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ON A PROOF OF THE JSJ THEOREM

Abstract. In this article we expose a proof of the Canonical Decomposition Theorem of irreducible 3-manifolds along tori and annuli, also known as JSJ Theorem. This proof will be based on the ideas that S. Matveev used in his article [5] which we extend from the closed to the compact case. The result is equivalent to the one proved by W.D. Neumann and G.A. Swarup in [6]. Moreover, before doing that, we discuss the relations between parallelism and isotopy of two surfaces embedded in a 3-manifold and we establish a result that has been extensively used in many published proofs of the JSJ Theorem without explicit mentioning.

Riassunto. In questo articolo esporremo una dimostrazione del Teorema di Decomposizione Canonica di 3-varietà irriducibili lungo tori ed anelli, anche noto come Teorema JSJ. Questa dimostrazione è basata su alcune idee che S. Matveev ha usato nel suo articolo [5] e che estenderemo dal caso chiuso al caso compatto. Il risultato è equivalente a quello ottenuto da W.D. Neumann e G.A. Swarup in [6]. Inoltre, prima di dimostrare il teorema, discuteremo le relazioni che intercorrono fra parallelismo e isotopia di due superfici embedded in una 3-varietà. Otterremo un risultato che pur essendo stato usato estesamente in molte dimostrazioni già pubblicate del Teorema JSJ, non sembra esistere in letteratura.

In the first section of this work we recall some definition and study the relations between parallelism and isotopy of surfaces embedded in an irreducible 3-manifold. The second section contains two lemmas which will be useful in the last section of this work, where the JSJ Theorem is proved. In this final section we introduce the concepts of rough annulus and torus, which generalize a definition given by S. Matveev in [5] (we now know that S. Matveev proved the theorem in the general case using slightly different techniques, see [4]). Then, using properties of these surfaces, we prove the JSJ Theorem.

I wish to warmly thank Prof. Carlo Petronio for many helpful conversations and support during my work.

1. Parallelism and isotopy

We consider only 3-manifold which are compact and orientable, and very often we use the irreducibility hypothesis. Recall that a 3-*pair* is a polyhedral pair (M, T) where M

is a 3-manifold and T is a (not necessarily connected) surface contained in ∂M. We will say that a 3-*pair* (M, T) is *irreducible* if M is irreducible and T is incompressible in M; we will say that it is compact when both M and T are compact. Let us recall the definition of parallelism:

DEFINITION 1. *Given a* 3*-pair* (M, T)*, let* W *and* W' *be disjoint surfaces embedded in* M *whose boundaries are contained in* T*. We say that* W *and* W' *are* parallel *in* (M, T) *if there exists a* 3*-manifold* $Q \subset M$ *such that* $(Q, W \cup W')$ *is homeomorphic to* $(W \times I, W \times \partial I)$ *and* $\partial Q - (\mathring{W} \cup \mathring{W}') \subset T$*. Such a manifold* Q *is said to be a* parallelism *in* (M, T) *between* W *and* W'*. If* $T = \partial M$ *we simply call* Q *a parallelism in* M*. The surface* W *is said to be* T*-parallel in* M *if it is parallel in* (M, T) *to a surface contained in* T*. If* $T = \partial M$*, we say that* W *is parallel to the boundary in* M*.*

It is clear that parallelism implies isotopy. The converse is false, as one easily sees for instance in S^3, by taking the boundary of a regular neighborhood of a suitable two component link. But if we add the incompressibility hypothesis, we are able to obtain the following result:

THEOREM 1. *Let* F *and* G *be connected, incompressible, bilateral and disjoint surfaces properly embedded in a* 3*-pair* (K, T)*. If* F *and* G *are isotopic in* (K, T) *then they are parallel in* (K, T)*.*

As previously mentioned, this theorem has been extensively used in many proofs of the JSJ Theorem but without an explicit mentioning, this is the reason why we concentrate on it here and we prove it. We deduce it by starting from the articles of F. Waldhausen [8, 7], and using a lemma whose proof is contained in [1] (Jaco and Shalen's memoir containing one of the first proofs of the JSJ Theorem). One can also prove the same result by using Waldhausen's Theorem of h-cobordism. To be able to apply such a theorem one could carefully use the propositions and the techniques exposed in F. Laudenbach's book [3], Chapter II, paragraph IV (see in particular Corollary 4.2). The first of the lemma we will use is Lemma 1.1 of [8]. Let us recall it:

LEMMA 1. *Let* K *be a (not necessarily finite) polyhedron, and let* L *be a (not necessarily connected) subpolyhedron of* K*. Suppose we have an embedding of* $L \times I$ *in* K*, such that* $L = L \times \{1/2\}$*, the subspace* $L \times I$ *is closed in* K *and* $L \times I$ *is a neighborhood of* $L \times \{t\}$ *for some* $t \in \mathring{I}$*. Moreover suppose that:*

$$\ker(\pi_j(L) \to \pi_j(K)) = 0, \;\; for\ j = 1, 2\ and\ \pi_2(K - L) = \pi_3(K) = 0.$$

Let Q *be an orientable and compact* 3*-manifold and let* $f : Q \to K$ *be a map. Then there exists a map* g*, homotopic to* f *and enjoying the following properties:*

1. g *is transversal to* L*;*

2. $g^{-1}(L)$ *is a compact surface, properly embedded and incompressible in* Q*.*

Moreover if $f|_{\partial Q}$ is transversal to L, one can suppose that the homotopy between g and f is constant on ∂Q.

We will use this lemma mostly when K is a compact, irreducible and sufficiently large 3-manifold, L is a bilateral and incompressible surface embedded in K, the manifold Q is homeomorphic to $F \times I$ where F is a surface and f is a homotopy between two embeddings of F in K. It can be checked that if $f(F \times \{0\})$ is incompressible in K, the hypotheses of the lemma are verified.

Another result we will need is the following (the proof of which is contained in the proof of Lemma V.4.6 of [1]).

LEMMA 2. *Let (M, T) be a compact and irreducible 3-pair and let (M', T') be a 3-pair satisfying the following conditions:*

1. *(M', T') is embedded in (M, T) (a pair);*
2. *$\mathrm{Fr_M M'}$ is incompressible in M;*
3. *no component of $\mathrm{Fr_M M'}$ is T-parallel in M.*

Let $(W, \partial W) \subset (M, T)$ be a connected, bilateral and incompressible surface in M such that $W \cap M' = \emptyset$. Suppose that the inclusion $i : (W, \partial W) \to (M, T)$ is homotopic as a map of pairs to a map $i' : (W, \partial W) \to (M, T)$ such that $i'(W) \subset \mathring{M}'$. Then there exists a homotopy of pairs $F : (W \times I, \partial W \times I) \to (M, T)$ between i and i' such that $F^{-1}(\mathrm{Fr_M M'})$ is a union of horizontal leaves (surfaces of the type $W \times \{t\}$ in $W \times I$).

Another useful result is Lemma 3.4 of [7]:

LEMMA 3. *Let F be an orientable surface. Let $S \subset F \times I$ be a properly embedded surface whose components are of the two following types:*

1. *discs intersecting $\partial F \times I$ only in two vertical arcs;*
2. *incompressible annuli with one boundary component in $\mathring{F} \times \{0\}$ and the other one in $\mathring{F} \times \{1\}$.*

Then there exists an isotopy, constant on $(F \times \{0\}) \cup (\partial F \times I)$, which puts S in a vertical position.

Finally, let us come to the key tool of our proof: Lemma 5.1 of [7].

LEMMA 4. *Let M be an irreducible 3-manifold. Let F be a closed, incompressible boundary component of M. In $\partial M - F$, let F' be an incompressible surface which need neither be closed nor compact. Suppose: if k is any closed curve in F, then some non-zero multiple of k is homotopic to a curve in F'. Then, M is homeomorphic to $F \times I$.*

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Let us prove it first when F and G are closed. In this case we will apply Lemma 1 with K the ambient manifold and $L = F \cup G$. Moreover we call $Q = F \times I$ and f the isotopy between F and G (then $f(F \times \{0\}) = F$). We modify now f to obtain a new isotopy $\tilde{f}$ as follows. Using regular neighborhoods of F and G we can first extend the isotopy f to $F \times [-1, 2]$ and then renormalize it to $F \times [0, 1]$, thus assuming that $F = f(F \times 1/3)$, $G = f(G \times [2/3])$ and that $f|_{F\times([0,1/3]\cup[2/3,1])}$ is an embedding. The isotopy $\tilde{f}$ thus obtained, is such that $\tilde{f}|_{\partial Q}$ is transversal to L. Applying Lemma 1 we obtain that, up to homotopy fixed on ∂Q, the preimage $\tilde{f}^{-1}(L)$ is an incompressible surface in Q, and so by Proposition 3.1 of [8], up to isotopy in Q, it is a union of "horizontal" leaves in Q (i.e. surfaces parallel in Q to $F \times \{0\}$), and in particular parallel between themselves. We can choose two of them, F' and G', which are consecutive and such that F' is a preimage of F and G' of G. Let us now call Q' the manifold contained in Q between F' and G'. Such a Q' is homeomorphic to $F \times [0, 1]$ and so it is a parallelism in Q between F' and G'. The map $\tilde{f}|_{Q'}$ is a homotopy between $(F, \partial F = \emptyset)$ and $(G, \partial G = \emptyset)$ in (K, T) such that ${\tilde{f}|_{Q'}}^{-1}(L) = F' \cup G'$ (there are no other preimages of L in Q'). Now cut (K, T) along $(L, \partial L) = (F \cup G, \emptyset)$. Since L is incompressible in K, we obtain an irreducible and compact pair. Let us call it (K', T'). Now we can apply Lemma 4 to the component of (K', T') containing the image of $\tilde{f}|_{Q'}$; in that component that map is the homotopy between two incompressible component of $\partial K'$; it is easily checked that the hypotheses of the lemma are verified in this case. So K' is homeomorphic to $F \times I$ and its inclusion in K is the parallelism we were looking for.

When F and G have a non-empty boundary, let us call $U(F)$, $U(G)$, $U(L) = U(F) \cup U(G)$ regular neighborhoods in K respectively of F, G, and L. Construct an isotopy $\tilde{f}$ as follows. Using a regular neighborhood of F in K we can first extend the isotopy f to $F \times [0, 2]$ and then renormalize it to $F \times [0, 1]$, thus assuming that $F = f(F \times \{1/2\})$ and $G = f(F \times \{1\})$, and that $f|_{F\times[0,1/2]}$ is an embedding. The map $\tilde{f}$ thus obtained, is an isotopy between a surface parallel to $(F, \partial F)$ in (K, T) and $G \subset U(L)$. It is simple to check that the hypotheses of Lemma 2 are verified if W is a component of $\mathrm{Fr_K}U(G)$. So there exist two consecutive horizontal leaves of $\tilde{f}^{-1}(\mathrm{Fr_M}U(L))$, one corresponding to a component F_1 of $\mathrm{Fr_K}U(F)$, and the other one to a component G_1 of $\mathrm{Fr_K}U(G)$. If we restrict $\tilde{f}$ to the cylinder contained between these two leaves we obtain a homotopy between $(F_1, \partial F_1)$ and $(G_1, \partial G_1)$ which does not intersect $\mathring{U}(L)$. So we have a homotopy of pairs between $(F_1, \partial F_1)$ and $(G_1, \partial G_1)$ whose image is contained in a component $(K_1, K_1 \cap T)$ of $(\overline{K - U(L)}, T \cap \overline{K - U(L)})$. Note now that the components of $T \cap \overline{K - U(L)}$ which contain ∂F_1 are (by the incompressibility of F_1) incompressible annuli (they are surfaces with 2 homotopic boundary components).

Now we reduce to the preceding case by constructing the manifold K_2 obtained by glueing 2 copies of K_1 along the annuli we already mentioned. The manifold K_2 is compact and irreducible since these annuli are incompressible in K_1. In K_2 the doubles F_2 and G_2 of F_1 and G_1 respectively, are parallel. To obtain a parallelism of

F and G in K we eventually use Lemma 3. By applying that lemma to the parallelism Q (homeomorphic to $F_2 \times I$) between F_2 and G_2 in K_2, with S the union of the incompressible annuli along which we glued, we can suppose S vertical in Q. So we can cut Q along S obtaining that a component of the what is left after cutting is a parallelism between F_1 and G_1 in K. Finally F and G are parallel in K since they are parallel by construction to respectively F_1 and G_1, and $F_1,\ F,\ G_1,\ G$ are pairwise disjoint (if one of F or G is contained in the parallelism between F' and G', recall that parallelism is a transitive relation between incompressible disjoint surfaces properly embedded in an irreducible 3-manifold as shown in [1]; this is an easy consequence of Proposition 3.1 of [8]).

□

2. The JSJ Theorem

Let us recall some definitions.

DEFINITION 2. *A pair (Σ, Φ) is called an I-pair if Σ is an I-bundle over a (possibly non orientable) compact surface and Φ is the corresponding ∂I-bundle. The pair is called product or twisted depending on whether the bundle is trivial or not.*

A pair (Σ, Φ) is called an S^1-pair if Σ can be given the structure of a Seifert fibered manifold so that Φ is saturated. Such a structure is called a Seifert structure of the pair.

A 3-pair is said to be a Seifert pair if it is an S^1-pair or an I-pair.

DEFINITION 3. *A 3-pair (M, T) is said to be* simple *if each incompressible torus or annulus $W \subset M$ such that $\partial W \subset \mathring{T}$ is T-parallel in M or parallel in (M, T) to a component of $(\partial M) - T$. A 3-manifold is said to be simple if $(M, \emptyset)$ is a simple pair.*

We will prove the following theorem:

THEOREM 2 (JSJ THEOREM). *Let M be a compact, orientable, irreducible and boundary-incompressible 3-manifold. Then there exists a unique (up to isotopy in M) finite set F of incompressible, pairwise disjoint tori and annuli properly embedded in M, which satisfies the following properties:*

1. *no component of F is parallel to the boundary in M;*
2. *each component of $(\sigma_F(M), \sigma_{\partial F}(\partial M))$ (the pair obtained by cutting M along F) is a Seifert pair or a simple pair;*
3. *F is minimal with respect to inclusion among all the sets which satisfy 1) and 2).*

To prove this theorem, we will use some of the ideas of the proof of S. Matveev given in [5]. The difference here is that we do not require the boundary to consist of tori. Let us recall the following definition:

DEFINITION 4. *A surface S, properly embedded in a 3-pair (M, T), is said to be* essential *if it satisfies the following conditions:*

1. *S is incompressible in M;*
2. *S is not parallel in (M, T) to a surface contained in T or in $(\partial M) - T$;*
3. *there is no disc D embedded in M, such that ∂D is the union of two arcs α and β with $\alpha \subset S$ and $\beta \subset T$ and α is not homotopic in $(S, \partial S)$ to an arc contained in ∂S.*

Here are some facts we will use:

LEMMA 5. *If M is an irreducible 3-manifold and L is a torus embedded and compressible in M, then L bounds a solid torus in M or it is contained in a ball.*

LEMMA 6. *If (M, T) is an irreducible I-pair and $(A, \partial A)$ is an essential annulus embedded in (M, T), then $(A, \partial A)$ is isotopic to an annulus which is saturated in the I-fibration of (M, T).*

Proof. If (M, T) is a product I-pair, then this lemma is a consequence of Lemma 3. Otherwise M is an I-bundle over a compact, non-orientable surface Σ which is embedded and not bilateral in M. Note that M has a horizontal foliation whose leaves, besides Σ, are two-fold cover of Σ and are parallel in $(M, \partial M - T)$ to T; these surfaces are homeomorphic to a two-fold covering Σ_2 of Σ.

By a simple Morse theory argument, it is easy to see that we can isotope A so to put it in a "quasi-vertical position", i.e. such that A is transversal to the already cited foliation of M. Cutting M along Σ, we obtain a product I-bundle $(M', T \cup T')$ over Σ_2 which contains two incompressible, disjoint annuli A_1 and A_2 obtained by cutting A. It is easy to show that, since these two annuli are disjoint and incompressible, we can isotope the surface $A_1 \cup A_2$ so to put it in a vertical position respect to the I-fibration of M', and by an isotopy which is constant on $(A_1 \cup A_2) \cap T'$. Then we can lift this isotopy to M obtaining an isotopy which puts A in a vertical position.

□

Let us consider first the following:

THEOREM 3. *Let M be an irreducible and boundary-irreducible, compact and orientable 3-manifold. It is possible to find a set F of essential, pairwise disjoint and non-parallel tori and annuli such that each component of the pair $(\sigma_F(M), \sigma_{\partial F}(\partial M))$ is either a Seifert pair or a simple pair.*

Proof. Choose a set F of essential, pairwise disjoint and non-parallel tori and annuli which is maximal with respect to inclusion. Using the Haken-Kneser finiteness theorem, we know that it is finite; call $M_i,\ i = 1, .., n$ the components of $\sigma_F(M)$. By maximality of F all the pairs $(M_i, \sigma_{\partial F}(\partial M) \cap M_i)$ are simple.

□

Now by Theorem 3, we can choose a set F of essential tori and annuli which cuts M in pairs which are either Seifert or simple, and which is minimal with respect to inclusion among all the sets which have this property. From now on we will call $(M_i, \partial M \cap M_i)$ the components of the pair obtained by cutting $(M, \partial M)$ along F. We set $\partial_0 M_i = M_i \cap \partial M$ and $\partial_1 M_i = \partial M_i \cap F$.

To proceed, we will need the following fact (see [2]):

THEOREM 4. *If S is a Seifert 3-manifold with non-empty boundary, then the following holds:*

1. *an incompressible torus embedded in S is isotopic to one which is saturated respect to the Seifert fibration of S;*
2. *if S is not homeomorphic to $S^1 \times S^1 \times I$ or to the twisted I-bundle over the Klein bottle, then each incompressible and non boundary parallel annulus in S is isotopic to a saturated one.*

We are now ready to prove a first theorem which is useful to distinguish the simple pairs from the Seifert pairs in the decomposition along F.

THEOREM 5. *If there exists an incompressible annulus A, properly embedded in $(M_i, \partial_1 M_i)$ and not parallel in $(M_i, \partial_1 M_i)$ to an annulus contained in $\partial_1 M_i$, then $(M_i, \partial_0 M_i)$ is an S^1-pair.*

Proof. Note that the components of $\partial_1 M_i$ are annuli and tori. We distinguish three cases:

1. both the components of ∂A are contained in annular components of $\partial_1 M_i$;
2. a component of ∂A is contained in an annular component of $\partial_1 M_i$ and the other one in a toric component;
3. both the components of ∂A are contained in toric components of $\partial_1 M_i$.

Case 1: Let S_1 and S_2 be the two annuli connected by A. We can distinguish two subcases:

Case 1.a: $S_1 \neq S_2$. Let N be a regular neighborhood of $A \cup S_1 \cup S_2$ in M_i, and call A_1 and A_2 the two annuli constituting $\mathrm{Fr}_{\mathrm{M_i}} N$ (see figure 1). If both A_1 and A_2 (which are properly embedded in the pair $(M_i, \partial_0 M_i)$) are not essential in $(M_i, \partial_0 M_i)$, then it is easy to see that, since they are parallel to annuli contained either in $\partial_0 M_i$ or in $\partial_1 M_i$, the manifold M_i is a solid torus and $\partial_0 M_i$ is made of at most 4 annuli. Otherwise, suppose that A_1 is essential, therefore $(M_i, \partial_0 M_i)$ is not a simple pair. If it is an S^1-pair we are finished; otherwise it is an I-pair, and we will now show that it also has a structure of S^1-pair. If it is a product I-pair, $(M_i, \partial_0 M_i)$ is an I-bundle over an orientable, compact surface X and the projection of A over X is a homotopy between two components (one corresponding to $A \cap S_1$ and the other one to $A \cap S_2$) of ∂X. So X is homeomorphic to an annulus, M_i is a solid torus and the pair is also an S^1-pair. If

$(M_i, \partial_0 M_i)$ is a twisted I-pair, the base of the I-fibration is a non-orientable compact surface X and the projection of A is again a homotopy between two components of ∂X, but there exists no compact non-orientable surface having two different boundary components which are homotopic to each other, so this case cannot happen.

Case 1.b: $S_1 = S_2$**.** In this case, consider a regular neighborhood N of $A \cup S_1$ in M_i. The surface $\mathrm{Fr}_{\mathrm{M_i}} N$ can consist either of:

1. a union of an incompressible annulus A_1 and a torus T
2. an incompressible annulus.

See figure 2.

Case 1.b.1: If T is not essential, it is either parallel to a component of $\partial_0 M_i$ or to one of $\partial_1 M_i$, or it is compressible in M_i. In the first two cases then $(M_i, \partial_0 M_i)$ is an S^1-pair, and we are finished. In fact, if A_1 is essential then $(M_i, \partial_0 M_i)$ is a Seifert pair (it cannot be simple) and actually an S^1-pair since there is a toric component of $\partial_0 M_i$ or of $\partial_1 M_i$. If A_1 is not essential, then M_i is homeomorphic to $S^1 \times S^1 \times I$ and it is easy to check that $(M_i, \partial_0 M_i)$ is an S^1-pair.

If T is compressible in M_i, then the boundary of the disc compressing it is not homotopic to the boundary of A (since A is incompressible). Moreover, we further can distinguish two cases: A_1 is either parallel to an annulus contained in $\partial_0 M_i$ or in $\partial_1 M_i$, and in this case it is easy to check that M_i is a solid torus and $(M_i, \partial_0 M_i)$ is an S^1-pair, or A_1 is essential. In the latter case $(M_i, \partial_0 M_i)$ is a Seifert pair (it is not simple). We will now show that more precisely it has a structure of S^1-pair. Suppose that it is an I-pair. By Lemma 6, we can suppose that A_1 is saturated in the I-fibration of M_i. Cutting $(M_i, \partial_0 M_i)$ along A_1, we get two I-pairs, one of which is homeomorphic to $(S^1 \times I \times I, S^1 \times I \times \{0, 1\})$ (note that $\partial S_1 \cup \partial A_1$ is the boundary of two annuli in $\partial_0 M_i$); this is absurd however, since A_1 would be parallel in $(M_i, \partial_0 M_i)$ to S_1.

The only case remaining therefore is the case in which T is essential; $(M_i, \partial_0 M_i)$ is then a Seifert pair. It can be checked that if an I-pair (K, L) contains an incompressible torus T which is not parallel to a torus contained in L nor to one contained in $\partial K - L$, then (K, L) is also an S^1-pair. In fact, if (K, L) is a product I-pair, by Proposition 3.1 of [7] we obtain that it is homeomorphic to $(S^1 \times S^1 \times I, S^1 \times S^1 \times \{0, 1\})$. If it is a twisted I-pair, we can reduce ourselves to the preceding case by considering the two fold cover of (K, L) which is a product I-pair; then, using Lemma 4, we obtain that the twisted I-pair over a Klein bottle contains no essential tori.

Case 1.b.2: In this case $\mathrm{Fr}_{\mathrm{M_i}} N$ consists of a single incompressible annulus A_1. If A_1 is parallel to an annulus contained either in $\partial_0 M_i$ or in $\partial_1 M_i$ then M_i is homeomorphic to $S^1 \tilde{\times} M$ (where M is a Moebius strip), and it easy to check that $(M_i, \partial_0 M_i)$ is an S^1-pair. Otherwise if A_1 is essential, $(M_i, \partial_0 M_i)$ is a Seifert pair. If it were an I-pair, by Lemma 6 we could suppose A_1 saturated with respect to the I-fibration of the pair. But then, cutting $(M_i, \partial_0 M_i)$ along A_1 we would obtain two I-pairs, one of which homeomorphic to the product I-pair over an annulus (since $\partial A_1 \cup \partial S_0$ is the boundary of two annuli in $\partial_0 M_i$), so A_1 would be parallel to an annulus contained in $\partial_1 M_i$. But this is absurd since we supposed A_1 essential.

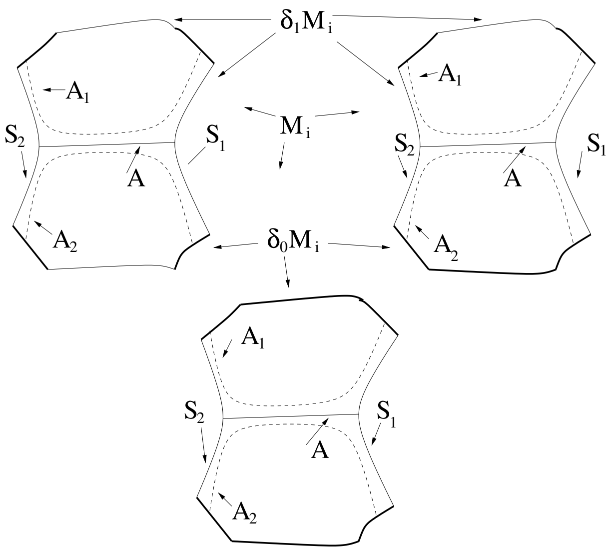

Figure 1: Here we draw some transversal section to $S_1 \cup A \cup S_2$, and distinguish three cases which are possible when $S_1 \neq S_2$. The thickened parts correspond to $\partial_0 M_i$.

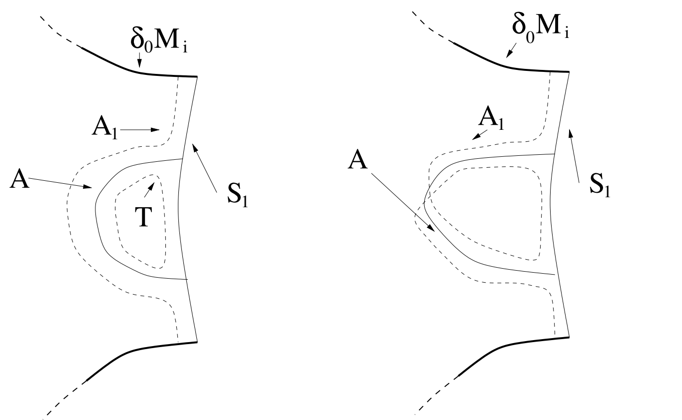

Figure 2: Here we draw the transversal sections to $A \cup S_1$ in the two subcases distinguished in case 1.b.

Case 2: Let us call T and S respectively the torus and the annulus which contain ∂A. Let N be a regular neighborhood of $A \cup T \cup S$ in M_i. The surface $F = \mathrm{Fr}_{\mathrm{M_i}} N$ is an incompressible annulus embedded in $(M_i, \partial_0 M_i)$. If F is not essential then it is easy to see that M_i is homeomorphic to $S^1 \times S^1 \times I$ and that $(M_i, \partial_0 M_i)$ is an S^1-pair. If F is essential, we arrive to the same conclusion because $(M_i, \partial_0 M_i)$ is not simple and there exists a toric component of $\partial_1 M_i$ (there are no I-pairs (K, L) such that $\partial K - L$ contains a torus).

Case 3: Let us call T_0 and T_1 the tori which contain ∂A. As in the first case, we can distinguish two subcases.

Case 3.a: $T_0 \neq T_1$**.** Let N be a regular neighborhood in M_i of $A \cup T_0 \cup T_1$, and let $F = \mathrm{Fr}_{\mathrm{M_i}} N$ (F being a torus). If F is parallel to a torus contained either in $\partial_0 M_i$ or in $\partial_1 M_i$, then M_i is homeomorphic to $S^1 \times X$ where X is a disc with two holes, and $(M_i, \partial_0 M_i)$ is an S^1-pair. If F is compressible, the natural Seifert fibration of N (in which the fiber is homotopic to ∂A) extends to a Seifert fibration of $N \cup R$ where R is the solid torus bounded by F (note that F is not contained in a ball since otherwise T_0 could not be incompressible), since the meridinal disc of R has a boundary which is not homotopic to ∂A. Otherwise F is essential and then $(M_i, \partial_0 M_i)$ is a Seifert pair and in particular an S^1-pair since $\partial_1 M_i$ contains a torus.

Case 3.b: $T_0 = T_1$**.** In this case we can distinguish two subcases: F (defined as above) can consist of one or two tori.

Case 3.b.1: F **consists of two tori.** If both the tori constituting F are parallel to tori contained either in $\partial_0 M_i$ or in $\partial_1 M_i$, then M_i is homeomorphic to $S^1 \times X$, where X is a disc with two holes, and $(M_i, \partial_0 M_i)$ is an S^1-pair. If one of these two tori is compressible, it bounds a solid torus R in M_i (also in this case it cannot be contained in a ball since otherwise T_0 could not be incompressible, moreover, for the same reason, T_0 is not contained in R), and it is easy to see that the Seifert fibration of N extends to $N \cup R$, since A is incompressible and therefore the boundary of the meridinal disc of R cannot be homotopic to ∂A.

There remains the case where at least one of those two tori is essential: in this case, $(M_i, \partial_0 M_i)$ is an S^1-pair since it is not simple and $\partial_1 M_i$ contains a torus.

Case 3.b.2: F **consists of one torus.** In this case, by distinguishing the analogous subcases (F is essential or not) it can be checked that $(M_i, \partial_0 M_i)$ is an S^1-pair.

□

Before proving another lemma useful to distinguish the I-pairs from the S^1 and simple pairs, we give the following definition.

DEFINITION 5. *An essential square embedded in a pair* $(M_i, \partial_0 M_i)$*, is a disc* D *embedded in* M_i *such that:*

1. $\partial D \subset M_i$*;*
2. $\partial D \cap \partial_0 N =: \partial_0 D$ *consists of two disjoint arcs;*
3. $(D, \partial_0 D, \partial_1 D)$ *is not isotopic in* $(N, \partial_0 N, \partial_1 N)$ *to a disc contained either in* $\partial_0 N$ *or in* $\partial_1 N$*; here, of course,* $\partial_1 D = \partial D - \partial_0 D$.

The next theorem is the analogous for I-pairs of Theorem 5:

THEOREM 6. *If there exists an essential square D embedded in $(M_i, \partial_0 M_i)$ then $(M_i, \partial_0 M_i)$ is an I-pair.*

Proof. We can distinguish two cases:

1. the two arcs of $\partial_1 D$ are contained in two different annuli A_1 and A_2 of $\partial_1 M_i$;

2. the two arcs of $\partial_1 D$ are contained in the same annulus A_1 of $\partial_1 M_i$.

Case 1: Let N be a regular neighborhood in M_i of $A_1 \cup A_2 \cup D$. The surface $\mathrm{Fr}_{\mathrm{M_i}} N$ is an annulus S embedded in $(M_i, \partial_0 M_i)$. If S is compressible, then $(M_i, \partial_0 M_i)$ is a product I-pair over an annulus. If S is parallel to an annulus contained in $\partial_1 M_i$, then $(M_i, \partial_0 M_i)$ is a product I-pair over a disc with two holes (see figure 3). Note that S cannot be parallel to an annulus contained in $\partial_0 M_i$ since otherwise an arc $\alpha \subset A_1$ would exist whose endpoints would be contained in different components of ∂A_1, and which would be parallel to an arc contained in $\partial_0 M_i$. But then, using the irreducibility of M_i and the incompressibility of $\partial_0 M_i$, we would obtain that A_1 also is parallel to an annulus contained in $\partial_0 M_i$, and this is absurd since $\partial_1 M_i$ has at least two components (A_1 and A_2). Hence only the case where S is essential remains. Then the pair $(M_i, \partial_0 M_i)$ is a Seifert pair (it is not simple). It cannot be an S^1-pair since otherwise we could isotope S so to make it saturated in a Seifert fibration of the pair and then also N would be. But this is not possible since $N \cap \partial_0 M_i$ cannot be saturated in any Seifert fibration of M_i, since it has a negative Euler characteristic. So $(M_i, \partial_0 M_i)$ is an I-pair.

Case 2: Let N be defined as above. The surface $S = \mathrm{Fr}_{\mathrm{M_i}} N$ can consist of one or two annuli. In the latter case, both the annuli, denoted S_1 and S_2, must be incompressible given the hypothesis that D is essential. For the same reason explained in the preceding case, if one of the annuli S_1 and S_2 were parallel to an annulus contained in $\partial_0 M_i$, then A_1 would also be, and so D would not be essential. On the contrary if both of them are parallel to an annulus contained in $\partial_1 M_i$, then $(M_i, \partial_0 M_i)$ is a product I-pair over a disc with two holes. If both S_1 and S_2 are essential, then $(M_i, \partial_0 M_i)$ is a Seifert pair and by the same reason explained in the preceding case, it is not an S^1-pair.

Finally, suppose that $\mathrm{Fr}_{\mathrm{M_i}} N$ consists of a single annulus S. As in the case seen above, S cannot be parallel to an annulus contained in $\partial_0 M_i$. If it is parallel to an annulus contained in $\partial_1 M_i$, then $(M_i, \partial_0 M_i)$ is an I-pair over a Moebius strip with a hole. Otherwise S is essential and the pair $(M_i, \partial_0 M_i)$ is a Seifert pair. It cannot be an S^1-pair since $N \cap \partial_0 M_i$ cannot be saturated in any Seifert fibration of M_i.

□

3. Rough annuli and tori

Now we can define the notion of rough annulus or torus, which generalizes that of rough torus given by S. Matveev in [5].

DEFINITION 6. *Let $(M, \partial M)$ be an irreducible, compact 3-pair. An incompressible surface $(S, \partial S)$ which can be a torus or an annulus embedded in $(M, \partial M)$, not parallel to the boundary, is said to be* thin *if there exists a Seifert pair (N, T) embedded in $(M, \partial M)$ such that:*

1. *no component of* $\mathrm{Fr_M} N$ *is compressible or parallel to the boundary in M;*
2. *$(S, \partial S) \subset (N, T)$ and $(S, \partial S)$ is saturated in a Seifert structure of (N, T);*
3. *$(S, \partial S)$ is not parallel in (N, T) neither to a surface contained in* $\mathrm{Fr_M} N$ *nor to one contained in ∂T.*

If $(S, \partial S)$ is not thin we will call it rough.

So a rough torus or annulus is basically one which cannot be contained in an essential way in any Seifert pair embedded in $(M, \partial M)$. Rough tori and annuli have the following important property:

THEOREM 7. *Let $(S, \partial S)$ be a rough torus or annulus, and let $(L, \partial L)$ be an incompressible torus or annulus. Then it is possible to isotope $(S, \partial S)$ off $(L, \partial L)$.*

Proof. We can distinguish two cases:

Case 1: $(S, \partial S)$ and $(L, \partial L)$ are two annuli. Up to isotopy, we can suppose that S and L intersect only in a finite number of disjoint arcs or closed curves. Moreover, if between all these positions we choose one in which the number of components of $S \cap L$ is minimal, then it is easy to see there are no arc whose endpoints are contained in the same boundary component of either S or L, and that if there is a closed curve then there is no arc and viceversa. So we can distinguish two subcases (see figure 4):

Case 1.a: $\partial S \cap \partial L \neq \emptyset$. Let $(R, R \cap \partial M)$ be a regular neighborhood of $(L \cup S, \partial S \cup \partial L)$ in $(M, \partial M)$. The pair $(R, R \cap \partial M)$ is an I-pair embedded in $(M, \partial M)$ (the fibers are isotopic to the arcs composing $L \cap S$), and it can easily be checked that no component of $\mathrm{Fr_M} R$ is compressible (by minimality of the number of arcs in $L \cap S$) or parallel to a surface contained in ∂M, otherwise S also would be; in fact, in this case S would contain an arc parallel to an arc contained in ∂M and since M is irreducible and boundary-incompressible, it is easily checked that S would be parallel to an annulus contained in ∂M. So, since S is saturated in $(R, R \cap \partial M)$ and it is rough, it can be isotoped on $\mathrm{Fr_M} R$ and so it can be isotoped away from L (which is embedded in $Int(R)$).

Case 1.b: $\partial S \cap \partial L = \emptyset$. Define R as above. The surface $\mathrm{Fr_M} R$ consists of annuli and tori; these tori can be compressible or not, parallel to the boundary in M or not. If a torus T of $\mathrm{Fr_M} R$ is parallel to the boundary in M, we extend R by attaching to it the parallelism between T and a torus in ∂M. If T is compressible, the boundary of the disc compressing T (note that T cannot be contained in a ball since S is incompressible) is not homotopic to ∂S (by incompressibility of S), so the natural Seifert fibration of R extends to the solid torus bounded by T in M. Moreover $\mathrm{Fr_M} R$ can contain annuli which are parallel to annuli contained in ∂M; in this case, as in the previous case, we

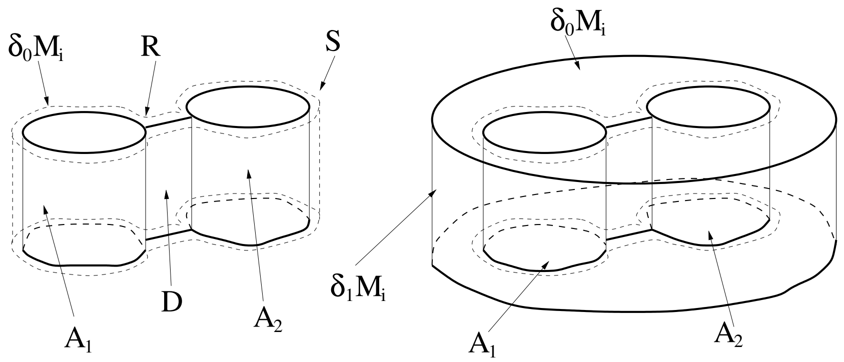

Figure 3: S can be either essential or parallel to $\partial_1 M_i$ in $(M_i, \partial_0 M_i)$.

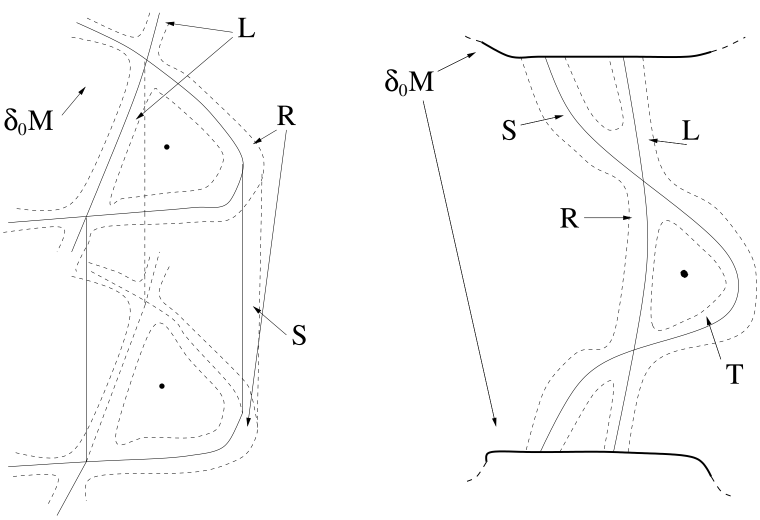

Figure 4: Here we draw a scheme of the two possibilities distinguished in the first case of the proof of Theorem 7.

attach the parallelisms to R. So, modifying R, by glueing some solid torus or some parallelism to $\mathrm{Fr_M} R$, we can construct an S^1-pair $(R, R \cap \partial M)$ embedded in $(M, \partial M)$ and such that each component of $\mathrm{Fr_M} R$ is incompressible in M and not parallel to a surface contained in ∂M. Moreover, $(S, \partial S)$ is saturated in the Seifert fibration of the pair, so since it is rough, it can be isotoped on $\mathrm{Fr_M} R'$, away from L.

Case 2: At least one of the two surfaces $(S, \partial S)$ and $(L, \partial L)$ is a torus. In this case we can proceed in a perfectly analogous way to that used in Case 1.b.

□

REMARK 1. The same proof shows that given a finite set of pairwise disjoint rough annuli and tori, and a family of pairwise disjoint incompressible annuli and tori, it is possible to isotope the first family away from the second family.

Let us now give the following definition:

DEFINITION 7. *A finite family F of rough annuli and tori which are pairwise disjoint, is* maximal *if any other rough torus or annulus disjoint from each element of F is parallel in $(M, \partial M)$ to an element of F.*

Another important step is the following:

THEOREM 8. *Let $(M, \partial M)$ be an irreducible, compact 3-pair. Then there exists, and it is unique up to isotopy in $(M, \partial M)$, a maximal family of pairwise disjoint rough tori and annuli such that its elements are pairwise non-parallel in $(M, \partial M)$.*

Proof. Take a maximal family of pairwise disjoint and non-parallel rough tori and annuli embedded in M; it is finite by the Haken-Kneser finiteness theorem. To prove its uniqueness, use Remark 1. In fact, given two maximal family's $T_1, ..T_n$ and $T'_1, ..T'_m$ of pairwise disjoint and non-parallel rough tori and annuli, we can isotope these two family's so that their intersection is empty. So by maximality, each T'_i is parallel to a T_j for some $j \leq n$ and viceversa. So the two family's are isotopic.

□

We are now ready to prove the JSJ decomposition theorem. The strategy we will use will consist in proving that a minimal splitting family (whose existence is given by Theorem 3) is isotopic to a maximal family of pairwise disjoint and non-parallel rough tori and annuli; then, using Theorem 8, we obtain the uniqueness of the splitting family.

Proof. Let $S = \{S_1, S_2, ..., S_n\}$ be a family of incompressible tori and annuli embedded in $(M, \partial M)$, minimal with respect to inclusion between all the family's of incompressible tori and annuli cutting $(M, \partial M)$ in pairs $(M_i, \partial_0 M_i)$ which are either Seifert or simple. Let $F = \{F_1, F_2, ..., F_m\}$ be a maximal family of pairwise non-parallel and disjoint rough tori and annuli embedded in $(M, \partial M)$. By Remark 1 we can suppose that the two family's are disjoint and therefore that each F_i is contained in one pair $(M_j, \partial_0 M_j)$ which can be either simple or Seifert.

If $(M_j, \partial_0 M_j)$ is simple, according to the definition of simple pair, F_i can be isotoped on $\partial_1 M_j$ (not on $\partial_0 M_j$ since F_i is rough), so it is parallel to a certain S_t for some $t \leq n$. Suppose now that $(M_j, \partial_0 M_j)$ is a Seifert pair. We claim that, up to isotopy, F_i is saturated in one of the Seifert fibrations of $(M_j, \partial_0 M_j)$.

To prove the claim, first note that if $\partial M_j = \emptyset$, then M_j is a closed component of M and it is a closed Seifert 3-manifold, so it cannot contain rough annuli. If it contains a saturated, incompressible torus L, we can suppose it is disjoint from the rough torus F_i. Then, by Theorem 4, F_i is isotopic to a saturated torus in the manifold obtained by cutting M_i along L, and therefore to a saturated torus in M_i and this is absurd. If M_i does not contain any saturated incompressible torus, then it can be checked that it cannot contain rough tori. So $\partial M_j \neq \emptyset$.

Now let us suppose that $(M_j, \partial_0 M_j)$ is an S^1-pair. By Theorem 4, if F_i is a torus, then it is isotopic to a torus which is saturated in a Seifert fibration of the pair. Otherwise it is an annulus, and it is easy to examine the two particular cases of Theorem 4, and in the other cases, by applying the same theorem, one concludes that the annulus F_i is isotopic to an annulus which is saturated in a Seifert structure of the pair $(M_j, \partial_0 M_j)$.

If $(M_j, \partial_0 M_j)$ is an I-pair, and if it contains an essential torus, it is also an S^1-pair (see the proof of Theorem 5). Otherwise, if F_i is an annulus, up to isotopy, we can suppose it to be vertical in a structure of I-fiber bundle of $(M_j, \partial_0 M_j)$ (see Lemma 3).

Our claim is now established, so we can suppose that each element of F which is contained in a Seifert pair $(M_j, \partial_0 M_j)$ is saturated in one of its Seifert fibrations. But since each F_i is rough, and it is not parallel to a surface contained in $\partial_0 M$, we deduce that it is parallel to a surface contained in $\partial_1 M_j$ for some $j \leq n$ and therefore also to an element of S. So we showed that each element of F is parallel to an element of S (obtaining a map from the family F to S) and, since two different elements of F are not parallel, this map is injective (otherwise two elements of F would be isotopic and disjoint hence parallel by Theorem 1), moreover we can say that $m \leq n$. Now we will show that each element of S is rough and, since F is maximal, that the two family's coincide up to isotopy.

Suppose that S_1 is thin, and therefore that there exists a Seifert pair (N, T) which satisfies the conditions of Definition 6. We can distinguish two cases: S_1 is an annulus, or a torus.

Case 1: S_1 is an annulus. In this case we can distinguish two subcases: the pair (N, T) is an S^1-pair or an I-pair.

Case 1.a: (N, T) is an S^1-pair. In this case, let us cut (N, T) along S_1, and obtain one or two S^1-pairs (depending on when $N - S_1$ is connected or not). We will call these pairs (N_-, T_-) and (N_+, T_+). Both $\partial_1 N_-$ and $\partial_1 N_+$ contain S_1. In the decomposition of $(M, \partial M)$ along S, there are two pairs containing S_1; let us call these pairs $(P_-, \partial_0 P_-)$ and $(P_+, \partial_0 P_+)$ (it can happen that $P_- = P_+$). We will now show that these two pairs, which touch themselves at least along S_1, are S^1-pairs and therefore that S_1 is not useful for the decomposition of M, i.e. that the family $S' = S - S_1$ splits M into pairs which are either simple or Seifert, and so verifies the hypotheses of the JSJ Theorem, and this is absurd by minimality of S.

Let B_+ be the base surface of the Seifert fibration of (N_+, T_+). The projection in $\pi(S_1)$ of S_1 in B_+ is an arc which is contained in ∂B_+. It is easy to see that it is almost always possible to find an arc in B_+ whose endpoints are contained in $\pi(S_1)$ and which is not homotopic to an arc contained in $\pi(S_1)$ by a homotopy which does not pass through the projection of the exceptional fibers. The only case when this is not possible is when B_+ is a disc and there is at most one exceptional fiber. In the other cases, the preimage of the arc we have found is an incompressible annulus whose boundary is contained in S_1, and non-parallel to a surface contained in $\partial_1 N_+$. Applying Theorem 5, we obtain that $(P_+, \partial_0 P_+)$ is an S^1-pair. In the case where B_+ is a disc, N_+ is a solid torus and $\partial_1 N_+$ is composed by at least 3 annuli (not only 2 because otherwise S_1 would be parallel to an annulus of $\partial_1 N$), one of which is S_1. We distinguish two more cases:

Case 1.a.1: N_+ intersects another annulus or torus S_2 of S. It is easy to prove that $S_2 \cap N_+$ is an annulus or torus which is saturated in the Seifert structure of N_+. In this case as well, one finds an incompressible annulus which connects S_1 and S_2 and which is not $\partial_1 P_+$-parallel. So one can again apply Theorem 5 obtaining that $(P_+, \partial_0 P_+)$ is an S^1-pair. The same can be done for $(P_-, \partial_0 P_-)$ and hence $S' = S - S_1$ is a family which still splits M in pairs which are either Seifert or simple (since the Seifert manifolds P_- and P_+ are glued along a saturated annulus). This contradicts the minimality of S.

Case 1.a.2: N_+ does not intersect any other element of S and no annulus of $\partial_1 N_+$ is parallel to an element of S. Any annulus of $\partial_1 N_+$ different from S_1, is essential in P_+, so $(P_+, \partial_0 P_+)$ is a Seifert pair. Let us now suppose that $(P_+, \partial_0 P_+)$ is an I-pair and deduce a contradiction. Then it will follow that $(P_+, \partial_0 P_+)$ is an S^1-pair. Up to isotopy, we can suppose that an annulus L of $\partial_1 N_+$ is saturated in an I-fibration of $(P_+, \partial_0 P_+)$. Note that we can choose L so that a component of ∂L and a component of ∂S_1 bound an annulus in $\partial_0 P_+$. Then a component of the pair obtained by cutting the I-pair $(P_+, \partial_0 P_+)$ along L is an I-pair product over an annulus, so S_1 would be parallel to L and this is absurd since it is essential in (N_+, T_+).

Case 1.b: (N, T) is an I-pair. Since S_1 is not parallel to a surface contained in $\partial_1 N$ in (N, T) and it is incompressible, it is easy to check that the pairs $(N_-, T \cap N_-)$ and $(N_+, T \cap N_+)$ are I-pairs with a base surface which is different from an annulus or a disc. Hence in each of them we can find an essential square whose two vertical arcs are contained in S_1, so the pairs $(P_-, \partial_0 P_-)$ and $(P_+, \partial_0 P_+)$ are I-pairs by Theorem 6. Hence also $P_- \cup P_+$ can be given a structure of I-pair, so also $S' = S - S_1$ is a family which splits M in pairs which are either Seifert or simple. This is absurd by minimality of S.

Case 2: S_1 is a torus. We already noted that if an I-pair contains an incompressible torus, then it is also an S^1-pair. So, in this case, we only need to examine the case where (N, T) is an S^1-pair. We still can distinguish two subcases:

Case 2.a: S_1 does disconnect N. Let $(N_-, T \cap N_-)$ and $(N_+, T \cap N_+)$ be the pairs obtained by cutting (N, T) along S_1; and let $(P_-, \partial_0 P_-)$ and $(P_+, \partial_0 P_+)$ (which can also be coincident) be the pairs of the decomposition of M along S containing S_1. Let B_- and B_+ be the base surface of the Seifert fibration respectively of $(N_-, T \cap N_-)$ and

$(N_+, T \cap N_+)$. Since S_1 is essential in (N, T), it is possible to find an essential arc in B_+ whose endpoints are contained in the projection of S_1 and which is not homotopic to an arc contained in ∂B_+ by a homotopy which does not intersect the projection of the exceptional fibers of $(N_+, T \cap N_+)$. The preimage of this arc is an incompressible annulus whose boundary is contained in S_1 and is not $\partial_1 N_+$ parallel. According to Theorem 5, $(P_+, \partial_0 P_+)$ is an S^1-pair. We can conclude the same for $(P_-, \partial_0 P_-)$. It remains to be proved that the two Seifert fibration of the pairs $(P_-, \partial_0 P_-)$ and $(P_+, \partial_0 P_+)$ are compatible, in the sense that their fibers on S_1 are homotopic. This is true since the annuli we found (which are saturated in P_- and P_+) have a homotopic boundary in S_1, since they are saturated in the Seifert fibration of (N, T). Hence we can since in this case we can exclude S_1 from S therefore obtaining another splitting family, and this is absurd since we supposed S minimal.

Case 2.b: S_1 does not disconnect N. This case is similar to the preceding one; the difference is this we cannot exclude that $(N_+, T \cap N_+)$ (which in that case coincides with $(N_-, T \cap N_-)$) is homeomorphic to $(S^1 \times S^1 \times I, S^1 \times S^1 \times \{0, 1\})$. But in this last case (N, T) coincides with a component of $(M, \partial M)$ and this is absurd since $S - S_1$ would be another splitting family for M.

□

References

[1] JACO W. AND SHALEN P., *Seifert fibered spaces in 3-manifolds*, Mem. Amer. Math. Soc. **220**, Providence RI 1979.

[2] FOMENKO A.T. AND MATVEEV S., *Algorithmic and Computer Methods for Three-Manifolds*, Mathematics and its Applications, Kluwer Academic Publishers 1997.

[3] LAUDENBACH F., *Topologie de la dimension trois: homotopie et isotopie*, Astérisque **12**, Société Mathématique de France 1974.

[4] MATVEEV S., *Algorithmic methods in* 3*-manifolds topology*, to be published.

[5] MATVEEV S., *Generalized Graph manifolds and their effective recognition*, Sb. Math. **189** (1998), 1517–1531.

[6] NEUMANN W.D. AND SWARUP G.A., *Canonical Decompositions of 3-Manifolds*, Geometry and Topology **1** (1997), 21–40.

Princeton University Press, Princeton, New Jersey 1997.

[7] WALDHAUSEN F., *On irreducible three manifolds which are sufficiently large*, Ann. of Math. **87** (1968), 56–88.

[8] WALDHAUSEN F., *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten I*, Invent. Math. **3** (1967), 308–333.

[9] WALDHAUSEN F., *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten II*, Invent. Math. **4** (1967), 87–117.

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