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M. Mancini*

PROJECTIVELY NORMAL CURVES DEFINED BY QUADRICS

Abstract. We study the projectively normal embeddings of small degree of smooth curves of genus g such that the ideal of the embedded curves is generated by quadrics.

1. Introduction

Let C be a smooth curve of genus g and $\mathcal{L}$ a very ample invertible sheaf on C, defining an embedding $\varphi_{\mathcal{L}}$ of C in $\mathbb{P}^{d-g+h^1(\mathcal{L})}$ with degree d (we always work on the field of complex numbers $\mathbb{C}$).

The problem to determine the value of d as small as possible for which $\varphi_{\mathcal{L}}(C)$ is projectively normal and its ideal is generated by forms of minimal degree has been the subject of study of many authors in the past.

We recall specifically D. Mumford (see [12]), who proved that every line bundle $\mathcal{L}$, with $h^1(\mathcal{L}) = 0$, is normally generated on C if $d \geq 2g + 1$ and that the ideal of $\varphi_{\mathcal{L}}(C)$ is generated by quadrics if $d \geq 3g + 1$; B. St. Donat (see [13]), who improved Mumford's result by showing that $\varphi_{\mathcal{L}}(C)$ is the intersection of quadrics in $\mathbb{P}^{d-g}$ if $d \geq 2g + 2$ and that it is defined by forms of degree ≤ 3 if $d \geq 2g + 1$; M. Homma (see [10]), who showed that, when C is a curve of genus $g \geq 5$ neither hyperelliptic, nor trigonal, nor double covering of an elliptic curve ($g \geq 6$), nor non-singular plane quintic, almost all the non-special line bundles of degree $d = 2g$ define an embedding of C in $\mathbb{P}^{d-g}$ which is projectively normal and whose ideal is generated in degree ≤ 3.

In the present paper we are interested in studying this problem for a generic curve C in M_g.

Given $g \in \mathbb{Z}$ such that $\binom{r-1}{2} \leq g \leq \binom{r}{2}$, with $r \geq 4$, we prove that a generic line bundle $\mathcal{L}$ with $d \geq g + 2r - 1$ embeds C in $\mathbb{P}^{d-g}$ as a projectively normal curve whose ideal is generated by quadrics, thus improving a result in [5].

2. Projectively normal curves defined by quadratic forms.

Let $r, s \in \mathbb{Z}$ be such that $r \geq 4$ and $s = \binom{r+1}{2} + 2$.

Let $Z = \{P_1, \ldots P_s\}$ be a set of generic points of $\mathbb{P}^2$ such that no $r + 1$ of them are collinear and let X_s be the blowing-up of $\mathbb{P}^2$ with center $P_1, \ldots P_s$.

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On X_s we consider the complete linear system

$$|D_{r+1}| = |(r+1)E_0 - E_1 - ... - E_s|.$$

Since the least t for which $\Delta H_Z(t) = 0$ is just $\sigma = r+1$, where $\Delta H_Z(t)$ denotes the first difference of the Hilbert function of Z, the linear system $|D_{r+1}|$ is very ample (see [3], Theorem 3.1) and embeds X_s in $\mathbb{P}^{2r}$ as a smooth rational surface V of degree $degV = \binom{r+2}{2} - 2$.

The surface V is arithmetically Cohen-Macaulay (see [7], Proposition 2.1) and its ideal $\mathcal{I}_V$ is given via two matrices of linear forms X an Y of size $3 \times (r-1)$ and 2×3, respectively. Namely the generators of $\mathcal{I}_V$ are the 2×2 minors of X and the entries of the $2 \times (r-1)$ matrix $Y \cdot X$ (see [9], Section 4).

Hence $\mathcal{I}_V$ is generated by quadrics.

In the divisor class of $(r+1)E_0 - 2E_1 - ... - 2E_n - E_{n+1} - ... - E_s$ on X_s, with $0 \le n \le r-1$, we consider the strict transform $\tilde{\mathrm{C}}$ of an irreducible plane curve of degree $r+1$ with n nodes as its singular locus.

One can refer to the constructions in [11], p.172 or in [6], Section 1, for a proof of the existence of such curves.

Via the very ample linear system $|D_{r+1}|$, $\tilde{\mathrm{C}}$ is embedded in $\mathbb{P}^{2r}$ as a smooth curve C of genus $g = \binom{r}{2} - n$, with $0 \le n \le r-1$, and degree $d = g + 2r - 1$.

Let $(r+1)E_0 - 2E_1 - ... - 2E_n - E_{n+1} - ... - E_s$ be the divisor class of C on V.

Notice that $C + E_1 + ... + E_n = H$, where H is the hyperplane divisor class $(r+1)E_0 - E_1 - ... - E_s$ on V, hence $C \subset \mathbb{P}^{2r-1}$. Furthermore $h^0(V, \mathcal{I}_{C,V}(1)) = h^0(V, O_V(H - C)) = h^0(V, O_V(E_1 + ... + E_n)) = 1$, thus C is contained in only one hyperplane section of V, so C is nondegenerate in $\mathbb{P}^{2r-1}$.

Now, with these assumptions made, we show the following statements:

a) $h^1(C, O_C(1)) = 0$.

Consider the exact sequence

$$0 \to \mathcal{I}_V(1) \to \mathcal{I}_C(1) \to \mathcal{I}_{C,V}(1) \to 0.$$

Since V is arithmetically Cohen-Macaulay in $\mathbb{P}^{2r}$ we have $h^1(\mathbb{P}^{2r}, \mathcal{I}_C(1)) = h^1(V, \mathcal{I}_{C,V}(1))$.

On the other hand, $h^1(V, \mathcal{I}_{C,V}(1)) = h^1(V, O_V(E_1 + ... + E_n)) = 0$, by the Riemann-Roch Theorem on V, so we have that $h^1(\mathbb{P}^{2r}, \mathcal{I}_C(1)) = 0$.

Thus $h^0(C, O_C(1)) = h^0(\mathbb{P}^{2r}, O_{\mathbb{P}^{2r}}(1)) - h^0(\mathbb{P}^{2r}, \mathcal{I}_C(1)) + h^1(\mathbb{P}^{2r}, \mathcal{I}_C(1)) = (2r+1) - 1 = 2r$.

Applying the Riemann-Roch Theorem on C, we get

$$2r = h^0(C, O_C(1)) = h^1(C, O_C(1)) + degC + 1 - g = h^1(C, O_C(1)) + 2r,$$

and so $h^1(C, O_C(1)) = 0$, as required.

b) *The curve C is projectively normal.*

From above we know that the surface V is arithmetically Cohen-Macaulay in $\mathbb{P}^{2r}$, hence $h^1(\mathbb{P}^{2r}, \mathcal{I}_V(\lambda)) = h^2(\mathbb{P}^{2r}, \mathcal{I}_V(\lambda)) = 0$ for all $\lambda \in \mathbb{Z}$.

Thus, from the exact sequence

$$0 \to \mathcal{I}_V(\lambda) \to \mathcal{I}_C(\lambda) \to \mathcal{I}_{C,V}(\lambda) \to 0,$$

we deduce that for all $\lambda > 0$

$$h^1(\mathbb{P}^{2r}, \mathcal{I}_C(\lambda)) = 0 \;\Leftrightarrow\; h^1(V, \mathcal{I}_{C,V}(\lambda)) = h^1(V, O_V(\lambda H - C)) = 0.$$

Since $h^1(C, O_C(1)) = 0$, the curve C is projectively normal if, and only if, it is linearly and quadratically normal (e.g. see [2], p. 222]).

In a) we have already proved that $h^1(\mathbb{P}^{2r}, \mathcal{I}_C(1)) = 0$, so it remains to show that C is quadratically normal in $\mathbb{P}^{2r}$.

Consider $2H - C = (r+1)E_0 - E_{n+1} - ... - E_s$; by construction, the points $P_{n+1}, ..., P_s$ impose independent conditions to plane curves of degree $r+1$, so

$$0 = h^1(\mathbb{P}^2, \mathcal{I}(r+1)) = h^1(X_s, O_{X_s}((r+1)E_0 - E_{n+1} - ... - E_s)) =$$

$$= h^1(V, O_V(2H - C)),$$

where $\mathcal{I}$ is the ideal sheaf in $O_{\mathbb{P}^2}$ associated to the P_i's, $i = n+1, ..., s$.

Thus $h^1(\mathbb{P}^{2r}, \mathcal{I}_C(2)) = 0$, as required.

c) *The ideal of C is generated by quadrics.*

Since the ideal of V is generated by quadrics, it is enough to prove that the curve C has its ideal generated in degree 2 in the homogeneous coordinate ring of V.

LEMMA 1. *Let $\Gamma \subseteq \mathbb{P}^N$ be a projectively normal curve of degree d and genus g, with $N \geq 4$ and $0 \leq i \leq N$, where $i = h^1(\Gamma, O_\Gamma(1))$. Then:*

(i) *if $i = 0$, Γ is defined by forms of degree ≤ 3;*

(ii) *if $i > 0$, Γ is defined by forms of degree ≤ 4.*

Proof. Consider the exact sequence

$$0 \to \mathcal{I}_\Gamma(m) \to O_{\mathbb{P}^N}(m) \to O_\Gamma(m) \to 0.$$

According to Mumford's definition of regularity of a sheaf on $\mathbb{P}^N$, we say that the ideal sheaf $\mathcal{I}_\Gamma$ of the curve $\Gamma \subseteq \mathbb{P}^N$ is m-regular if $H^j(\mathbb{P}^N, \mathcal{I}_\Gamma(m-j) = 0$ for all $j > 0$.

Since the curve Γ is projectively normal, when

$$i = h^1(\Gamma, O_\Gamma(1)) = h^2(\mathbb{P}^N, \mathcal{I}_\Gamma(1)) = 0,$$

$\mathcal{I}_\Gamma$ is 3-regular, and this implies that Γ is generated by forms of degree ≤ 3 (e.g. see [4], p. 516).

When $i = h^1(\Gamma, O_\Gamma(1)) > 0$, with similar arguments we prove that $\mathcal{I}_\Gamma$ is 4-regular, hence that $\mathcal{I}_\Gamma$ is generated by forms of degree ≤ 4. In fact, we have $h^1(\mathbb{P}^N, \mathcal{I}_\Gamma(3)) = 0$, $h^2(\mathbb{P}^N, \mathcal{I}_\Gamma(2)) = h^1(\Gamma, O_\Gamma(2)) = 0$ for degree reasons (using that $i \leq N$), while $h^3(\mathbb{P}^N, \mathcal{I}_\Gamma(1)) = h^2(\Gamma, O_\Gamma(2))$ is trivially zero.

□

From Lemma 1, we know that the ideal of C can be always generated by forms of degree ≤ 3, hence what we have to show is equivalent to prove that the map

$$\eta : H^0(V, O_V(2H - C)) \otimes H^0(V, O_V(H)) \to H^0(V, O_V(3H - C))$$

is surjective, i.e. that $H^0(V, \mathcal{I}_{C,V}(2)) \otimes H^0(V, O_V(1))$ surjects on $H^0(V, \mathcal{I}_{C,V}(3))$.

We work by induction on the number n of nodes of the plane model of C. If $n = 0$, the assertion is obvious, as V is generated by quadrics and C coincides with a hyperplane section of V.

Suppose the statement is true for $n - 1$ and consider the exact sequence

$$0 \to O_V(2H - C - E_n) \to O_V(2H - C) \to O_{E_n}(2H - C) \to 0, \tag{1}$$

where $2H - C - E_n = H + E_1 + ... + E_{n-1} = (r + 1)E_0 - E_n - E_{n+1} - ... - E_s$.

Since $h^1(V, O_V(2H-C-E_n)) = 0$ (the points $P_n, ..., P_s$ impose, by construction, independent conditions to plane curves of degree $r + 1$), we can apply the following lemma.

LEMMA 2 (SEE [12], P.46). *Let $\mathcal{L}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be coherent sheaves on a scheme X. If*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is an exact sequence and $H^1(X, \mathcal{F}_1) = 0$, then the sequence

$$S(\mathcal{F}_1, \mathcal{L}) \to S(\mathcal{F}_2, \mathcal{L}) \to S(\mathcal{F}_3, \mathcal{L})$$

is exact, where each $S(\mathcal{F}_i, \mathcal{L})$ is defined by the exact sequence

$$H^0(X, \mathcal{F}_i) \otimes H^0(X, \mathcal{L}) \to H^0(X, \mathcal{F}_i \otimes \mathcal{L}) \to S(\mathcal{F}_i, \mathcal{L}).$$

Thus, from (1), we get the exact sequence:

$$S(O_V(2H - C - E_n), O_V(H)) \to S(O_V(2H - C), O_V(H)) \to$$

$$\to S(O_{E_n}(2H - C), O_V(H)).$$

By the induction hypothesis $S(O_V(2H - C - E_n), O_V(H)) = 0$, hence we will have $S(O_V(2H - C), O_V(H)) = 0$ whenever $S(O_{E_n}(2H - C), O_V(H)) = 0$, i.e. if the map

$$\psi : H^0(E_n, O_{E_n}(2H - C)) \otimes H^0(V, O_V(H)) \to H^0(E_n, O_{E_n}(3H - C))$$

is surjective.

Since $E_n.(2H - C) = E_n.(H + E_1 + ... + E_n) = 0$, we have $O_{E_n}(3H - C) = O_{E_n}(2H + E_1 + ... + E_n) \cong O_{E_n}(H) \otimes O_{E_n}(H + E_1 + ... + E_n) \cong O_{E_n}(H) \otimes O_{E_n} \cong O_{E_n}(H)$, thus ψ multiplies the sections of $O_V(H)$ by constants and then it restricts them to E_n. So ψ is trivially surjective, as we wanted to show, and this implies $S(O_V(2H - C), O_V(H)) = 0$ and η is surjective as required

Now we can prove our main result.

PROPOSITION 1. *Let $r, g \in \mathbf{Z}$ be such that $r \geq 4$ and $\binom{r-1}{2} \leq g \leq \binom{r}{2}$. Let C be a generic curve of genus g and $\mathcal{L} \in W_d^{d-g}(C)$ a generic line bundle of degree $d \geq g + 2r - 1$. Then the embedding of C in $\mathbb{P}^{d-g}$ via the line bundle $\mathcal{L}$ is projectively normal and its ideal is generated by quadrics.*

Proof. We recall that, if a generic line bundle $\mathcal{L}$ of degree d on a smooth curve C is normally generated, then also the generic line bundle of degree $d + 1$ on C is normally generated (see [8], p. 129), hence it is enough to prove the statement of the proposition in the case $d = g + 2r - 1$.

We reconsider the curve $C \subset \mathbb{P}^{2r-1}$ of genus $\binom{r-1}{2} \leq g \leq \binom{r}{2}$ and degree $d = g + 2r - 1$ on the projective embedding of X_s, $s = \binom{r+1}{2} + 2$, via the very ample linear system $|(r + 1)E_0 - E_1 - ... - E_s|$.

By [5], Lemma 4.7, $C \in V'$, where V' is the open subset of the Hilbert scheme $Hilb_{d,g}^{2r-1}$ whose points parametrize smooth projectively normal curves defined by quadratic forms.

Since $h^0(C, O_C(-1) \otimes \omega_C) = h^1(C, O_C(1)) = 0$, the map

$$\mu_0 : H^0(O_C(1)) \otimes H^0(O_C(-1) \otimes \omega_C) \to H^0(\omega_C)$$

is the 0-map, which is trivially injective, hence, by the following proposition, V' has general moduli.

PROPOSITION 2 (SEE [14] AND [1]). *Let $C \subset \mathbb{P}^r$, with $r \geq 3$, be a nondegenerate curve of degree d and genus g. Assume that*

(i) $h^0(C, O_C(1)) = r + 1$

(ii) *the natural map* $\mu_0 : H^0(O_C(1)) \otimes H^0(O_C(-1) \otimes \omega_C) \to H^0(\omega_C)$ *is injective.*

Then $H^1(N_C) = 0$, where N_C is the normal line bundle of C in $\mathbb{P}^r$, i.e. C is parametrized by a smooth point of $Hilb_{d,g}^r$, which will belong to an unique open set V parametrizing nondegenerate curves of genus g and degree d, and V has general moduli.

This implies that the generic curve of genus g can be embedded in $\mathbb{P}^{2r-1}$ as a projectively normal curve defined by quadrics, with the same degree as the curve C, by some very ample invertible sheaf $\mathcal{L} \in W_d^{2r-1}(C)$.

We conclude the proof by recalling that, for any smooth curve C of V', $W_{g+2r-1}^{2r-1}(C)$ is irreducible, so the generic element of $W_{g+2r-1}^{2r-1}(C)$ will embed C as a projectively normal curve whose ideal is generated by quadrics (see [6], Theorem 3.1 or [5]).

□

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Marina MANCINI
Dipartimento di Matematica
Università di Bologna
P.za S. Donato 5
40127 Bologna, ITALIA
e-mail: mancinm@libero.it

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