|_\infty \leq R + \rho$, set $\tilde{u}^n = u^n$. If $\|u^n\|_\infty > R + \rho$, then by the assumption ($\mathbf{h_1}$), u^n solves $\Delta u = 2H_\infty u_x \wedge u_y$. Let $p_n \in \operatorname{range} u^n$ be such that $|p_n| = \|u^n\|_\infty$. Set $q_n = \left(1 - \frac{R+\rho}{|p_n|}\right) p_n$ and $\tilde{u}^n = u^n - q_n$. Then $\|\tilde{u}^n\|_\infty \leq R + \rho$, and $|\tilde{u}^n(z)| \geq R$ for every $z \in \mathbb{R}^2$. Hence, also $\tilde{u}^n \in \mathcal{B}_H$, and $E_H(\tilde{u}^n) = E_{H_\infty}(\tilde{u}^n) = E_H(u^n)$. Therefore $(\tilde{u}^n)$ is a minimizing sequence of H-bubbles satisfying the required uniform estimates.

Step 2 (Local "ε-regularity" estimates): there exist $\varepsilon > 0$ and, for every $s \in (1, +\infty)$ a constant $C_s > 0$ (depending only on $\|H\|_\infty$), such that if u is a weak solution of $(B)_H$, then

$$\|\nabla u\|_{L^2(D_R(z))} \leq \varepsilon \ \Rightarrow\ \|\nabla u\|_{H^{1,s}(D_{R/2}(z))} \leq C_s \|\nabla u\|_{L^2(D_R(z))}$$

for every $R \in (0, 1]$ and for every $z \in \mathbb{R}^2$.

These ε-regularity estimates are an adaptation of a similar result obtained by Sacks and Uhlenbeck in their celebrated paper [37]. We omit the quite technical proof of this step and we refer to [15] for the details.

Step 3 (Passing to the limit): there exists $u \in H^1 \cap C^1(\mathbb{R}^2, \mathbb{R}^3)$ such that, for a subsequence, $u^n \to u$ weakly in H^1 and strongly in $C^1_{loc}(\mathbb{R}^2, \mathbb{R}^3)$.

By the uniform estimates stated in the step 1, we may assume that the sequence (u^n) is bounded in H^1. Hence, there exists $u \in H^1$ such that, for a subsequence, still denoted (u^n), one has that $u^n \to u$ weakly in H^1. Now, fix a compact set K in $\mathbb{R}^2$. Since $\|\nabla \omega^n\|_\infty = 1$, there exists $R > 0$ and a finite covering $\{D_{R/2}(z_i)\}_{i \in I}$ of K such that $\|\nabla u^n\|_{L^2(D_R(z_i))} \le \varepsilon$ for every $n \in \mathbb{N}$ and $i \in I$. Using the ε-regularity estimates stated in the step 2, and since (u^n) is bounded in L^∞, we have that $\|u^n\|_{H^{2,s}(D_{R/2}(z_i))} \le \bar{C}_{s,R}$ for some constant $\bar{C}_{s,R} > 0$ independent of $i \in I$ and $n \in \mathbb{N}$. Then the sequence (u^n) is bounded in $H^{2,p}(K, \mathbb{R}^3)$. For $s > 2$ the space $H^{2,s}(K, \mathbb{R}^3)$ is compactly embedded into $C^1(K, \mathbb{R}^3)$. Hence $u^n \to u$ strongly in $C^1(K, \mathbb{R}^3)$. By a standard diagonal argument, one concludes that $u^n \to u$ strongly in $C^1_{loc}(\mathbb{R}^2, \mathbb{R}^3)$.

Step 4: u is an H-bubble.

For every $n \in \mathbb{N}$ one has that if $\varphi \in C^\infty_c(\mathbb{R}^2, \mathbb{R}^3)$ then

$$\int_{\mathbb{R}^2} \nabla u^n \cdot \nabla \varphi + 2 \int_{\mathbb{R}^2} H(u^n)\varphi \cdot u^n_x \wedge u^n_y = 0 \, .$$

By step 3, passing to the limit, one immediately infers that u is a weak solution of $(B)_H$. According to Remark 8, u is a classical, conformal solution of $(B)_H$. In addition, u is nonconstant, since $|\nabla u(0)| = \lim |\nabla u^n(0)| = 1$. Hence $u \in \mathcal{B}_H$.

Step 5 (Semicontinuity inequality): $E_H(u) \le \liminf E_H(u^n)$.

By the strong convergence in $C^1_{loc}(\mathbb{R}^2, \mathbb{R}^3)$, for every $R > 0$, one has

$$E_H(u^n, D_R) \to E_H(u, D_R) \tag{56}$$

where we denoted

$$E_H(u^n, \Omega) = \frac{1}{2} \int_\Omega |\nabla u^n|^2 + 2 \int_\Omega m_H(u^n) u^n \cdot u^n_x \wedge u^n_y$$

(and similarly for $E_H(u, \Omega)$). Now, fixing $\epsilon > 0$, let $R > 0$ be such that

$$E_H(u, \mathbb{R}^2 \setminus D_R) \le \epsilon \tag{57}$$

$$\int_{\mathbb{R}^2 \setminus D_R} |\nabla u|^2 \le \epsilon \, . \tag{58}$$

By (57) and (56) we have

$$\begin{aligned} E_H(u) &\le E_H(u, D_R) + \epsilon \\ &= E_H(u^n, D_R) + \epsilon + o(1) \\ &= E_H(u^n) - E_H(u^n, \mathbb{R}^2 \setminus D_R) + \epsilon + o(1) \end{aligned} \tag{59}$$

with $o(1) \to 0$ as $n \to +\infty$. Since every u^n is an H-bubble, using the divergence theorem, for any $R > 0$ one has

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^2\setminus D_R} |\nabla u^n|^2 &= 3E_H(u^n, \mathbb{R}^2 \setminus D_R) - \int_{\partial D_R} u^n \cdot \frac{\partial u^n}{\partial \nu} \\ &\quad +2\int_{\mathbb{R}^2\setminus D_R} (H(u^n) - 3m_H(u^n))u^n \cdot u^n_x \wedge u^n_y \,.\end{aligned}$$

We can estimate the last term as in (55), obtaining that

$$\begin{aligned}-E_H(u^n, \mathbb{R}^2 \setminus D_R) &\le -\frac{1}{3}\int_{\partial D_R} u^n \cdot \frac{\partial u^n}{\partial \nu} - \left(\frac{1}{6} - \frac{\bar{M}_H}{3}\right)\int_{\mathbb{R}^2\setminus D_R} |\nabla u^n|^2 \\ &\le -\frac{1}{3}\int_{\partial D_R} u^n \cdot \frac{\partial u^n}{\partial \nu}\,,\end{aligned} \tag{60}$$

because of the assumption ($\mathbf{h_2}$). Using again the C^1_{loc} convergence of u^n to u, as well as the fact that u is an H-bubble, we obtain that

$$\begin{aligned}\lim_{n\to+\infty}\left|\int_{\partial D_R} u^n \cdot \frac{\partial u^n}{\partial \nu}\right| &= \left|\int_{\partial D_R} u \cdot \frac{\partial u}{\partial \nu}\right| \\ &= \left|\int_{\mathbb{R}^2\setminus D_R} \Big(u \cdot \Delta u + |\nabla u|^2\Big)\right| \\ &= \left|\int_{\mathbb{R}^2\setminus D_R} \Big(2H(u)u \cdot u_x \wedge u_y + |\nabla u|^2\Big)\right| \\ &\le (\|u\|_\infty \|H\|_\infty + 1)\int_{\mathbb{R}^2\setminus D_R} |\nabla u|^2 \\ &\le (\|u\|_\infty \|H\|_\infty + 1)\,\epsilon\end{aligned} \tag{61}$$

thanks to (58). Finally, (59), (60) and (61) imply

$$E_H(u) \le E_H(u^n) + C\epsilon + o(1)$$

for some positive constant C independent of ϵ and n. Hence, the conclusion follows. □

7.2. Existence of minimal H-bubbles

Here we study the case of a prescribed mean curvature function $H \in C^1(\mathbb{R}^3)$ asymptotic to a constant at infinity and, in particular, we discuss a result obtained in [15] about the existence of H-bubbles with minimal energy, under global assumptions on the prescribed mean curvature H.

Before stating this result, we need some preliminaries. First, we observe that, by the generalized isoperimetric inequality stated in Theorem 11 and since E_H is invariant under dilation, for a nonzero, bounded function H, the volume functional V_H turns out to be essentially cubic and $u \equiv 0$ is a strict local minimum for E_H in the space of smooth functions $C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$. Moreover, if H is nonzero on a sufficiently large set (as it happens if H is asymptotic to a nonzero constant at infinity), $E_H(v) < 0$ for some $v \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$. Hence E_H has a mountain pass geometry on $C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$. Let us introduce the value

$$c_H = \inf_{\substack{u \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3) \\ u \neq 0}} \sup_{s>0} E_H(su) ,$$

which represents the mountain pass level along radial paths. Now, the existence of minimal H-bubbles can be stated as follows.

THEOREM 18. *Let $H \in C^1(\mathbb{R}^3)$ satisfy*

(**h₃**) *$H(u) \to H_\infty$ as $|u| \to \infty$, for some $H_\infty \in \mathbb{R}$,*

(**h₄**) *$\sup_{u \in \mathbb{R}^3} |\nabla H(u) \cdot u\, u| =: M_H < 1$,*

(**h₅**) *$c_H < \frac{4\pi}{3H_\infty^2}$.*

Then there exists an H-bubble $\bar{u}$ with $E_H(\bar{u}) = c_H = \inf_{u \in \mathcal{B}_H} E_H(u)$.

The assumption (**h₄**) is a stronger version of the condition (**h₂**) (indeed $2\bar{M}_H \le M_H$), and it essentially guarantees that the value c_H is an admissible minimax level.

The assumption (**h₅**) is variational in nature, and it yields a comparison between the radial mountain pass level c_H for the energy functional E_H and the corresponding level for the problem at infinity, in the spirit of concentration-compactness principle by P.-L. Lions [35]. Indeed, the problem at infinity corresponds to the constant curvature H_∞ and, in this case, one can evaluate $c_{H_\infty} = \frac{4\pi}{3H_\infty^2}$.

The hypothesis (**h₅**) can be checked in terms of H in some cases. For instance, (**h₅**) holds true when $|H(u)| \ge |H_\infty| > 0$ for all $u \in \mathbb{R}$ but $H \not\equiv H_\infty$, or when $|H(u)| > |H_\infty| > 0$ for $|u|$ large, or when $H_\infty = 0$ and $E_H(v) < 0$ for some $v \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$. On the other hand, one can show that if $H \in C^1(\mathbb{R}^3)$ satisfies (**h₃**), (**h₄**), and $|H(u)| \le |H_\infty|$ for all $u \in \mathbb{R}^3$, then (**h₅**) fails and, in this case, Theorem 18 gives no information about the existence of H-bubbles.

As a preliminary result, we state some properties about the value c_H, which make clearer the role of the assumption (**h₄**).

LEMMA 12. *Let $H \in C^1(\mathbb{R}^3)$ be such that $M_H < 1$. The following properties hold:*

(i) *if $u \in \mathcal{B}_H$ then $E_H(u) \ge c_H$;*

(ii) *if $\lambda \in (0, 1]$ then $c_{\lambda H} \ge c_H$;*

(iii) *if $(H_n) \subset C^1(\mathbb{R}^3)$ is a sequence converging uniformly to H and $M_{H_n} < 1$ for all $n \in \mathbb{N}$, then* $\limsup c_{H_n} \le c_H$.

Proof. (i) Let $u \in \mathcal{B}_H$ and consider the mapping $s \mapsto f(s) := E_H(su)$ for $s \geq 0$. We know that $s = 1$ is a stationary point for f since u is a critical point of E_H. Moreover, if $\bar{s} > 0$ is a stationary point for f, then

$$0 = f'(\bar{s}) = \bar{s} \int_{\mathbb{R}^2} |\nabla u|^2 + 2\bar{s}^2 \int_{\mathbb{R}^2} H(\bar{s}u) u \cdot u_x \wedge u_y$$

and consequently

$$\begin{aligned} f''(\bar{s}) &= \int_{\mathbb{R}^2} |\nabla u|^2 + 4\bar{s} \int_{\mathbb{R}^2} H(\bar{s}u) u \cdot u_x \wedge u_y + 2\bar{s}^2 \int_{\mathbb{R}^2} \nabla H(\bar{s}u) \cdot u \; u \cdot u_x \wedge u_y \\ &= -\int_{\mathbb{R}^2} |\nabla u|^2 + 2 \int_{\mathbb{R}^2} \nabla H(\bar{s}u) \cdot \bar{s}u \; \bar{s}u \cdot u_x \wedge u_y \\ &\leq -(1 - M_H) \int_{\mathbb{R}^2} |\nabla u|^2 . \end{aligned}$$

Hence, there exists only one stationary point $\bar{s} > 0$ for f and $\bar{s} = 1$. Moreover $\max_{s \geq 0} E_H(su) = E_H(u)$. Since $C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$ is dense in H^1 with respect to the Dirichlet norm, for every $\epsilon > 0$ there exists $v \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$ such that $|E_H(sv) - E_H(su)| < \epsilon$ for all $s \geq 0$ in a compact interval. This is enough to obtain the desired estimate.

The statements (ii) and (iii) follow by the definition of c_H, and by using arguments similar to the proof of (i).

□

Proof of Theorem 18. . We just give an outline of the proof and we refer to [15] for all the details.

First part: The case H constant far out.

Firstly one proves the result under the additional condition $(\mathbf{h_1})$. Since $\bar{M}_H \leq \frac{1}{2} M_H < \frac{1}{2}$ one can apply Theorem 17 to infer the existence of an H-bubble at the minimal level μ_H. Then one has to show that $c_H = \mu_H$, which is an essential information in order to give up the extra assumption $(\mathbf{h_1})$, performing an approximation procedure on the prescribed mean curvature function H. From Lemma 12, part (i), one gets $\mu_H \geq c_H$. The opposite inequality needs more work and its proof is obtained in few steps.

Step 1: Approximating compact problems.

Let us introduce the family of Dirichlet problems given by

$$(D)_{H,\alpha} \qquad \begin{cases} \operatorname{div}((1 + |\nabla u|^2)^{\alpha - 1} \nabla u) = 2H(u) u_x \wedge u_y & \text{in } D^2 \\ u = 0 & \text{on } \partial D^2 , \end{cases}$$

where $\alpha > 1$, α close to 1. This kind of approximation is in essence the same as in a well known paper by Sacks and Uhlenbeck [37] and it turns out to be particularly helpful in order to get uniform estimates. Solutions to $(D)_{H,\alpha}$ can be obtained as critical points of the functional

$$E_H^\alpha(u) = \frac{1}{2\alpha}\int_{D^2}((1+|\nabla u|^2)^\alpha - 1) + 2V_H(u)$$

defined on $H_0^{1,2\alpha} := H_0^{1,2\alpha}(D^2,\mathbb{R}^3)$. Since $H_0^{1,2\alpha}$ is continuously embedded into $H_0^1 \cap L^\infty$, the functional E_H^α is of class C^1 on $H_0^{1,2\alpha}$. Moreover, for $\alpha > 1$, α close to 1, E_H^α admits a mountain pass geometry at a level $c_H^\alpha > 0$, and it satisfies the Palais-Smale condition, because the embedding of $H_0^{1,2\alpha}$ into L^∞ is compact. Then, an application of the mountain pass lemma (Theorem 13) gives the existence of a critical point $u^\alpha \in H_0^{1,2\alpha}$ for E_H^α at level c_H^α, namely a nontrivial weak solution to $(D)_{H,\alpha}$.

Step 2: Uniform estimates on u^α. The family of solutions (u^α) turn out to satisfy the following uniform estimates:

$$\limsup_{\alpha\to 1} E_H^\alpha(u^\alpha) \le c_H\ , \tag{62}$$

$$C_0 \le \|\nabla u^\alpha\|_2 \le C_1 \text{ for some } 0 < C_0 < C_1 < +\infty\ , \tag{63}$$

$$\sup_\alpha \|u^\alpha\|_\infty < +\infty\ . \tag{64}$$

The inequality (62) is proved by showing that $\limsup_{\alpha\to 1} c_H^\alpha \le c_H$, which can be obtained using ($\mathbf{h_5}$), the definitions of c_H^α and c_H, and the fact that $E_H^\alpha(u) \to E_H(u)$ as $\alpha \to 1$ for every $u \in C_c^\infty(D^2,\mathbb{R}^3)$. As regards (63), the upper bound follows by an estimate similar to (55), whereas the lower bound is a consequence of the generalized isoperimetric inequality. In both the estimates one uses the bound $\bar{M}_H < \frac{1}{2}$. Finally, (64) is proved with the aid of a nice result by Bethuel and Ghidaglia [8] which needs the condition that H is constant far out (here we use the additional assumption ($\mathbf{h_1}$)).

Now, taking advantage from the previous uniform estimates, one can pass to the limit as $\alpha \to 1$ and one finds that the weak limit u of (u^α) is a solution of

$$\begin{cases} \Delta u = 2H(u)u_x \wedge u_y & \text{in } D^2 \\ u = 0 & \text{on } \partial D^2\ . \end{cases} \tag{$(D)_H$}$$

A nonexistence result by Wente [48] implies that $u \equiv 0$. Hence a lack of compactness occurs by a blow up phenomenon.

Step 3: Blow-up.

Let us define

$$v^\alpha(z) = u^\alpha(z_\alpha + \epsilon_\alpha z)$$

with $z_\alpha \in \mathbb{R}^2$ and $\epsilon_\alpha > 0$ chosen in order that $\|\nabla v^\alpha\|_\infty = |\nabla v^\alpha(0)| = 1$. Notice that $\epsilon_\alpha \to 0$ and the sets $\Omega_\alpha := \{z \in \mathbb{R}^2 \ :\ |z_\alpha + \epsilon_\alpha z| < 1\}$ are discs which become larger and larger as $\alpha \to 1$. Moreover $v^\alpha \in C_c(\mathbb{R}^2,\mathbb{R}^3) \cap H^1$ is a weak solution to

$$\begin{cases} \Delta v^\alpha = -\frac{2(\alpha-1)}{\epsilon_\alpha^2+|\nabla v^\alpha|^2}(\nabla^2 v^\alpha, \nabla v^\alpha)\nabla v^\alpha + \frac{2\epsilon_\alpha^{2(\alpha-1)}H(v^\alpha)}{(\epsilon_\alpha^2+|\nabla v^\alpha|^2)^{\alpha-1}} v_x^\alpha \wedge v_y^\alpha & \text{in } D_\alpha \\ v = 0 & \text{on } \partial D_\alpha\ , \end{cases}$$

satisfying the same uniform estimates as u^α for the Dirichlet and L^∞ norms, as well as the previous normalization conditions on its gradient. Using a refined version (adapted to the above system) of the ε-regularity estimates similar to the step 2 in the proof of Theorem 17, one can show that there exists $u \in H^1$ such that $v^\alpha \to u$ weakly in H^1 and strongly in $C^1_{loc}(\mathbb{R}^2, \mathbb{R}^3)$, and u is a λH-bubble for some $\lambda \in (0, 1]$. Here the value λ comes out as limit of $\epsilon_\alpha^{2(\alpha-1)}$ when $\alpha \to 1$. It remains to show that actually $\lambda = 1$. Indeed, one can show that $E_{\lambda H}(u) \leq \lambda \liminf E^\alpha_H(u^\alpha)$. Using (62) and Lemma 12, parts (i) and (ii), one infers that $c_H \leq c_{\lambda H} \leq E_H(u) \leq \lambda c_H$. Therefore $\lambda = 1$ and u is an H-bubble, with $E_H(u) = c_H$. In particular $\mu_H \leq c_H$ and actually, by Lemma 12, part (i), $\mu_H = c_H$, which was our goal.

Second part: Removing the extra assumption ($\mathbf{h_1}$).

It is possible to construct a sequence $(H_n) \subset C^1(\mathbb{R}^3)$ converging uniformly to H and satisfying ($\mathbf{h_1}$) and $M_{H_n} \leq M_H$. By the first part of the proof, for every $n \in \mathbb{N}$ there exists an H_n-bubble u^n with $E_{H_n}(u^n) = \mu_{H_n} = c_{H_n}$. Since $M_{H_n} \leq M_H < 1$, by an estimate similar to (55), one deduces that the sequence (u^n) is uniformly bounded with respect to the Dirichlet norm. Moreover one has that that $\limsup E_{H_n}(u^n) = \limsup c_{H_n} \leq c_H$, because of Lemma 12, part (iii). In order to get also a uniform L^∞ bound, one argues by contradiction. Suppose that (u^n) is unbounded in L^∞. Using Theorem 7, one can prove that the sequence of values diam u^n is bounded. Consequently, the sequence (u^n) moves at infinity and, roughly speaking, it accumulates on a solution u^∞ of the problem at infinity, that is on an H_∞-bubble. In addition, as in the proof of Theorem 17, the semicontinuity inequality $\liminf E_{H_n}(u^n) \geq E_{H_\infty}(u^\infty)$ holds true. Since the problem at infinity corresponds to a constant mean curvature H_∞, by Theorem 15, one has that $E_{H_\infty}(u^\infty) \geq \mu_{H_\infty} = \frac{4\pi}{3H^2_\infty}$. On the other hand, $E_{H_n}(u^n) = c_{H_n}$, and then $c_H \geq \limsup c_{H_n} \geq \frac{4\pi}{3H^2_\infty}$, in contradiction with the assumption ($\mathbf{h_5}$). Therefore (u^n) satisfies the uniform bounds

$$\sup \|\nabla u^n\|_2 < +\infty \ , \ \sup \|u^n\|_\infty < +\infty \ .$$

Now one can repeat essentially the same argument of the proof of Theorem 17 to conclude that, after normalization, u^n converges weakly in H^1 and strongly in $C^1_{loc}(\mathbb{R}^2, \mathbb{R}^3)$ to an H-bubble $\bar{u}$. Moreover

$$E_H(\bar{u}) \leq \liminf E_H(u^n) = \liminf c_{H_n} \leq c_H \ .$$

Since $E_H(\bar{u}) \geq c_H$ (see Lemma 12, (i)), the conclusion follows.

□

In [17] it is proved that the existence result about minimal H-bubbles stated in Theorem 18 is stable under small perturbations of the prescribed curvature function. More precisely, the following result holds.

THEOREM 19. *Let $H \in C^1(\mathbb{R}^3)$ satisfy ($\mathbf{h_3}$)–($\mathbf{h_5}$), and let $H_1 \in C^1(\mathbb{R}^3)$. Then there is $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon})$ there exists an $(H + \varepsilon H_1)$-bubble u^ε. Furthermore, as $\varepsilon \to 0$, u^ε converges to some minimal H-bubble u in $C^{1,\alpha}(\mathbb{S}^2, \mathbb{R}^3)$.*

We remark that the energy of u^ε is close to the (unperturbed) minimal energy of H-bubbles. However in general we cannot say that u^ε is a minimal $(H+\varepsilon H_1)$-bubble.

Finally, we notice that Theorem 19 cannot be applied when the unperturbed curvature H is a constant, since assumption (h_3) is not satisfied. That case is studied in the next subsection.

7.3. H-bubbles in a perturbative setting

Here we study the H-bubble problem when the prescribed mean curvature is a perturbation of a nonzero constant. More precisely we investigate the existence and the location of nonconstant solutions to the problem

$$(B)_{H_\varepsilon} \qquad \begin{cases} \Delta u = 2H_\varepsilon(u)u_x \wedge u_y & \text{on } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} |\nabla u|^2 < +\infty. \end{cases}$$

where

$$H_\varepsilon(u) = H_0 + \varepsilon H_1(u)$$

being $H_0 \in \mathbb{R} \setminus \{0\}$, $H_1 \in C^2(\mathbb{R}^3)$ and $\varepsilon \in \mathbb{R}$, with $|\varepsilon|$ small. All the results of this subsection are taken from [18].

To begin, we observe that the unperturbed problem $(B)_{H_0}$ is invariant under translations on the image, since the mean curvature is the constant H_0. It admits a fundamental solution

$$\omega^0 = \frac{1}{H_0}\omega$$

(with ω defined by (51)), and a corresponding family of solutions of the form $\omega^0 \circ g + p$ where g is any conformal diffeomorphism of $\mathbb{R}^2 \cup \{\infty\}$ and p runs in $\mathbb{R}^3$.

Notice that the translation invariance on the image is broken for $\varepsilon \neq 0$, when the perturbation H_1 is switched on, but problem $(B)_{H_\varepsilon}$ maintains the conformal invariance for every ε.

An important role for the existence of H_ε-bubbles is played by the following Poincaré-Melnikov function:

$$\Gamma(p) := -\int_{B_{1/|H_0|}(p)} H_1(q)\, dq$$

which measures the H_1-weighted volume of a ball centered at an arbitrary $p \in \mathbb{R}^3$ and with radius $1/|H_0|$. For future convenience, we point out that:

$$\Gamma(p) = V_{H_1}(\omega^0 + p)\ , \tag{65}$$

$$\nabla\Gamma(p) = \int_{\mathbb{R}^2} H_1(\omega^0 + p)\omega^0_x \wedge \omega^0_y\ . \tag{66}$$

The first equality is like (52), the second one can be obtained in a similar way, noting that div $(H_1(\cdot + p)e_i) = \partial_i H_i(\cdot + p)$ (e_1, e_2, e_3 denotes that canonical basis in $\mathbb{R}^3$, ∂_i means differentiation with respect to the i-th component).

The next result yields a necessary condition, expressed in terms of Γ, in order to have the existence of H_ε-bubbles approaching a sphere, as $\varepsilon \to 0$.

PROPOSITION 4. *Assume that there exists a sequence u^{ε_k} of H_{ε_k}-bubbles, with $\varepsilon_k \to 0$, and a point $p \in \mathbb{R}^3$ such that*

$$\|u^{\varepsilon_k} - (\omega^0 + p)\|_{C^1(\mathbb{S}^2,\mathbb{R}^3)} \to 0 \ \text{ as } k \to \infty.$$

Then p is a stationary point for Γ.

Proof. The maps u^{ε_k} solve $\Delta u^{\varepsilon_k} = 2H_0 u_x^{\varepsilon_k} \wedge u_y^{\varepsilon_k} + 2\varepsilon_k H_1(u^{\varepsilon_k}) u_x^{\varepsilon_k} \wedge u_y^{\varepsilon_k}$. Testing with the constant functions e_i $(i = 1, 2, 3)$ and passing to the limit, we get

$$0 = \int_{\mathbb{R}^2} H_1(u^{\varepsilon_k}) e_i \cdot u_x^{\varepsilon_k} \wedge u_y^{\varepsilon_k} \ = o(1) + \int_{\mathbb{R}^2} H_1(\omega^0 + p) e_i \cdot \omega_x^0 \wedge \omega_y^0 \ = o(1) + \partial_i \Gamma(p),$$

thanks to (66). Then the Proposition is readily proved.

□

In the next result we consider the case in which Γ admits nondegenerate stationary points.

THEOREM 20. *If $\bar{p} \in \mathbb{R}^3$ is a nondegenerate stationary point for Γ, then there exists a curve $\varepsilon \mapsto u^\varepsilon$ of class C^1 from a neighborhood $I \subset \mathbb{R}$ of 0 into $C^{1,\alpha}(\mathbb{S}^2, \mathbb{R}^3)$ such that $u^0 = \omega^0 + \bar{p}$ and, for every $\varepsilon \in I$, u^ε is an H_ε-bubble, without branch points.*

In the case of extremal points for Γ, we can weaken the nondegeneracy condition. More precisely, we have the following result.

THEOREM 21. *If there exists a nonempty compact set $K \subset \mathbb{R}^3$ such that*

$$\max_{p \in \partial K} \Gamma(p) < \max_{p \in K} \Gamma(p) \ \text{ or } \ \min_{p \in \partial K} \Gamma(p) > \min_{p \in K} \Gamma(p) \ ,$$

then for $|\varepsilon|$ small enough there exists an H_ε-bubble u^ε, without branch points, and such that

$$\|u^\varepsilon - (\omega^0 + p_\varepsilon)\|_{C^{1,\alpha}(\mathbb{S}^2,\mathbb{R}^3)} \to 0 \ \text{ as } \varepsilon \to 0,$$

where $p_\varepsilon \in K$ is such that $\Gamma(p_\varepsilon) \to \max_K \Gamma$, or $\Gamma(p_\varepsilon) \to \min_K \Gamma$, respectively.

To prove Theorems 20 and 21 we adopt a variational-perturbative method introduced by Ambrosetti and Badiale in [1] and subsequently used with success to get existence and multiplicity results for a wide class of variational problems in some perturbative setting (see, e.g., [2] and [3]).

Firstly, we observe that solutions to problem $(B)_{H_\varepsilon}$ can be obtained as critical points of the energy functional

$$E_{H_\varepsilon}(u) = E_{H_0}(u) + 2\varepsilon V_{H_1}(u) \ .$$

Notice that E_{H_0} is the energy functional corresponding to the unperturbed problem $(B)_{H_0}$. Since in our argument we will need enough regularity for E_{H_ε}, a first (technical) difficulty concerns the functional setting (see Remark 10, 2). We can overcome this problem, either multiplying H_1 by a suitable cut-off function and proving some a priori estimates on the solutions we will find, or taking as a domain of E_{H_ε} a Sobolev space smaller than H^1, like for instance the space

$$W^{1,s} = \{v \circ \omega \ : \ v \in W^{1,s}(\mathbb{S}^2, \mathbb{R}^3)\}$$

with $s > 2$ fixed. Let us follow this second strategy, taking for simplicity $s = 3$. Hence E_{H_ε} is of class C^2 on $W^{1,3}$, since $H_1 \in C^2$ and $W^{1,3}$ is compactly embedded into L^∞.

Secondly, we point out that the unperturbed energy functional E_{H_0} admits a manifold Z of critical points that can be parametrized by $G \times \mathbb{R}^3$, where G is the conformal group of $\mathbb{S}^2 \approx \mathbb{R}^2 \cup \{\infty\}$, having dimension 6, and $\mathbb{R}^3$ keeps into account of the translation invariance on the image.

Thanks to some key results already known in the literature, see e.g. [32], Z is a nondegenerate manifold, that is

$$T_u Z = \ker E''_{H_0}(u) \quad \text{for every } u \in Z \ ,$$

where $T_u Z$ denotes the tangent space of Z at u, whereas $\ker E''_{H_0}(u)$ is the kernel of the second differential of E_{H_0} at u. This allows us to apply the implicit function theorem to get, taking account also of the G-invariance of E_{H_ε}, for $|\varepsilon|$ small, a 3-dimensional manifold $\mathcal{Z}_\varepsilon$ close to Z, constituting a natural constraint for the perturbed functional E_{H_ε}. More precisely, defining

$$(T_{\omega^0} Z)^\perp := \{v \in H^1 \mid \int_{\mathbb{R}^2} \nabla v \cdot \nabla u = 0 \ \ \forall u \in T_{\omega^0} Z\} \ ,$$

we can prove the following result.

LEMMA 13. *Let $R > 0$ be fixed. Then there exist $\bar{\varepsilon} > 0$, and a map $\eta^\varepsilon(p) \in W^{1,3}$ defined and of class C^1 on $(-\bar{\varepsilon}, \bar{\varepsilon}) \times B_R \subset \mathbb{R} \times \mathbb{R}^3$, such that $\eta^0(p) = 0$ and*

$$\begin{aligned} E'_{H_\varepsilon}(\omega + p + \eta^\varepsilon(p)) &\in T_{\omega^0} Z \\ \eta^\varepsilon(p) &\in (T_{\omega^0} Z)^\perp \\ \int_{\mathbb{S}^2} \eta^\varepsilon(p) &= 0. \end{aligned}$$

Moreover, for every fixed $\varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon})$ the set $\mathcal{Z}_\varepsilon^R := \{\omega^0 + p + \eta^\varepsilon(p) \mid |p| < R\}$ is a natural constraint for E_{H_ε}, that is, if $u \in \mathcal{Z}_\varepsilon^R$ is such that $dE_{H_\varepsilon}\big|_{\mathcal{Z}_\varepsilon^R}(u) = 0$, then $E'_{H_\varepsilon}(u) = 0$.

We refer to [18] for the proof of Lemma 13. Now, the problem is reduced to look for critical points of the function $f_\varepsilon \colon B_R \to \mathbb{R}$ defined by

$$f_\varepsilon(p) = E_{H_\varepsilon}(\omega^0 + p + \eta^\varepsilon(p)) \quad (p \in B_R) \ . \tag{67}$$

This step gives the finite dimensional reduction of the problem. The proofs of Theorems 20 and 21 can be completed as follows.

Proof of Theorem 20. Let $\bar{p} \in \mathbb{R}^3$ be a nondegenerate critical point of Γ and let $R > |\bar{p}|$. One can show that the function f_ε defined in (67) satisfies:

$$\nabla f_\varepsilon(p) = 2\varepsilon G(\varepsilon, p) \tag{68}$$

where

$$G(\varepsilon, p) = \int_{\mathbb{R}^2} H_1(\omega^0 + p + \eta^\varepsilon(p))(\omega^0 + \eta^\varepsilon(p))_x \wedge (\omega^0 + \eta^\varepsilon(p))_y \,.$$

By (66), one has that $G(0, p) = \nabla\Gamma(p)$ and, in addition, $\partial_i G_k(0, p) = \partial^2_{ik}\Gamma(p)$. Hence $G(0, \bar{p}) = 0$, because $\bar{p}$ is a stationary point of Γ. Moreover, since $\bar{p}$ is nondegenerate, $\nabla_p G(0, \bar{p})$ is invertible. Therefore by the implicit function theorem, there exists a neighborhood I of 0 (in $\mathbb{R}$) and a C^1 mapping $\varepsilon \mapsto p^\varepsilon \in \mathbb{R}^3$ defined on I, such that $p^0 = \bar{p}$ and $G(\varepsilon, p^\varepsilon) = 0$ for all $\varepsilon \in I$. Hence, by (67), (68) and by Lemma 13, we obtain that the function

$$\varepsilon \mapsto u^\varepsilon := \omega^0 + p^\varepsilon + \eta^\varepsilon(p^\varepsilon) \qquad (\varepsilon \in I)$$

defines a C^1 curve from I into $W^{1,3}$ of H_ε-bubbles, passing through $\omega^0 + \bar{p}$ when $\varepsilon = 0$. It remains to prove that the curve $\varepsilon \mapsto u^\varepsilon$ is of class C^1 from I into $C^{1,\alpha}(\mathbb{S}^2, \mathbb{R}^3)$. This can be obtained by a boot-strap argument. Indeed u^ε solves $\Delta u^\varepsilon = F^\varepsilon$ on $\mathbb{R}^2$, where $F^\varepsilon = 2H_\varepsilon(u^\varepsilon)u^\varepsilon_x \wedge u^\varepsilon_y$. Since $\varepsilon \mapsto u^\varepsilon$ is of class C^1 from I into $W^{1,3}$ we have that $\varepsilon \mapsto F^\varepsilon$ is of class C^1 from I into $L^{3/2}$. Now, regularity theory yields that the mapping $\varepsilon \mapsto u^\varepsilon$ turns out of class C^1 from I into $W^{2,3/2}$. This implies that $\varepsilon \mapsto du^\varepsilon$ is C^1 from I into L^6, by Sobolev embedding. Hence $\varepsilon \mapsto F^\varepsilon$ belongs to $C^1(I, L^3)$. Consequently, again by regularity theory, $\varepsilon \mapsto u^\varepsilon$ is of class C^1 from I into $W^{2,3}$. By the embedding of $W^{2,3}$ into $C^{1,\alpha}(\mathbb{S}^2, \mathbb{R}^3)$, the conclusion follows. Lastly, we point out that u^ε has no branch points because $u^\varepsilon \to \omega^0 + \bar{p}$ in $C^{1,\alpha}(\mathbb{S}^2, \mathbb{R}^3)$ as $\varepsilon \to 0$, and ω^0 is conformal on $\mathbb{R}^2$.

$\square$

Proof of Theorem 21. Since $\eta^\varepsilon(p)$ is of class C^1 with respect to the pair (ε, p), and $\eta^0(p) = 0$, we have that

$$\|\eta^\varepsilon(p)\|_{W^{1,3}} = O(\varepsilon) \text{ uniformly for } p \in B_R, \text{ as } \varepsilon \to 0\,. \tag{69}$$

Now we show that

$$f_\varepsilon(p) = E_{H_0}(\omega^0) + 2\varepsilon\Gamma(p) + O(\varepsilon^2) \text{ as } \varepsilon \to 0, \text{ uniformly for } p \in B_R\,. \tag{70}$$

Indeed, set

$$\begin{aligned} R^\varepsilon(p) &:= f_\varepsilon(p) - E_{H_0}(\omega^0) - 2\varepsilon\Gamma(p) \\ &= E_{H_0}(\omega^0 + \eta^\varepsilon(p)) - E_{H_0}(\omega^0) \\ &\quad +2\varepsilon\Big(V_{H_1}(\omega^0 + p + \eta^\varepsilon(p)) - V_{H_1}(\omega^0 + p)\Big)\,. \end{aligned}$$

Using $E'_{H_0}(\omega^0) = 0$ and the decomposition $V_{H_0}(u+v) = V_{H_0}(u)+V_{H_0}(v)+H_0\int_{\mathbb{R}^2} u \cdot v_x \wedge v_y + H_0\int_{\mathbb{R}^2} v \cdot u_x \wedge u_y$ we compute

$$\begin{aligned} E_{H_0}(\omega^0 + \eta^\varepsilon(p)) - E_{H_0}(\omega^0) &= E_{H_0}(\eta^\varepsilon(p)) + 2V_{H_0}(\eta^\varepsilon(p)) \\ &\quad +2H_0\int_{\mathbb{R}^2} \omega^0 \cdot \eta^\varepsilon(p)_x \wedge \eta^\varepsilon(p)_y \\ &= O(\|d\eta^\varepsilon(p)\|_3^2) \end{aligned}$$

Therefore, using also (69), we infer that

$$\begin{aligned} R^\varepsilon(p)\varepsilon^{-2} &= O(\|d\eta^\varepsilon(p)\|_3^2)\varepsilon^{-2} + 2\Big(V_{H_1}(\omega^0 + p + \eta^\varepsilon(p)) - V_{H_1}(\omega^0 + p)\Big)\varepsilon^{-1} \\ &= O(1) + 2(dV_{H_1}(\omega^0 + p)\eta^\varepsilon(p) + \|\eta^\varepsilon(p)\|_{W^{1,3}}o(1))\varepsilon^{-1} = O(1), \end{aligned}$$

and (70) follows. Now, let K be given according to the assumption and take $R > 0$ so large that $K \subset B_R$. The hypothesis on K and (70) imply that for $|\varepsilon|$ small, there exists $p_\varepsilon \in K$ such that $u^\varepsilon := \omega^0 + p_\varepsilon + \eta^\varepsilon(p_\varepsilon)$ is a stationary point for E_{H_ε} constrained to $\mathcal{Z}_\varepsilon^R$. According to Lemma 13, $E'_{H_\varepsilon}(u^\varepsilon) = 0$, namely u^ε is an H_ε-bubble. Moreover, $\Gamma(p_\varepsilon) \to \max_K \Gamma$ (or $\Gamma(p_\varepsilon) \to \min_K \Gamma$) as $\varepsilon \to 0$. To prove that $\|u^\varepsilon - (p_\varepsilon + \omega^0)\|_{C^{1,\alpha}(\mathbb{S}^2,\mathbb{R}^3)} \to 0$ as $\varepsilon \to 0$ one can follow a boot-strap argument, as in the last part of the proof of Theorem 20.

□

The assumptions on Γ in Theorems 20 and 21 can be made explicit in terms of H_1 when $|H_0|$ is large. In particular, as a first consequence of the above existence theorems we obtain the following result, which says that nondegenerate critical points as well as topologically stable extremal points of the perturbation term H_1 are concentration points of H_ε-bubbles, in the double limit $\varepsilon \to 0$ and $|H_0| \to \infty$.

THEOREM 22. *Assume that one of the following conditions is satisfied:*

(i) *there exists a nondegenerate stationary point $\bar{p} \in \mathbb{R}^3$ for H_1;*

(ii) *there exists a nonempty compact set $K \subset \mathbb{R}^3$ such that $\max_{p\in\partial K} H_1(p) < \max_{p\in K} H_1(p)$ or $\min_{p\in\partial K} H_1(p) > \min_{p\in K} H_1(p)$.*

Then, for every $H_0 \in \mathbb{R}$ with $|H_0|$ large, there exists $\varepsilon_{H_0} > 0$ such that for every $\varepsilon \in [-\varepsilon_{H_0}, \varepsilon_{H_0}]$ there is a smooth H_ε-bubble $u^{H_0,\varepsilon}$ without branch points. Moreover

$$\lim_{|H_0|\to\infty} \lim_{\varepsilon\to 0} \|u^{H_0,\varepsilon} - p_\varepsilon\|_{C^{1,\alpha}(\mathbb{S}^2,\mathbb{R}^3)} = 0$$

where $p_\varepsilon \equiv \bar{p}$ if (i) *holds, or $p_\varepsilon \in \mathbb{R}^3$ is such that $p_\varepsilon \in K$ and $H_1(p_\varepsilon) \to \max_K H_1$, or $H_1(p_\varepsilon) \to \min_K H_1$ if* (ii) *holds. In addition, under the condition* (i)*, the map $\varepsilon \mapsto u^{H_0,\varepsilon}$ defines a C^1 curve in $C^{1,\alpha}(\mathbb{S}^2, \mathbb{R}^3)$.*

As a further application of Theorem 21, we consider a perturbation H_1 having some decay at infinity.

THEOREM 23. *If* $H_1 \in L^1(\mathbb{R}^3) + L^2(\mathbb{R}^3)$*, then for* $|\varepsilon|$ *small enough there exist* $p_\varepsilon \in \mathbb{R}^3$ *and a smooth* H_ε*-bubble* u^ε*, without branch points, such that* $\|u^\varepsilon - (\omega^0 + p_\varepsilon)\|_{C^{1,\alpha}(\mathbb{S}^2,\mathbb{R}^3)} \to 0$ *as* $\varepsilon \to 0$*, and* (p_ε) *is uniformly bounded with respect to* ε.

We refer to [18] for the proofs of Theorems 22 and 23.

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