Rend. Sem. Mat. Univ. Pol. Torino
Vol. 60, 2 (2002)

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ON THE SOLVABILITY IN GEVREY CLASSES OF A LINEAR OPERATOR IN TWO VARIABLES

Abstract. We show non solvability results in Gevrey spaces $\mathcal{G}^s$ for a linear partial differential operator with a single real characteristic of constant multiplicity m, $m \geq 3$, provided $s > m/(m-2)+\delta$, where $\delta > 0$ depends on the order of the degeneracy of a suitable lower order term. In particular, $\delta \to 0$ as the order of the degeneracy tends to $+\infty$.

1. Introduction

The main aim of the present paper is to study in detail the local solvability of a model linear partial differential operator in two variables. We recall that although most of the well known classical operators, appearing in the basic theory of PDEs and in Mathematical Phisics, are solvable, non solvable operators exist, as proved first by Lewy [9], and often of a very simple form. The example of Lewy was generalized by Hörmander [6], who proved a necessary condition for the local solvability of partial differential operators, given by the following

THEOREM 1. *Let the linear partial differential operator P with coefficients in $C^\infty(\Omega)$ be solvable in Ω, in the sense that for every $f \in C_0^\infty(\Omega)$ we can find a solution $u \in \mathcal{D}'(\Omega)$ of $Pu = f$. Then, for every compact set $K \subset \Omega$ there exist a positive constant C and an integer $M \geq 0$ such that*

$$(1) \qquad \Big| \int f(x)\varphi(x)\,dx \Big| \leq \sum_{|\alpha|\leq M} \sup_{x\in K} |D^\alpha f(x)| \sum_{|\alpha|\leq M} \sup_{x\in K} |D^\alpha\, {}^tP\varphi(x)|$$

for all $f, \varphi \in C_0^\infty(K)$.

If an operator P is non locally solvable in the previous sense it is natural to analyze its behaviour in Gevrey spaces, being intermediate classes between the analytic and the C^∞ functions. More precisely, from now on we will refer to the following functional setting.
Let K be a compact subset of Ω and C a positive fixed constant. Let us consider the subspace $\mathcal{G}_0^s(\Omega, K, C)$, $1 < s < +\infty$, of $C_0^\infty(\Omega)$, given by all the functions $f(x)$ with

*The author is thankful to Professor Luigi Rodino and Professor Petar Popivanov for useful discussions and hints on the subject of this paper.

support contained in K such that, for some $R \geq 0$,

$$\sup_{x \in K} |\partial^\alpha f(x)| \leq R\, C^{|\alpha|} (\alpha!)^s. \tag{2}$$

$\mathcal{G}^s_0(\Omega, K, C)$ is a Banach space endowed with the norm:

$$\| f \|_{s,K,C} := \sup_\alpha \left(C^{-|\alpha|} (\alpha!)^{-s} \sup_{x \in K} |\partial^\alpha f(x)| \right) \tag{3}$$

or equivalently

$$\| f \|_{s,K,C} := \sum_\alpha C^{-|\alpha|} (\alpha!)^{-s} \| \partial^\alpha f \|_{L^p(K)} \tag{4}$$

with $p \geq 1$ fixed. From now on we consider (4) with $p = 2$.

DEFINITION 1. *$\mathcal{G}^s_0(\Omega) = \bigcup_{K,C} \mathcal{G}^s_0(\Omega, K, C)$ where K and C run respectively over the set of all the compact sets contained in Ω and over the set of the positive real numbers.*

Therefore it is natural to endow $\mathcal{G}^s_0(\Omega)$ with the inductive limit topology:

$$\operatorname{ind} \lim_{\substack{K \nearrow \Omega \\ C \nearrow +\infty}} \mathcal{G}^s_0(\Omega, K, C). \tag{5}$$

Similarly we define $\mathcal{G}^s(\Omega)$ as the projective limit topology of the spaces $\mathcal{G}^s(\Omega, K, C)$ of all the functions $f \in C^\infty(\Omega)$ for which the norm (3) of the restriction to K is finite. The main result that we will handle in the following, concerning with the solvability in a Gevrey frame of partial differential operators with $\mathcal{G}^s$ coefficients, is the Gevrey version of Theorem 1 proved by Corli [2].

THEOREM 2. *Let s be a fixed real number, $1 < s < +\infty$. Let the linear partial differential operator P be s-solvable in Ω, i.e. for all $f \in \mathcal{G}^s_0(\Omega)$ there exists u in $\mathcal{D}'_s(\Omega)$, space of the s-ultradistributions in Ω, solution of $Pu = f$. Then for every compact subset K of Ω, for every $\eta > \epsilon > 0$, there exists a positive constant C such that:*

$$\left(\max_{x \in K} |f(x)| \right)^2 \leq C\, \| f \|_{s,K,\frac{1}{\eta-\epsilon}} \, \| {}^t P f \|_{s,K,\frac{1}{\eta-\epsilon}} \tag{6}$$

for every $f \in \mathcal{G}^s_0(\Omega, K, \frac{1}{\eta})$.

The previous theorem will be used in the following manner: an operator P is not s-locally solvable in x_0 if the associated transposed equation ${}^t P f = 0$ admits a suitable sequence of approximated solutions which make the right hand side of (6) arbitrarily small and let the left hand side bounded from below.

Now let us analyze the problem we are interested in.
Let us consider the operator

$$P = D_{x_1}^m - A(x_1)D_{x_2}^{m-1} - B(x_1)D_{x_2}^{m-2}D_{x_1} \tag{7}$$

with m odd, $m \geq 3$; $A(x_1)$, $B(x_1)$ are analytic functions of x_1, defined in a neighborhood of $x_1 = 0$.
The operator P is weakly hyperbolic in x_1 with a single characteristic of constant multiplicity m and therefore it is always locally $\mathcal{G}^s$ solvable for $s < \frac{m}{m-1}$ without any restrictions on $A(x_1)$ and $B(x_1)$. This well known fact follows from the $\mathcal{G}^s$ well posedness of the Cauchy problem for P or from more general results about the $\mathcal{G}^s$ solvability of linear PDEs with multiple characteristics (e.g., cf. Corollary 5.1.3 in Mascarello-Rodino [13], see also [1], [5], [17]). Next if $\Im m(A(x_1))$ vanishes of odd order at $x_1 = 0$, changing its sign from - to +, the results of Corli [3] imply that P is not locally solvable in $\mathcal{G}^s$ for $s > \frac{m}{m-1}$. We will investigate the case of $\Im m A(x_1)$ vanishing of even order at $x_1 = 0$.
Let us suppose that for some integer $h > 0$:

$$\Re e\, A(0) \neq 0, \tag{8}$$

$$\Im m\, A(x_1) = cx_1^{2h} + o(x_1^{2h}) \quad \text{for} \quad x_1 \to 0,\ c \neq 0. \tag{9}$$

Moreover for a fixed $l > 0$:

$$\Im m\, B(x_1) = dx_1^l + o(x_1^l) \quad \text{for} \quad x_1 \to 0,\ d \neq 0. \tag{10}$$

Popivanov [14] (see also Popivanov-Popov [15]) proved that P is not locally solvable, in the C^∞ sense, if h is sufficiently large with respect to l.
Moreover, if the conditions (8), (9), hold, as a particular case of Theorem 3.7 in [10] or Theorem 3.2 in [4] we obtain that the operator (7) is s-solvable for

$$s < \frac{m}{m-2}. \tag{11}$$

At this point a natural question arises: what can we say about the behaviour of P for indexes $s \geq \frac{m}{m-2}$? We get that P is non s-solvable for

$$s > \frac{m}{m-2} + \theta(h), \tag{12}$$

with $\theta(h) \to 0$ for $h \to 0$.
The proof of this result, developed in the next section, is based on Theorem 2, applied to a function $f_\lambda(x)$ of the following form:

$$f_\lambda(x) = v(x) \cdot e^{i\left(\lambda\psi_0(x)+\lambda^{\frac{m-1}{m}}\psi_1(x)+\lambda^{\frac{m-2}{m}}\psi_2(x)+\ldots+\lambda^{\frac{1}{m}}\psi_{m-1}(x)\right)}.$$

We will choose the phase functions $\psi_i(x)$ according to a standard proceeding (see [3]). At this point, solving suitable transport equations, we are able to find an amplitude function $v(x)$ such that the approximate solution $f(x)$ of the homogeneous equation ${}^tPf_\lambda = 0$ makes the right hand side of (6) arbitrarily small for λ sufficiently large and s satisfying (12), but leaves the left hand side greater than or equal to 1. We observe that the previous estimates need also a suitable version for this case of the results of Ivrii [8]. For more details we also address to [12].

2. The main result

THEOREM 3. *Let us suppose that A (respectively, B) satisfies* (8), (9) *(respectively,* (10)*) and*

$$h \geq \begin{cases} 2(2l+2) & \text{if} \quad m = 3 \\ 2l+2 & \text{if} \quad m \geq 5. \end{cases} \tag{13}$$

Then P is non s-solvable at the origin for every s satisfying (12) *where*

$$\theta(h) = \frac{m(l+1)}{(m-2)[(2m-4)h-(m-1)l-1]}.$$

Proof. We will reason ab absurdo.
First let us observe that

$$\begin{aligned} {}^tP = &-(-i)^m\partial_{x_1}^m - (-i)^{m-1}A(x_1)\partial_{x_2}^{m-1} - (-i)^{m-1}B(x_1)\partial_{x_2}^{m-2}\,\partial_{x_1} + \\ &-(-i)^{m-1}B'(x_1)\partial_{x_2}^{m-2}. \end{aligned} \tag{14}$$

Let us define

$$f_\lambda(x) := e^{i\left(\lambda\psi_0(x)+\lambda^{\frac{m-1}{m}}\psi_1(x)+\lambda^{\frac{m-2}{m}}\psi_2(x)+\ldots+\lambda^{\frac{1}{m}}\psi_{m-1}(x)\right)} \tag{15}$$

where $x = (x_1, x_2)$.
1^{st} STEP: *choice of the phase functions $\psi_i(x)$.*
The basic idea is to apply the operator ${}^tP$ to $f_\lambda(x)$ and choose the phase functions $\psi_0(x)$, $\psi_1(x)$, $\psi_2(x)$ in order to make equal to zero the higher powers of λ.
Following a standard approach for constructing formal asymptotic solutions (e.g., cf. [2], see also [5], Chapter IV) we choose $\psi_0 = x_2$ and obtain

$${}^tPf_\lambda(x) = \Big(l_{m-1}(x)\lambda^{m-1} + o(\lambda^{m-1})\Big)\, f_\lambda(x), \;\; \lambda \to +\infty.$$

where the coefficient is given by

$$l_{m-1} := -\left(\frac{\partial\psi_1(x)}{\partial x_1}\right)^m - A(x_1). \tag{16}$$

Letting $l_{m-1} = 0$, we obtain m complex solutions and we choose

$$\psi_1(x) = \int_0^x \widetilde{A}(x_1)\, dx_1, \tag{17}$$

where $\widetilde{A}(x_1) = [A(x_1)]^{\frac{1}{m}}$ is determined in such a way that $[A(0)]^{\frac{1}{m}}$ is real-valued. Note that properties (8) and (9) keep valid for $\widetilde{A}(x_1)$, for a new constant c.
Then

$$\Im m\ \psi_1(x) = \frac{c}{2h+1} x_1^{2h+1} + o(x_1^{2h+1}). \tag{18}$$

Repeating the previous arguments with this choice of $\psi_1(x)$ finally we find:

$$\psi_2 = \frac{1}{m} \int \widetilde{B}(x_1)\, dx_1 + i x_2^2 \tag{19}$$

with $\widetilde{B}(x_1)$ satisfying again (10) for a new constant d. Therefore

$$\Im m\ \psi_2(x) = \frac{1}{m}\left(\frac{d}{l+1} x_1^{l+1} + o(x_1^{l+1})\right) + x_2^2. \tag{20}$$

The other phase functions are determined recursively, according to a standard proceeding, cf. [2]. We need not precise information on them, but they vanish at $x_1 = 0$. We then write

$$f_\lambda(x) = e^{i\Phi(x)} \tag{21}$$

where

$$\begin{aligned} \Im m\ \Phi(x) = \lambda^{\frac{m-1}{m}} \left\{ \frac{c}{2h+1} x_1^{2h+1} + o(x_1^{2h+1}) \right\} + \lambda^{\frac{m-2}{m}} \left\{ \frac{1}{m}\left(\frac{d}{l+1} x_1^{l+1} + \right.\right. \\ \left.\left. + o(x_1^{l+1}) \right) + x_2^2 \right\} + \lambda^{\frac{m-3}{m}} o(x_1). \end{aligned} \tag{22}$$

2^{nd} STEP: *the action of ^tP on $u(x) = f_\lambda(x)\, v(x)$.*
According to our choices we find:

$$^tPu(x) = f_\lambda(x)\, {}^tP_\lambda v(x)$$

setting

$$^tP_\lambda := \lambda^{\frac{m-1}{m}} \left\{ Q_0(x, D) + \sum_{j=1}^{m-1} \lambda^{-j} Q_j(x, D) \right\} \tag{23}$$

where:

- $Q_0(x, D) = (-1)^m \big(\sum_i c_i(x) D_i + S(x)\big)$, where $c_i(x)$ are polynomial in x and $S(x)$ is an analytic function;

- $Q_j(x, D)$ are differential operators of order less than or equal to m, with analytic coefficients.

3^{rd} STEP: *minimum of* $\Im m\ \Phi(x)$.
We follow here the arguments in Popivanov [14].
Obviously

$$\Im m\ \Phi(0,0) = 0. \tag{24}$$

We are interested in the solutions of the following equations:

$$\frac{\partial \Im m\ \Phi(x)}{\partial x_1} = \lambda^{\frac{m-1}{m}}(cx_1^{2h} + o(x_1^{2h})) + \lambda^{\frac{m-2}{m}}\frac{1}{m}(dx_1^l + o(x_1^l)) + \tag{25}$$
$$+\lambda^{\frac{m-3}{m}}o(1) = 0,$$

$$\frac{\partial \Im m\ \Phi(x)}{\partial x_2} = 2\lambda^{\frac{m-2}{m}}x_2 + \lambda^{\frac{m-3}{m}}o(x_1) = 0. \tag{26}$$

The second equation is solved by

$$x_2 = o(\lambda^{-\frac{1}{m}}x_1), \ \text{ for } x_1 \to 0. \tag{27}$$

Let us try to solve the first equation

$$\lambda^{\frac{m-1}{m}}\big(cx_1^{2h} + \frac{1}{m}d\lambda^{-\frac{1}{m}}x_1^l + o(x_1^{2h}) + \lambda^{-\frac{1}{m}}o(x_1^l) + \lambda^{-\frac{2}{m}}o(1)\big) = 0$$

Let us begin by considering the case l odd, $c > 0$, $d < 0$, $2h \geq 2l + 2$.
Taking large λ, in this way we have to solve

$$cx_1^{2h} + \frac{1}{m}d\lambda^{-\frac{1}{m}}x_1^l = o(\lambda^{-\frac{2}{m}}), \ \text{ for } \ \lambda \to +\infty. \tag{28}$$

Let us define

$$\epsilon := \lambda^{-\frac{1}{m}\frac{1}{2h-l}} \to 0 \text{ if } \lambda \to 0. \tag{29}$$

From the previous definition it follows immediately that $\lambda^{-\frac{1}{m}} = \epsilon^{2h-l}$; thus, replacing this quantity in (28) we obtain the following equation:

$$cx_1^{2h} + \frac{1}{m}d\epsilon^{2h-l}x_1^l = o(\epsilon^{2(2h-l)}), \ \epsilon \to 0. \tag{30}$$

Let us set:

$$x_1 = (1 + y_1)\left(-\frac{d}{mc}\right)^{\frac{1}{2h-l}}\epsilon; \tag{31}$$

then (30) transforms into

$$(1 + y_1)^l\{(1 + y_1)^{2h-l} - 1\} = o(\epsilon^{2h-2l}).$$

Now let us consider the corresponding function

$$g(y_1,\epsilon) = (1+y_1^l)\{(1+y_1)^{2h-l}-1\}+o(\epsilon^{2h-2l}). \tag{32}$$

Clearly

$$g(0,0)=0,\ \left(\frac{\partial g}{\partial y_1}\right)(0,0)=2h-l>0,\ \left(\frac{\partial g}{\partial \epsilon}\right)(0,0)=0.$$

Then, by the Implicit Function Theorem there exists a function $y_1 = y_1(\epsilon)\in C^2$, such that $g(y_1,\epsilon)=0$, $y_1(0)=0$, $y_1'(0)=0$. This implies that

$$y_1(\epsilon)=o(\epsilon^2)\ \text{for}\ \epsilon\to 0.$$

Thus

$$x_1 = (1+o(\epsilon^2))\left(-\frac{d}{mc}\right)^{\frac{1}{2h-l}}\epsilon = \left(-\frac{d}{mc}\right)^{\frac{1}{2h-l}}\epsilon + o(\epsilon^3).$$

Therefore, considering (29) and (27) we conclude that a critical point is given by

$$x_{1\,\lambda} = \left(-\frac{d}{mc}\right)^{\frac{1}{2h-l}}\lambda^{-\frac{1}{m}\frac{1}{2h-l}} + o\big(\lambda^{-\frac{3}{m(2h-l)}}\big),\ \lambda\to+\infty \tag{33}$$

$$x_{2\lambda} = o\big(\lambda^{-\frac{2h-l+1}{m(2h-l)}}\big),\ \lambda\to+\infty. \tag{34}$$

Now, we may give a complete picture of the behaviour of $\Im m\ \Phi(x)$ varying l and sign of d:

1. l odd;

   - $d>0$: $\Im m\,\Phi(x)$ assumes its minimum at the origin and we have $\Im m\ \Phi(0,0)=0$;
   - $d<0$: $\Im m\ \Phi(x)$ assumes its minimum value in $(x_{1\,\lambda}, x_{2\,\lambda})$ and, setting,

   $$C_0\lambda^{\delta_1} := \left(-\frac{d}{mc}\right)^{\frac{l+1}{2h-l}}\left(\frac{d}{m}\right)\frac{2h-l}{(l+1)(2h+1)}\lambda^{\frac{m-1}{m}-\frac{2h+1}{m(2h-l)}}, \tag{35}$$

   we have

   $$\Im m\ \Phi(x_{1\lambda},x_{2\lambda}) = \widetilde{C}(\lambda) = C_0\lambda^{\delta_1}+o(\lambda^{\delta_1}). \tag{36}$$

   where, being $2h\ge 2l+2$ and $m\ge 3$, we immediately get:

   $$C_0<0,\quad 0<\delta_1=\frac{m-1}{m}-\frac{2h+1}{m(2h-l)}<1. \tag{37}$$

2. l even;

   - $d\cdot c<0$: the expression of the critical point $(x_{1\,\lambda}, x_{2\,\lambda})$ and the behaviour of $\Im m\ \Phi(x)$ are the same described before;

- $d \cdot c > 0$: if $d, c > 0$ $\Im m\, \Phi(x)$ is an increasing function of x_1; otherwise $\Im m\, \Phi(x)$ is a decreasing function of x_1.

We may limit our attention to the case l odd and $d < 0$ (the case l even, $c > 0, d < 0$ is treated in the same way. Note moreover that the change of variable $x_2' = -x_2$ gives a change of sign for the constant d).
We want to apply Taylor's formula in order to analyze the behaviour of $\Im m\, \Phi(x)$ near the point $(x_{1\,\lambda}, x_{2\,\lambda})$. To this aim let us observe that:

$$\frac{\partial^2 \Im m\, \Phi(x_1, x_2)}{\partial x_2^2} = 2\lambda^{\frac{m-2}{m}} x_2 + \lambda^{\frac{m-3}{m}} o(x_1),$$

$$\frac{\partial^2 \Im m\, \Phi(x_1, x_2)}{\partial x_2 \partial x_1} = \lambda^{\frac{m-3}{m}} o(1),$$

$$\frac{\partial^2 \Im m\, \Phi(x_1, x_2)}{\partial x_1^2} = \lambda^{\frac{m-1}{m}} \{2hcx_1^{2h-1} + \tfrac{1}{m} dl\lambda^{-\frac{1}{m}} x_1^{l-1} + o(x_1^{2h-1}) + \\ + \lambda^{-\frac{1}{m}} o(x_1^{l-1}) + \lambda^{-\frac{2}{m}} o(1)\}.$$

Therefore

$$\begin{aligned} \Im m\, \Phi(x) &= \widetilde{C}(\lambda) + C_0'(x_1 - x_{1\,\lambda})^2 \lambda^{\delta_2} + (x_2 - x_{2\,\lambda})^2 \lambda^{\frac{m-2}{m}} + \\ &+ \sum_{j=3}^{2h+1} (x_1 - x_{1\,\lambda})^j \lambda^{\frac{m-1}{m} - \frac{2h+1-j}{m(2h-l)}} + o((x_2 - x_{2\,\lambda})^3) \lambda^{\frac{m-1}{m}}, \end{aligned} \tag{38}$$

where

$$C_0' := 2c(2h - l)\left(-\frac{d}{mc}\right)^{\frac{2h-1}{2h-l}} > 0$$

and

$$0 < \delta_2 := \frac{m-1}{m} - \frac{2h-1}{m(2h-l)} < 1.$$

Now let us suppose that

$$|x_1 - x_{1\,\lambda}| \leq \epsilon_1 x_{1\,\lambda} \quad \text{with} \quad 0 < \epsilon_1 \ll 1.$$

Being:

$$\sum_{j=3}^{2h+1} (x_1 - x_{1\,\lambda})^j \lambda^{\frac{m-1}{m} - \frac{2h+1-j}{m(2h-l)}} < \\ < (x_1 - x_{1\,\lambda})^2 \lambda^{\delta_2} \{\epsilon_1 x_{1\,\lambda} \lambda^{\frac{1}{m(2h-l)}} + (\epsilon_1 x_{1\,\lambda}^2 \lambda^{\frac{2}{m(2h-l)}} + \ldots \\ \ldots + (\epsilon_1 x_{1\,\lambda})^{2h-1} \lambda^{\frac{2h-1}{m(2h-l)}}\}$$

we get

$$C_0'(x_1 - x_{1\lambda})^2\lambda^{\delta_2} + \sum_{j=3}^{2h+1}(x_1 - x_{1\lambda})^j \lambda^{\frac{m-1}{m}-\frac{2h+1-j}{m(2h-l)}} = \tag{39}$$
$$= C_0'(x_1 - x_{1\lambda})^2\lambda^{\delta_2}\{1 + \mathcal{O}(\epsilon_1)\}.$$

Now let us define:

$$g_1(x_1) := \begin{cases} 1 & \text{if} \quad x_1 \in [1 - \frac{\epsilon_1}{2}, 1 + \frac{\epsilon_1}{2}] \\ 0 & \text{if} \quad x_1 \in [1 - \epsilon_1, 1 + \epsilon_1]. \end{cases} \tag{40}$$

Let us suppose that:

- $g_1(x_1) \in \mathcal{G}_0^s(\mathbb{R})$;
- $0 \le g_1(x_1) \le 1$, for every x_1 in $\mathbb{R}$.

By definition, it follows immediately that

$$g_1(x_1 x_{1\lambda}^{-1}) = 1 \quad \text{if} \quad |x_1 - x_{1\lambda}| \le \frac{\epsilon_1}{2} x_{1\lambda}$$

and

$$supp\, g_1(x_1 x_{1\lambda}^{-1}) \subset \{x_1 \in \mathbb{R} \ : \ |x_1 - x_{1\lambda}| \le \epsilon_1 x_{1\lambda}\}.$$

Moreover

$$supp\, \tfrac{\partial}{\partial x_1} g_1(x_1 x_{1\lambda}^{-1}) \subset \{x_1 \ : \ (1-\epsilon_1)x_{1\lambda} \le |x_1 - x_{1\lambda}| \le (1 - \tfrac{\epsilon_1}{2})x_{1\lambda}\} \bigcup$$
$$\{x_1 \ : \ (1+\tfrac{\epsilon_1}{2})x_{1\lambda} \le |x_1 - x_{1\lambda}| \le (1 + \epsilon_1)x_{1\lambda}\}.$$

Then on $supp g_1(x_1 x_{1\lambda}^{-1})$, and for $|x_2 - x_{2\lambda}| \le \epsilon_1$ we have

$$\Im m\ \Phi(x) \ge \widetilde{C}(\lambda) + C_0'\lambda^{\delta_2}(x_1 - x_{1,\lambda})^2[1 + \mathcal{O}(\epsilon_1)] + \frac{1}{2}|x_1 - x_{1\lambda}|^2\lambda^{\frac{m-2}{m}}. \tag{41}$$

In an analogous way on $supp\, \frac{\partial g_1}{\partial x_1}$ and for $|x_2 - x_{2\lambda}| \le \epsilon_1$ we get

$$\Im m\ \Phi(x) \ge \widetilde{C}(\lambda) + C_0''\lambda^{\delta_1} \tag{42}$$

with

$$C_0'' := \epsilon_1^2 C_0' \left(-\frac{d}{mc}\right)^{\frac{2}{2h-l}} > 0 \tag{43}$$

and $0 < \delta_1 < 1$, defined as before.
Moreover let us consider:

$$g_2(x_2) := \begin{cases} 1 & \text{if} \quad |x_2| \le \frac{\epsilon_1}{4} \\ 0 & \text{if} \quad |x_2| \ge \frac{\epsilon_1}{2} \end{cases} \tag{44}$$

satisfying the following conditions

- $g_2(x_2) \in \mathcal{G}_0^s(\mathbb{R})$;
- $0 \leq g_2(x_1) \leq 1$, for every x_1 in $\mathbb{R}$.

Let us suppose that $x_2 \in supp\, g_2$. Then

$$|x_2 - x_{2\,\lambda}| \leq \tfrac{\epsilon_1}{2} + |x_{2\,\lambda}|.$$

Considering that $x_{2\,\lambda} \to 0$ for $\lambda \to +\infty$, for λ sufficiently large we have

$$|x_2 - x_{2\,\lambda}| \leq \epsilon_1.$$

Moreover, by definition we have

$$supp\, \tfrac{\partial g_2(x_2)}{\partial x_2} \subset \{\tfrac{\epsilon_1}{4} \leq |x_2| \leq \tfrac{\epsilon_1}{2}\}.$$

Then, on $supp\, \frac{\partial g_2(x_2)}{\partial x_2}$, for $|x_1 - x_{1\,\lambda}| \leq \epsilon_1 x_{1\,\lambda}$ and for λ sufficiently large we obtain

$$\Im m\, \Phi(x) \geq \widetilde{C}(\lambda) + \epsilon_1^2 \lambda^{\frac{m-2}{m}}. \tag{45}$$

4^{th} STEP: *choice of* $v(x)$.
Let us define

$$\chi(x_1, x_2) := g_1(x_1 x_{1\,\lambda}^{-1})\, g_2(x_2). \tag{46}$$

Now let us observe that the Cauchy Kovalevsky Theorem assures us the existence of functions $v_\lambda^{(k)}$ defined by induction as solutions of the following transport equations:

$$\begin{cases} Q_0(x, D) v_\lambda^{(0)} = 0 \\ Q_0(x, D) v_\lambda^{(k)} = -\sum\limits_j \lambda^{-\frac{j}{m}} Q_j(x, D) v_\lambda^{(k-j)}, \quad k \geq 1 \\ v_\lambda^{(0)} = 1 \quad \text{on} \quad x_1 = x_{1\,\lambda} \\ v_\lambda^{(k)} = 0 \quad \text{on} \quad x_1 = x_{1\,\lambda} \quad \text{for} \quad k \geq 1 \end{cases} \tag{47}$$

where in the sum j runs from 1 to $\min\{k, m-1\}$.
Let us consider:

$$V_\lambda^{(N)} := \sum_{k=0}^{N} v_\lambda^{(k)}. \tag{48}$$

Let us define

$$\boxed{u_\lambda(x) := \chi(x)\, V_\lambda^{(N)}(x)\, e^{i\Phi(x)}} \tag{49}$$

5^{th} STEP: *the conclusion.*
We want to apply the necessary condition of Corli [3] to $u_\lambda(x)$. We recall that we want to contradict the estimate (6) for $f = u_\lambda$, $\lambda \to +\infty$, i.e.

$$\big(\max_K |u_\lambda(x)|\big)^2 \leq C\, \|u_\lambda\|_{s,K,\frac{1}{\eta-\epsilon}}\, \|{}^t P u_\lambda\|_{s,K,\frac{1}{\eta-\epsilon}} \tag{50}$$

where K is a suitable small compact neighborhood of the origin.
In view of (46), we may limit our attention to

$$K := \{(x_1, x_2)\ :\ |x_1 - x_{1\,\lambda}| \leq \epsilon_1 x_{1\,\lambda},\ \ |x_2| \leq \tfrac{\epsilon_1}{2}\}. \tag{51}$$

Let us start evaluating the left hand side.
Let us observe that

$$|u_\lambda(x_{1\,\lambda}, x_{2\,\lambda})| = e^{-\Im m\,\Phi(x_{1\,\lambda}, x_{2\,\lambda})} = e^{-\widetilde{C}(\lambda)}, \tag{52}$$

therefore

$$\big(\max_K |u_\lambda(x)|\big)^2 \geq e^{-2\widetilde{C}(\lambda)}. \tag{53}$$

Now let us analyze $\|u_\lambda\|_{s,K,\frac{1}{\eta-\epsilon}} \|{}^t P u_\lambda\|_{s,K,\frac{1}{\eta-\epsilon}}$.
Let us reason as in Ivrii [8]: there exist positive constants B and N such that, setting

$$\lambda = 4Be^L N, \quad L \geq 1,$$

the following estimate holds:

$$\sum_{|\alpha|\leq m} \|D^\alpha v_\lambda^{(k)}\|_{s,K,\frac{1}{\eta}} \leq B e^{-Lk + 4MN^{\frac{\nu}{m}}}, \quad k = 0, 1, 2, \ldots \tag{54}$$

with $\nu = \frac{m-1}{m}$. Therefore

$$\sum_{|\alpha|\leq m} \|D^\alpha V_\lambda^{(N)}\|_{s,K,\frac{1}{\eta}} \leq 2Be^{4MN^{\frac{\nu}{m}}}.$$

Since our choice of $v_\lambda^{(k)}$ implies

$${}^t P_\lambda V_\lambda^{(N)} = \sum_{h=0}^{m-2} \sum_{j=h}^{m-2} \lambda^{-\frac{h}{m}} Q_h v_\lambda^{N-m+2+j},$$

from (54) it follows that

$$\|{}^t P_\lambda V_\lambda^{(N)}\|_{s,K,\frac{1}{\eta}} \leq Ce^{-LN + 4MN^{\frac{\nu}{m}}}. \tag{55}$$

Now

$$\begin{aligned}\|u_\lambda\|_{s,K,\frac{1}{\eta-\epsilon}} &\leq \|\chi\|_{s,K,\frac{1}{\eta+\epsilon}} \|V_\lambda^{(N)}\|_{s,K,\frac{1}{\eta}} \|e^{i\Phi}\|_{s,K,\frac{1}{\eta+\epsilon}} \\ &\leq c\, e^{4MN^{\frac{\nu}{m}}} \|e^{i\Phi}\|_{s,K,\frac{1}{\eta+\epsilon}}.\end{aligned}$$

Let us recall now a useful result about Gevrey norms (Corli [2], [3], Marcolongo [11], Mascarello-Rodino [13], Gramchev-Popivanov [5]):

PROPOSITION 1. *Let Ψ be an analytic function in a neighborhood Ω of the origin of $\mathbb{R}^n$. Let us fix $s > 1$. Then for every compact $K \subset \Omega$ and for every $\eta > 0$ we get:*

$$\|e^{i\lambda\Psi(x)}\|_{s,K,\frac{1}{\eta}} \leq Ce^{a\lambda+d(\lambda\eta)^{\frac{1}{s}}+d\eta^{\frac{1}{s-1}}}, \tag{56}$$

where C and d are positive constants and $a := \sup_K(-\Im m\ \Psi(x))$.

Applying the previous proposition we obtain that:

$$\|e^{i\Phi}\|_{s,K,\frac{1}{\eta+\epsilon}} \leq Ce^{a+d\lambda^{\frac{1}{s}}(\eta+\epsilon)^{\frac{1}{s}}+d(\eta+\epsilon)^{\frac{1}{s-1}}}$$

with $a = -\widetilde{C}(\lambda)$. Thus

$$\|u_\lambda\|_{s,K,\frac{1}{\eta-\epsilon}} \leq Ce^{4MN^{\frac{\nu}{m}}-C_0\lambda^{\delta_1}+d\lambda^{\frac{1}{s}}(\eta+\epsilon)^{\frac{1}{s}}+d(\eta+\epsilon)^{\frac{1}{s-1}}}. \tag{57}$$

Regarding ${}^tPu_\lambda$ we get

$$\begin{aligned} \|{}^tPu_\lambda\|_{s,K,\frac{1}{\eta-\epsilon}} &\leq C\|{}^tPV_\lambda^{(N)}\|_{s,K,\frac{1}{\eta}}\|e^{i\Phi}\|_{s,K,\frac{1}{\eta}} \\ &\quad + C\lambda^{\frac{m-1}{m}}\sum_{|\alpha|\leq m}\|D^\alpha V_\lambda^{(N)}\|_{s,K,\frac{1}{\eta}}\|e^{i\Phi}\|_{s,K\cap supp\,\bigtriangledown\chi,\frac{1}{\eta}}. \end{aligned} \tag{58}$$

Since $K \cap supp\ \bigtriangledown\chi$ doesn't contain a small neighborhood of $x_2 = 0$, again by Proposition 1, considering also (42), we obtain that:

$$\|e^{i\Phi}\|_{s,K\cap supp\,\bigtriangledown\chi,\frac{1}{\eta}} \leq Ce^{-\widetilde{C}(\lambda)-C_0''\lambda^{\delta_1}+d\lambda^{\frac{1}{s}}\eta^{\frac{1}{s}}+d\eta^{\frac{1}{s-1}}}.$$

Therefore

$$\|{}^tPu_\lambda\|_{s,K,\frac{1}{\eta-\epsilon}} \leq Ce^{4MN^{\frac{\nu}{m}}-\widetilde{C}(\lambda)+d(\lambda\eta)^{\frac{1}{s}}+d\eta^{\frac{1}{s-1}}}\{e^{-LN}+\lambda^{\frac{m-1}{m}}e^{-C_0''\lambda^{\delta_1}}\}. \tag{59}$$

Summing up

$$\begin{aligned} \|u_\lambda\|_{s,K,\frac{1}{\eta-\epsilon}}\|{}^tPu_\lambda\|_{s,K,\frac{1}{\eta-\epsilon}} &\leq Ce^{8MN^{\frac{\nu}{m}}-2\widetilde{C}(\lambda)}\{C_1e^{-LN+d\lambda^{\frac{1}{s}}(\eta^{\frac{1}{s}}+(\eta+\epsilon)^{\frac{1}{s}})} \\ &\quad + C_2\lambda^{\frac{m-1}{m}}e^{-C_0''\lambda^{\delta_1}+d\lambda^{\frac{1}{s}}(\eta^{\frac{1}{s}}+(\eta+\epsilon)^{\frac{1}{s}})}\}. \end{aligned}$$

where C_1 and C_2 are positive constants depending on $d,\ \eta,\ \epsilon$. But, from the previous arguments,

$$\|u_\lambda\|_{s,K,\frac{1}{\eta-\epsilon}}\|{}^tPu_\lambda\|_{s,K,\frac{1}{\eta-\epsilon}} \geq e^{-2\widetilde{C}(\lambda)},$$

therefore:

$$1 \le C_1 e^{-LN+8MN^{\frac{\nu}{m}}+d\lambda^{\frac{1}{s}}(\eta^{\frac{1}{s}}+(\eta+\epsilon)^{\frac{1}{s}})}$$
$$+ C_2 \lambda^{\frac{m-1}{m}} e^{8MN^{\frac{\nu}{m}}-C_0''\lambda^{\delta_1}+d\lambda^{\frac{1}{s}}(\eta^{\frac{1}{s}}+(\eta+\epsilon)^{\frac{1}{s}})}, \quad \forall N.$$

Clearly the first addend of the right hand side tends to zero when $N \to +\infty$.
Regarding the second addend of the right hand side, in order to let it to zero when $N \to +\infty$, we must impose

$$\frac{\nu}{m} < \delta_1 \tag{60}$$

$$\frac{1}{s} < \delta_1. \tag{61}$$

Recalling that

$$\delta_1 = \frac{m-1}{m} - \frac{2h+1}{m(2h-l)}$$

we get that (60) is satisfied if the hypothesis (13) is valid and (61) is equivalent to require $s > \frac{m}{m-2} + \theta(h)$. This concludes the proof. □

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AMS Subject Classification: 35A07.

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Lavoro pervenuto in redazione il 03.05.2001 e, in forma definitiva, il 28.01.2002.