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CHARACTERIZATION, SPECTRAL INVARIANCE AND THE FREDHOLM PROPERTY OF MULTI-QUASI-ELLIPTIC OPERATORS

Abstract. The class $L^0_{\rho,\mathcal{P}}(\mathbb{R}^n)$ of pseudodifferential operators of zero order, modelled on a multi-quasi-elliptic weight, is shown to be a Ψ^*-algebra in the algebra $\mathcal{B}(L^2(\mathbb{R}^n))$ of all bounded operators on $L^2(\mathbb{R}^n)$. Moreover, the Fredholm property is proven to characterize the elliptic elements in this algebra. This is achieved through a characterization of these operators in terms of the mapping properties between the Sobolev spaces $H^s_{\mathcal{P}}(\mathbb{R}^n)$ of their iterated commutators with multiplication operators and vector fields. We also prove and make use of the fact that order reduction holds in the scale of the $H^s_{\mathcal{P}}(\mathbb{R}^n)$-Sobolev spaces, that is every $H^s_{\mathcal{P}}(\mathbb{R}^n)$ is homeomorphic to $L^2(\mathbb{R}^n)$ through a suitable multi-quasi-elliptic operator of order s.

1. Introduction. Statement of the results.

Multi-quasi-elliptic polynomials were introduced in the seventies as a natural generalization of elliptic and quasi-elliptic polynomials. They are an important subclass of the hypoelliptic polynomials of Hörmander [15] and were studied by many authors, among them Friberg [9], Cattabriga [8], Zanghirati [24], Pini [17] and Volevic-Gindikin [23].

In [5] this theory is used to develop a pseudodifferential calculus for a class of operators on weighted Sobolev spaces in $\mathbb{R}^{2n}$ based on the concept of a "Newton polyhedron".

DEFINITION 1. *A* complete Newton polyhedron $\mathcal{P}$ *is a polyhedron of dimension d in $\mathbb{R}^d_+ = \{r \in \mathbb{R}^d : r_j \geq 0, j = 1, \ldots, d\}$ with integer vertices and the following properties:*

(i) If $v^{(k)}, k = 1, \ldots, N$ are the vertices of $\mathcal{P}$, then $\{r \in \mathbb{R}^d_+ : r_j \leq v^{(k)}_j, j = 1, \ldots, d\} \subseteq \mathcal{P}$.

(ii) There are finitely many elements $a^{(l)}, l = 1, \ldots, M$, with $a^{(l)}_j > 0$ for $j = 1, \ldots, d$, such that

$$\mathcal{P} = \{r \in \mathbb{R}^d_+ : \langle a^{(l)}, r\rangle \leq 1, l = 1, \ldots, M\}.$$

In other words, $\mathcal{P}$ may contain only points with non-negative coordinates and must have one face on each coordinate hyperplane, while the other faces must have normal vectors with all components strictly positive; in particular these faces must not be parallel to the coordinate hyperplanes.

Given a polynomial $p = \sum c_\gamma z^\gamma$ in d complex variables, we associate with it the polyhedron $\mathcal{P}$ consisting of the convex hull of all $\gamma \in \mathbb{N}_0^d$ with $c_\gamma \neq 0$. It is called *multi-quasi-elliptic* if the associated polyhedron is a complete Newton polyhedron and there exist constants C, R, such that:

$$\sum_{\gamma \in \mathcal{P}} |z^\gamma| \leq C|p(z)| \qquad \text{for } |z| \geq R.$$

Here and in the following the notation $\sum_{\gamma \in \mathcal{P}}$ indicates that we sum over all multi-indices $\gamma \in \mathbb{N}_0^n \cap \mathcal{P}$, so that the sum is clearly finite.
The corresponding operators $P(D_z) = \sum_{\gamma \in \mathcal{P}} c_\gamma D_z^\gamma$ are also said to be multi-quasi-elliptic.

We now consider the case of even dimension $d = 2n$, $n \geq 1$, and split $z \in \mathbb{R}^{2n}$ into $z = (x, \xi)$, where $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$. Giving to ξ the role of a covariable, we recover the case of operators with polynomial cefficients:

$$p(x, D) = \sum_{(\alpha,\beta) \in \mathcal{P}} x^\alpha D_x^\beta.$$

Easy but significant examples are the operators on $\mathbb{R}$ of the form:

$$P = x^{2h_1} + x^{2h_0} D^{2k_0} + D^{2k_1}.$$

If $h_0, h_1, k_0, k_1 \in \mathbb{N}$ satisfy the conditions:

$$0 < h_0 < h_1, \qquad 0 < k_0 < k_1, \qquad \frac{h_0}{h_1} + \frac{k_0}{k_1} > 1,$$

then to P is associated the complete polyhedron $\mathcal{P}$ with vertices $\{(0,0), (2h_1, 0), (2h_0, 2k_0), (0, k_1)\}$, and P is multi-quasi-elliptic with respect to $\mathcal{P}$.

The symbols $\sigma(P) = x^{2h_1} + x^{2h_0}\xi^{2k_0} + \xi^{2k_1}$ associated with these operators were originally considered by Gorcakov [10] and Pini [17] and later studied by Friberg in connection with differential operators with constants coefficients.

As an example of multi-quasi-elliptic operators in dimension n let us consider the operators of the form

$$\Delta_x^k + \sum_{|\alpha+\beta|<k} x^{2\alpha} D_x^{2\beta} + |x|^{2k}.$$

They are multi-quasi-elliptic. The associated polyhedron is $\mathcal{P} = \{z \in \mathbb{R}_+^n \times \mathbb{R}_+^n : \sum_{j=1}^{2n} z_j \leq 2k\}$. Unlike the class of quasi-elliptic operators, the space of multi-quasi-elliptic operators is closed under composition: If A_1 and A_2 are multi-quasi-elliptic with respect to $w_{\mathcal{P}_1}$ and $w_{\mathcal{P}_2}$, then the operator $A_1 \circ A_2$ is multi-quasi-elliptic with respect to $w_{\mathcal{P}_1+\mathcal{P}_2}$ (where $\mathcal{P}_1+\mathcal{P}_2 = \{z \in \mathbb{R}^{2n} : z = z_1+z_2 \text{ for some } z_1 \in \mathcal{P}_1; z_2 \in \mathcal{P}_2\}$).

In particular, any product of quasi-elliptic operators is multi-quasi-elliptic although it is, in general, no more quasi-elliptic. On the other hand there are multi-quasi-elliptic operators that are not product of quasi-elliptic operators (see Friberg [9]).

More generally, for a fixed complete polyhedron $\mathcal{P}$, one can introduce pseudodifferential operators in classes suitably modelled on the polyhedron $\mathcal{P}$. Before stating our results we summarize here some facts about this theory following [5], [3].

DEFINITION 2. *Let* $w_{\mathcal{P}}(z) = (\sum_{\gamma \in \mathcal{P}} z^{2\gamma})^{1/2}$, $z = (x, \xi) \in \mathbb{R}^{2n}$, *be the standard weight function associated to* $\mathcal{P}$. *For* $m \in \mathbb{R}$, $\rho \in]0, 1]$, *we define the symbol classes:*

$$\Lambda^m_{\rho,\mathcal{P}}(\mathbb{R}^{2n}) = \{a \in C^\infty(\mathbb{R}^{2n}) : \ |\partial_z^\gamma a(z)| \le C_\gamma w_{\mathcal{P}}^{m-\rho|\gamma|}(z)\}.$$

The choice of the best constants defines a Fréchet topology for $\Lambda^m_{\rho,\mathcal{P}}(\mathbb{R}^{2n})$.

For technical reasons we will also assume that ρ does not exceed a certain value $1/\mu < 1$, where μ is the "formal order" of $p(z)$, depending on the polyhedron $\mathcal{P}$. This is of no importance for our purposes, see [3] for more details. The corresponding classes of operators are:

$$\mathrm{L}^m_{\rho,\mathcal{P}} = \{A = Op(a) : a \in \Lambda^m_{\rho,\mathcal{P}}(\mathbb{R}^{2n})\}.$$

Here $[Op(a)u](x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} a(x, \xi)\hat{u}(\xi)d\xi$, while $\hat{u}(\xi) = \int e^{-ix\xi} u(x)dx$ for $u \in S(\mathbb{R}^n)$, the Schwartz space of all rapidly decreasing functions.

DEFINITION 3. $E\Lambda^m_{\rho,\mathcal{P}}$ *is the space of all* $a \in \Lambda^m_{\rho,\mathcal{P}}(\mathbb{R}^{2n})$ *such that, with suitable constants* $C, R > 0$,

$$w^m_{\mathcal{P}}(z) \le C|a(z)|, \quad |z| \ge R.$$

The elements of $E\Lambda^m_{\rho,\mathcal{P}}$ *are called multi-quasi-elliptic symbols, the corresponding classes of operators are denoted with*

$$E\mathrm{L}^m_{\rho,\mathcal{P}} = \{A = Op(a) : \ a \in E\Lambda^m_{\rho,\mathcal{P}}\}.$$

Being a natural generalization of the classes considered by Friberg and Cattabriga, we call these operators multi-quasi-elliptic of order m.

For $m = 0$, multi-quasi-ellipticity of a symbol a requires that $|a(x, \xi)| \ge c > 0$, so multi-quasi-ellipticity then coincides with uniform ellipticity.

In case $\mathcal{P}$ is the simplex with vertices the origin and the points $\{e^{(k)}\}_{k=1}^{2n}$ of the canonical basis of $\mathbb{R}^{2n}$, we are reduced to Shubin's classes $\Gamma^m_\rho(\mathbb{R}^{2n})$, see Shubin [21], Chapter IV; Helffer [14], Chapter I.

DEFINITION 4. *Let* $\Lambda_s = Op(w^s_{\mathcal{P}})$. *For* $s \in \mathbb{R}$ *we define the Sobolev spaces*

$$H^s_{\mathcal{P}}(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) : \Lambda_s u \in L^2(\mathbb{R}^n)\}.$$

The pseudodifferential operators with symbols in $\Lambda^m_{\rho,\mathcal{P}}(\mathbb{R}^{2n})$ *act continuously on these spaces: For* $a \in \Lambda^m_{\rho,\mathcal{P}}(\mathbb{R}^{2n})$ *and* $s \in \mathbb{R}$,

$$Op(a) : H^s_{\mathcal{P}}(\mathbb{R}^n) \to H^{s-m}_{\mathcal{P}}(\mathbb{R}^n)$$

is bounded and the operator norm can be estimated in terms of the symbol semi-norms of a.

The first result we present is that the pseudodifferential operators in $\mathrm{L}^0_{\rho,\mathcal{P}}$ can be characterized by commutators, in fact we have

THEOREM 1. *A linear operator* $A : S(\mathbb{R}^n) \to S'(R^n)$ *belongs to* $\mathrm{L}^0_{\rho,\mathcal{P}}$ *if and only if, for all multi-indices,* $\alpha, \beta \in \mathbb{N}^n_0$, *the iterated commutators* $\mathrm{ad}^\alpha(x)\mathrm{ad}^\beta(D_x)A$ *have bounded extensions*

$$\mathrm{ad}^\alpha(x)\mathrm{ad}^\beta(D_x)A : L^2(\mathbb{R}^n) \to H^{\rho|\alpha+\beta|}_{\mathcal{P}}(\mathbb{R}^n).$$

Theorem 1 is the key tool in the proof of the spectral invariance result for $\mathrm{L}^0_{\rho,\mathcal{P}}$:

THEOREM 2. *Let* $A \in \mathrm{L}^0_{\rho,\mathcal{P}}$, *and suppose that* A *is invertible in* $\mathcal{B}(L^2(\mathbb{R}^n))$. *Then* $A^{-1} \in \mathrm{L}^0_{\rho,\mathcal{P}}$.

The idea of characterizing pseudodifferential operators by commutators and using this for showing the spectral invariance goes back to R. Beals. In [1], Beals had introduced a calculus with pseudodifferential operators having symbols in very general classes $S^\lambda_{\Phi,\varphi}$. In [1] he stated the analogs of Theorems 1 and 2 under the restriction $\varphi = \Phi$. Since the proof was not generally accepted, Ueberberg in 1988 published an article where he showed corresponding versions of Theorems 1 and 2 for the operators with symbols in Hörmander's classes $S^0_{\rho,\delta}$, $0 \leq \delta \leq \rho \leq 1, \delta < 1$. Let us point out, however, that the symbols we are considering are not contained in the class introduced by Beals, since our case corresponds to the choice

$$\varphi(x,\xi) = \Phi(x,\xi) = \omega^\rho_{\mathcal{P}}(x,\xi),$$

which does not satisfy Beals' condition $\varphi \leq$const. In 1994, Bony and Chemin [6] proved analogous of the above theorems for a large class of symbols using the Hörmander-Weyl quantization:

$$[Op^w a]u(x) = (2\pi)^{-n} \int\int e^{i(x-y)\xi} a(\frac{x+y}{2},\xi)u(y)dyd\xi.$$

Multi-quasi-elliptic operators can be naturally re-considered in the frame of the Weyl-calculus, see [4], so that if one can show that the metric

$$g_{x,\xi} = \omega^\rho_{\mathcal{P}}(x,\xi)(|dx|^2 + |d\xi|^2)$$

satisfies the conditions ”Lenteur”, ”Principe d'Incertitude” (which are actually obvious), and ”Tempérence (Forte)” (not evident in our case) of [6], then Theorems 1 and 2 could be deduced from [6], Théorèmes 5.5 and 7.6. It seems however that the present direct proof, which relies on the spectral invariance result for $S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$, shows an easier, alternative technique.

From Theorems 1 and 2 we will obtain

THEOREM 3. *$\mathrm{L}^0_{\rho,\mathcal{P}}$ is a submultiplicative Ψ^* -subalgebra of $\mathcal{L}(L^2(\mathbb{R}^n))$.*

Ψ^* algebras were introduced by Gramsch [11], Definition 5.1: A subalgebra $\mathcal{A}$ of $\mathcal{L}(H)$, H a Hilbert, space is called a Ψ^*-subalgebra, if

(i) it carries a Fréchet topology which is stronger than the norm topology of $\mathcal{L}(H)$,

(ii) it is symmetric, i.e., $\mathcal{A}^* = \mathcal{A}$, and

(iii) it is spectrally invariant, i.e., $\mathcal{A} \cap \mathcal{L}(H)^{-1} = \mathcal{A}^{-1}$.

Here, $\mathcal{A}^{-1}$ and $\mathcal{L}(H)^{-1}$ denote the groups of invertible elements in the respective algebras. Ψ^*-algebras have many important features: in Ψ^*-algebras there is a holomorphic functional calculus in several complex variables, there are results in Fredholm and perturtation theory [11] as well as for periodic geodesics. Concerning decompositions of inverses to analytic Fredholm functionals and the division problem for operator-valued distributions, see [12]. For further results on the Ψ^*-property see also [13], [14], [18], [19], [20].

The Fréchet topology on $\mathrm{L}^0_{\rho,\mathcal{P}}$ is the one induced from $\Lambda^0_{\rho,\mathcal{P}}(\mathbb{R}^{2n})$ via the injective mapping Op. A Fréchet algebra $\mathcal{A}$ is called submultiplicative if there is a system of semi-norms $\{q_j : j \in \mathbb{N}\}$ which defines the topology and satisfies

$$q_j(ab) \le q_j(a)q_j(b), \quad a, b \in \mathcal{A}.$$

We next show the existence of order reduction within this calculus, see Lemma 2. This allows us to extend the results on the characterization by commutators and spectral invariance to the case of arbitrary order.

Finally we characterize the Fredholm property by multi-quasi-ellipticity.

THEOREM 4. *Let $A \in \mathrm{L}^m_{\rho,\mathcal{P}}$, $s \in \mathbb{R}$. Then $A : H^s_{\mathcal{P}}(\mathbb{R}^n) \to H^{s-m}_{\mathcal{P}}(\mathbb{R}^n)$ is a Fredholm operator if and only if it is multi-quasi-elliptic of order m.*

Let $A \in \mathrm{L}^m_{\rho,\mathcal{P}}$, $s \in \mathbb{R}$. Then $A : H^s_{\mathcal{P}}(\mathbb{R}^n) \to H^{s-m}_{\mathcal{P}}(\mathbb{R}^n)$ is a Fredholm operator if and only if it is multi-quasi-elliptic of order m.

The links between pseudodifferential calculus and quantization are pointed out in Berezin-Shubin [7]. Before going on to the technical part of the paper let us recall two basic examples of multi-quasi-elliptic operators that come up in quantum mechanics. For a more detailed exposition, as well as for further examples and motivations, we refer to [5], [3], [4].

Let $p(x)$ be a positive multi-quasi-elliptic polynomial in the variables $x \in \mathbb{R}^n$. Then the Schrödinger operator $P = -\Delta_x + p(x)$, a generalization of the harmonic oscillator of quantum mechanics, is multi-quasi-elliptic in the sense of our previous definition.

A slight modification of the above operators $P = x^{2h_1} + x^{2h_0} D^{2k_0} + D^{2k_1}$ yields self-adjoint multi-quasi-elliptic operators of the form:

$$P = x^{2h_1} + D^{k_0}\left(x^{2h_0} D^{k_0}\right) + D^{2k_1}.$$

In both cases one would like to have spectral asymptotic estimates for these operators, in particular for the function $N(\lambda) = \sum_{\lambda_j \le \lambda} 1$ counting the number of the eigenvalues λ_j not exceeding λ ([3] can be considered a first step in this direction). The idea is to follow the approach taken by Helffer in [14] which is based on a thorough analysis of Shubin's classes, and this makes the results of spectral invariance and their consequences particularly interesting.

2. Pseudodifferential operators in $L^m_{\rho,\mathcal{P}}$

We now review the main properties of the multi-quasi-elliptic calculus assuming familiarity with the standard pseudodifferential calculus, cf. [16].

PROPOSITION 1. *Let* $m, m_1, m_2 \in \mathbb{R}$.

(a) $\Lambda^m_{\rho,\mathcal{P}}(\mathbb{R}^d)$ *is a vector space.*

(b) If $a_1 \in \Lambda^{m_1}_{\rho_1,\mathcal{P}}(\mathbb{R}^d)$, $a_2 \in \Lambda^{m_2}_{\rho_2,\mathcal{P}}(\mathbb{R}^d)$, *then* $a_1 a_2 \in \Lambda^{m_1+m_2}_{\min(\rho_1,\rho_2),\mathcal{P}}(\mathbb{R}^d)$.

(c) For every multi-index $\alpha \in \mathbb{N}^d$ *we have* $\partial^\alpha_\zeta a \in \Lambda^{m-\rho|\alpha|}_{\rho,\mathcal{P}}(\mathbb{R}^d)$.

(d) $\bigcap_{m\in\mathbb{R}} \Lambda^m_{\rho,\mathcal{P}}(\mathbb{R}^d) = S(\mathbb{R}^d)$, *the Schwartz space of rapidly decreasing functions.*

DEFINITION 5. *Let* $a_j \in \Lambda^{m_j}_{\rho,\mathcal{P}}(\mathbb{R}^d)$ *and* $m_j \to -\infty$ *for* $j \to +\infty$. *We write* $a \sim \sum_{j=1}^\infty a_j$ *if* $a \in C^\infty(\mathbb{R}^d)$ *and* $a - \sum_{j=1}^{r-1} a_j \in \Lambda^{\tilde{m}_r}_{\rho,\mathcal{P}}(\mathbb{R}^d)$ *where* $\tilde{m}_r = \max_{j\ge r} m_j$. *We then have* $a \in \Lambda^m_{\rho,\mathcal{P}}(\mathbb{R}^d)$, $m = \max_{j\ge 1} m_j$.

PROPOSITION 2. *Given* $a_j \in \Lambda^{m_j}_{\rho,\mathcal{P}}(\mathbb{R}^d)$ *with* $m_j \to -\infty$ *as* $j \to +\infty$ *there exists* $a \in C^\infty(\mathbb{R}^d)$ *such that* $a \sim \sum_{j=1}^\infty a_j$. *Furthermore, if b is another function such that* $b \sim \sum_{j=1}^\infty a_j$, *then* $a - b \in S(\mathbb{R}^d)$.

PROPOSITION 3. *Let* $A_1 = \mathrm{Op}\, a_1 \in \mathrm{L}^{m_1}_{\rho,\mathcal{P}}$ *and* $A_2 = \mathrm{Op}\, a_2 \in \mathrm{L}^{m_2}_{\rho,\mathcal{P}}$. *Then* $A_1 A_2 \in \mathrm{L}^{m_1+m_2}_{\rho,\mathcal{P}}$, *and the symbol* $\sigma(A_1 A_2)$ *of* $A_1 A_2$ *has the asymptotic expansion* $\sigma(A_1 A_2) \sim \sum_\alpha \frac{1}{\alpha!} \partial^\alpha_\xi a_1(x,\xi) D^\alpha_x a_2(x,\xi)$.

PROPOSITION 4. *(a) If $a \in E\Lambda^{m}_{\rho,\mathcal{P}}$ then $a^{-1} \in E\Lambda^{-m}_{\rho,\mathcal{P}}$ and $a^{-1}\partial^{\alpha}a \in E\Lambda^{-\rho|\alpha|}_{\rho,\mathcal{P}}$ for all α (possibly after a modification of $a(\zeta)$ on a compact set).*

(b) If $a_1 \in E\Lambda^{m_1}_{\rho,\mathcal{P}}$ and $a_2 \in E\Lambda^{m_2}_{\rho,\mathcal{P}}$ then $a_1 a_2 \in E\Lambda^{m_1+m_2}_{\rho,\mathcal{P}}$.

(c) If $A_1 \in E\mathrm{L}^{m_1}_{\rho,\mathcal{P}}$ and $A_2 \in E\mathrm{L}^{m_2}_{\rho,\mathcal{P}}$ then $A_1 A_2 \in E\mathrm{L}^{m_1+m_2}_{\rho,\mathcal{P}}$.

DEFINITION 6. *An operator $R \in \bigcap_{m\in\mathbb{R}} \mathrm{L}^{m}_{\rho,\mathcal{P}}$ is called* regularizing *or* smoothing. *Regularizing operators define continuous maps $R : S'(\mathbb{R}^n) \to S(\mathbb{R}^n)$. They have integral kernels $R = R(x, y) \in S(\mathbb{R}^n_x \times \mathbb{R}^n_y)$.*

PROPOSITION 5 (EXISTENCE OF THE PARAMETRIX). *Let $A \in E\mathrm{L}^{m}_{\rho,\mathcal{P}}$; then there exists an operator $B \in E\mathrm{L}^{-m}_{\rho,\mathcal{P}}$ such that the operators $R_1 = AB - I$ and $R_2 = BA - I$ both are regularizing. B is said to be a* parametrix *to A. If B' is another parametrix to the same operator A, then $B - B'$ is a regularizing operator.*

PROPOSITION 6 (ELLIPTIC REGULARITY). *Let $A \in E\mathrm{L}^{m}_{\rho,\mathcal{P}}$. If $Au \in S(\mathbb{R}^n)$ for some $u \in S'(\mathbb{R}^n)$ then necessarily $u \in S(\mathbb{R}^n)$.*

Definition 4 of the Sobolev spaces $H^m_{\mathcal{P}}(\mathbb{R}^n)$ can be rephrased.

PROPOSITION 7. *Let $\mathcal{P}$ be a fixed complete Newton polyhedron and let $A_m \in EL^{m}_{\rho,\mathcal{P}}$ be a multi-quasi-elliptic operator; then*

$$H^m_{\mathcal{P}}(\mathbb{R}^n) = A_m^{-1}(L^2(\mathbb{R}^n)).$$

Note that $H^m_{\mathcal{P}}(\mathbb{R}^n)$ depends neither on ρ nor on the particular operator A_m, but only on m and $\mathcal{P}$.

The main features of these spaces are the following.

PROPOSITION 8. *$H^m_{\mathcal{P}}(\mathbb{R}^n)$ has a Hilbert space structure given by the inner product $(u, v)_{\mathcal{P}} = (A_m u, A_m v)_{L^2} + (Ru, Rv)_{L^2}$. Here A_m is an elliptic operator defining the space $H^m_{\mathcal{P}}(\mathbb{R}^n)$ according to Proposition 7, and $R = I - \widetilde{A}_m A_m$, with a parametrix $\widetilde{A}_m$ of A_m. We denote by $||u||_m$ the norm of an element u in the space $H^m_{\mathcal{P}}(\mathbb{R}^n)$. Equivalently, we could define*

$$H^1_{\mathcal{P}}(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) : x^{\alpha} D^{\beta} u \in L^2(\mathbb{R}^n) \text{ for } (\alpha, \beta) \in \mathcal{P}\}.$$

with the inner product $(u, v)_{\Lambda_{\mathcal{P}}} = \sum_{(\alpha,\beta)\in\mathcal{P}} (x^{\alpha} D^{\beta} u, x^{\alpha} D^{\beta} v)_{L^2}$.

PROPOSITION 9. *(a) The topological dual $H^{m'}_{\mathcal{P}}(\mathbb{R}^n)$ of $H^m_{\mathcal{P}}(\mathbb{R}^n)$ is $H^{-m}_{\mathcal{P}}(\mathbb{R}^n)$.*

(b) We have continuous imbeddings $S(\mathbb{R}^n) \hookrightarrow H^m_{\mathcal{P}}(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n)$.

(c) We have compact imbeddings $H^t_{\mathcal{P}}(\mathbb{R}^n) \hookrightarrow H^s_{\mathcal{P}}(\mathbb{R}^n)$ if $t > s$.

(d) $\mathrm{proj} - \lim_{m\in\mathbb{R}} H^m_{\mathcal{P}}(\mathbb{R}^n) = S(\mathbb{R}^n), \qquad \mathrm{ind} - \lim_{m\in\mathbb{R}} H^m_{\mathcal{P}}(\mathbb{R}^n) = S'(\mathbb{R}^n).$

3. Abstract characterization, order reduction, spectral invariance and the Fredhold property

We now adress the central questions of this paper. We begin by proving that the operators of order zero can be characterized via iterated commutators.

DEFINITION 7. *Let* $A : S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ *be a linear operator. For* $j = 1, \dots, n$, *we set* $\mathrm{ad}^0(D_{x_j})A = A$; $\mathrm{ad}^0(x_j)A = A$; $\mathrm{ad}^k(D_{x_j})A = [D_{x_j}, \mathrm{ad}^{k-1}(D_{x_j})A]$; $\mathrm{ad}^k(x_j)A = [x_j, \mathrm{ad}^{k-1}(x_j)A]$. *For multi-indices* α, β *we let*

$$B^\alpha_\beta A = \mathrm{ad}^\alpha(x)\mathrm{ad}^\beta(D_x)A = \mathrm{ad}^{\alpha_1}(x_1)\dots\mathrm{ad}^{\alpha_n}(x_n)\mathrm{ad}^{\beta_1}(D_{x_1})\dots\mathrm{ad}^{\beta_n}(D_{x_n})A$$

THEOREM 5 (ABSTRACT CHARACTERIZATION). *A linear operator* $A : S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ *belongs to* $\mathrm{L}^0_{\rho,\mathcal{P}}$ *if and only if, for all multi-indices* α, β, *the iterated commutators* $B^\alpha_\beta A$ *have continuous extensions to linear maps:* $B^\alpha_\beta A : L^2(\mathbb{R}^n) \to H^{\rho|\alpha+\beta|}_{\mathcal{P}}(\mathbb{R}^n)$.

Proof. If $A \in \mathrm{L}^0_{\rho,\mathcal{P}}$ then the symbol of $B^\alpha_\beta A$ is $\partial^\alpha_\xi \partial^\beta_x a \in \Lambda^{-\rho|\alpha+\beta|}_{\rho,\mathcal{P}}(\mathbb{R}^{2n})$, so that clearly the commutators extend to continuous maps: $B^\alpha_\beta A : L^2(\mathbb{R}^n) \to H^{\rho|\alpha+\beta|}_{\mathcal{P}}(\mathbb{R}^n)$.

Conversely assume that A admits the required continuous extensions: $B^\alpha_\beta A : L^2(\mathbb{R}^n) \to H^{\rho|\alpha+\beta|}_{\mathcal{P}}(\mathbb{R}^n)$.

Let α_0, β_0 be arbitrary multi-indices and let $\Lambda_{\rho|\alpha_0+\beta_0|} = Op(w_{\mathcal{P}}^{\rho(|\alpha_0+\beta_0|)})$ be as in Definition 4. For all multi-indices α, β, we then have continuous maps:

$$B^\alpha_\beta[\Lambda_{\rho|\alpha_0+\beta_0|} \circ B^{\alpha_0}_{\beta_0} A] : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n). \tag{1}$$

This follows from Leibniz' rule:

$$B^\alpha_\beta[\Lambda_{\rho|\alpha_0+\beta_0|} \circ B^{\alpha_0}_{\beta_0} A] = \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} c_{\alpha_1,\alpha_2,\beta_1,\beta_2}\, B^{\alpha_2}_{\beta_2}\Lambda_{\rho|\alpha_0+\beta_0|} \circ B^{\alpha_1}_{\beta_1} B^{\alpha_0}_{\beta_0} A$$

and the continuity of the operators:

$$\begin{aligned}
&B^{\alpha_1}_{\beta_1} B^{\alpha_0}_{\beta_0} A : L^2(\mathbb{R}^n) \to H^{\rho(|\alpha_1+\alpha_0+\beta_1+\beta_0|)}_{\mathcal{P}}(\mathbb{R}^n); \\
&id : H^{\rho(|\alpha_1+\alpha_0+\beta_1+\beta_0|)}_{\mathcal{P}}(\mathbb{R}^n) \hookrightarrow H^{\rho(|\alpha_0|-|\alpha_2|+|\beta_0|-|\beta_2|)}_{\mathcal{P}}(\mathbb{R}^n); \\
&B^{\alpha_2}_{\beta_2}\Lambda_{\rho|\alpha_0+\beta_0|} : H^{\rho(|\alpha_0|-|\alpha_2|+|\beta_0|-|\beta_2|)}_{\mathcal{P}}(\mathbb{R}^n) \to L^2(\mathbb{R}^n).
\end{aligned}$$

The continuity of the first operator is due to the hypothesis and the equality $B^{\alpha_1}_{\beta_1} B^{\alpha_0}_{\beta_0} A = B^{\alpha_1+\alpha_0}_{\beta_1+\beta_0} A$ which is easily checked.

According to the characterization of the Hörmander class $S^0_{0,0}(\mathbb{R}^{2n})$, see Beals [2], Ueberberg [22], (1) implies that $\Lambda_{\rho|\alpha_0+\beta_0|} \circ B^{\alpha_0}_{\beta_0} A$ is a pseudodifferential operator with the symbol

$$b_{\alpha_0,\beta_0} = \sigma(\Lambda_{\rho|\alpha_0+\beta_0|} \circ B^{\alpha_0}_{\beta_0} A) \in S^0_{0,0}(\mathbb{R}^{2n}). \tag{2}$$

In particular, choosing $\alpha_0 = \beta_0 = 0$, we see that A is a pseudodifferential operator and $b_{0,0} = \sigma(A) = a \in S^0_{0,0}(\mathbb{R}^{2n})$.

Since $w^2_{\mathcal{P}}(x,\xi)$ is a polynomial, there exist $m \in \mathbb{R}$ and $\rho' > 0$ such that $\Lambda_{\rho|\alpha_0+\beta_0|} \in \mathrm{L}^m_{\rho',0}$. For the symbol b we therefore have the asympotic expansion:

$$b_{\alpha_0,\beta_0} \sim \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} \partial^\alpha_\xi \sigma(\Lambda_{\rho|\alpha_0+\beta_0|})\, \partial^\alpha_x \sigma(B^{\alpha_0}_{\beta_0} A),$$

where $\partial^\alpha_x \sigma(B^{\alpha_0}_{\beta_0} A) = \partial^{\alpha_0+\alpha}_x \partial^{\beta_0}_\xi a \in S^0_{0,0}(\mathbb{R}^{2n})$ and $\partial^\alpha_\xi \sigma(\Lambda_{\rho|\alpha_0+\beta_0|}) = \partial^\alpha_\xi (w^{\rho|\alpha_0+\beta_0|}_{\mathcal{P}}) \in S^{m-\rho'|\alpha|}_{\rho',0}(\mathbb{R}^{2n})$.

Next let us assume that $|\alpha_0+\beta_0| = 1$, i.e., $\partial^{\alpha_0}_\xi \partial^{\beta_0}_x a(x,\xi) = \partial_{z_j} a(z)$ with $z = (x,\xi)$ and suitable $j \in \{1,\dots,2n\}$. Using the asymptotic expansion of b_{α_0,β_0} we have, for sufficiently large $k \in \mathbb{N}$,

$$b_{\alpha_0,\beta_0} - \sum_{|\alpha|\le k} \frac{i^{|\alpha|}}{\alpha!} \partial^\alpha_\xi \sigma(\Lambda_{\rho|\alpha_0+\beta_0|}) \partial^\alpha_x \sigma(B^{\alpha_0}_{\beta_0} A) \in S^0_{0,0}(\mathbb{R}^{2n}). \tag{3}$$

In particular, the difference is a bounded function.

For $1 \le |\alpha| \le k$ the terms under the summation in (3) are products of derivatives of $a \in S^0_{0,0}(\mathbb{R}^{2n})$ and of $w^\rho_{\mathcal{P}} \in \Lambda^\rho_{\rho,\mathcal{P}}(\mathbb{R}^{2n})$. They are therefore bounded. By (2), b_{α_0,β_0} is also bounded, hence so is the term for $\alpha = 0$, namely $\partial_{z_j} a(z)\, w^\rho_{\mathcal{P}}(z)$. So we have the estimate $|\partial^{\alpha_0}_\xi \partial^{\beta_0}_x a(z)| = |\partial_{z_j} a(z)| \le C w^{-\rho}_{\mathcal{P}}(z)$, for $|\alpha_0 + \beta_0| = 1$, with a suitable constant $C \ge 0$.

We may now repeat the argument with $a(z)$ replaced by $\partial_{z_k} a(z)$, $k = 1,\dots,2n$. The operator with the symbol $\partial_{z_k}\alpha$ also satisfies the assymption of the theorem. Just as before, we see that $\partial^2_{z_j z_k} a(z) w^\rho_{\mathcal{P}}(z)$ for all $j \in \{1,\dots,2n\}$ is bounded. By iteration we conclude that $\partial^\gamma_z a(z) w^\rho_{\mathcal{P}}(z)$ is bounded for every multi-index γ. This shows that

$$\partial_{z_j} a(z) \in \Lambda^{-\rho}_{0,\mathcal{P}}(\mathbb{R}^{2n}). \tag{4}$$

Notice that we still have the subscript "0" instead of the desired "ρ". Let us now suppose $|\alpha_0 + \beta_0| = 2$. The terms of (3) with $1 \le |\alpha| \le k$ are now products of derivatives of $a(z)$ and of $w^{2\rho}_{\mathcal{P}}(z) \in \Lambda^{2\rho}_{\rho,\mathcal{P}}(\mathbb{R}^{2n})$, so that, thanks to (4), they are still bounded. Proceeding as before, we conclude that the second derivatives of $a(z)$ belong to $\Lambda^{-2\rho}_{0,\mathcal{P}}(\mathbb{R}^{2n})$. Iteration of the argument shows that $\partial^\gamma_z a(z) \in \Lambda^{-|\gamma|\rho}_{0,\mathcal{P}}(\mathbb{R}^{2n})$ for all γ, which implies $a \in \Lambda^0_{\rho,\mathcal{P}}(\mathbb{R}^{2n})$.

□

The following corollary is an immediate consequence of Theorem 5.

COROLLARY 1. *A linear operator $A : S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ belongs to $\mathrm{L}^0_{\rho,\mathcal{P}}$ if and only if, for all multi-indices α, β and for all $s \in \mathbb{R}$, the iterated commutators $B^\alpha_\beta A$ have continuous extensions to linear maps: $B^\alpha_\beta A : H^s_{\mathcal{P}}(\mathbb{R}^n) \to H^{s+\rho|\alpha+\beta|}_{\mathcal{P}}(\mathbb{R}^n)$.*

As a preparation for the proof of the spectral invariance we need the following lemma. The proof is just as in the standard case. The crucial identity one has to verify is that

$$[A^{-1}, \Lambda_\varepsilon] = -A^{-1}[A, \Lambda_\varepsilon]A^{-1}.$$

for all $A \in L^0_{\rho,\mathcal{P}}$ and a suitable $\varepsilon > 0$. As before, $\Lambda_\varepsilon = Op(\omega^\varepsilon_{\mathcal{P}})$. Equality holds, because $A \in S^0_{0,0}(\mathbb{R}^{2n})$ and $\omega_{\mathcal{P}} \in S^m_{\rho',0}(\mathbb{R}^2 n)$ for suitable $m, \rho' > 0$. For details see [22] or [21], Section I.6.

LEMMA 1. *Let $A \in \mathrm{L}^0_{\rho,\mathcal{P}}$ be invertible in the class $\mathcal{B}(L^2(\mathbb{R}^n))$ of bounded operators on $L^2(\mathbb{R}^n)$. Then A is invertible in $\mathcal{B}(H^s_{\mathcal{P}}(\mathbb{R}^n))$ for all $s \in \mathbb{R}$.*

THEOREM 6 (SPECTRAL INVARIANCE AND SUBMULTIPLICATIVITY)). *Let $A \in \mathrm{L}^0_{\rho,\mathcal{P}}$, and suppose that A is invertible in $\mathcal{B}(L^2(\mathbb{R}^n))$. Then $A^{-1} \in \mathrm{L}^0_{\rho,\mathcal{P}}$. Moreover, $\mathrm{L}^0_{\rho,\mathcal{P}}$ is a submultiplicative Ψ^*-subalgebra of $\mathcal{B}(L^2(\mathbb{R}^n))$.*

Proof. $\mathrm{L}^0_{\rho,\mathcal{P}}$ is a symmetric Fréchet subalgebra of $\mathcal{B}(L^2(\mathbb{R}^n))$ with a stronger topology. In order to show it is a Ψ^*-subalgebra, we only have to check spectral invariance. Since $S^0_{0,0}(\mathbb{R}^{2n})$ is a Ψ^*-algebra, A^{-1} necessarily belongs to $S^0_{0,0}(\mathbb{R}^{2n})$, hence

$$\begin{aligned} &[x_j, A^{-1}] = -A^{-1}[x_j, A]A^{-1}, \\ &[D_j, A^{-1}] = -A^{-1}[D_j, A]A^{-1}, \end{aligned} \tag{5}$$

cf. [20], Appendix. Using Leibniz' rule and Lemma 1, these identities show that

$$B^\alpha_\beta A^{-1} : L^2(\mathbb{R}^n) \to H^{\rho|\alpha+\beta|}_{\mathcal{P}}(\mathbb{R}^n)$$

is bounded. Hence $A^{-1} \in \mathrm{L}^0_{\rho,\mathcal{P}}$ by Theorem 5.

Finally let us check submultiplicativity. Corollary 1 suggests the following system of semi-norms $\{p_{\alpha,\beta,s} : \alpha, \beta \in \mathbb{N}^n_0, s \in \mathbb{N}\}$ for the topology of $\mathrm{L}^0_{\rho,\mathcal{P}}$:

$$p_{\alpha,\beta,s}(A) = \|B^\alpha_\beta A\|_{\mathcal{L}(H^s_{\mathcal{P}}(\mathbb{R}^n), H^{s+\rho|\alpha+\beta|}_{\mathcal{P}}(\mathbb{R}^n))}.$$

A priori, this topology is weaker than the topology induced from $\Lambda^0_{\rho,\mathcal{P}}(\mathbb{R}^{2n})$, since the operator norm can be estimated in terms of the symbol semi-norms. The open mapping theorem yields that both are equivalent. The construction in [13], 3.4 ff, eventually shows how to derive submultiplicative semi-norms from the system $\{p_{\alpha,\beta,s}\}$.

□

We proceed by constructing order reducing operators. They will be used in Corollary 2.

LEMMA 2 (ORDER REDUCTION). *For all $s \in \mathbb{R}$ there exists an operator $T \in E\mathrm{L}^{s}_{\rho,\mathcal{P}}$ such that $T : H^s_{\mathcal{P}}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bicontinuous bijection.*

Proof. Let us set $A = Op(w^{s/2}_{\mathcal{P}}) \in E\mathrm{L}^{s/2}_{\rho,\mathcal{P}}$. If A^* is its L^2-formal adjoint, then $AA^* \in E\mathrm{L}^{s}_{\rho,\mathcal{P}}$ and $AA^* : H^s_{\mathcal{P}}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is Fredholm. It can be shown easily that $\mathrm{Ker}AA^* = AA^*(H^s_{\mathcal{P}}(\mathbb{R}^n))^{\perp}$, where $\perp$ means orthocomplementation in $L^2(\mathbb{R}^n)$. In fact, $\mathrm{Ker}AA^* \subseteq S(\mathbb{R}^n)$, is independent of s, and $f \in \mathrm{Ker}AA^*$ is equivalent to $f \perp AA^*(S(\mathbb{R}^n))$ which in turn is equivalent to $f \perp AA^*(H^s_{\mathcal{P}}(\mathbb{R}^n))$ as $AA^*(S(\mathbb{R}^n))$ is dense in $AA^*(H^s_{\mathcal{P}}(\mathbb{R}^n))$. Suppose now that $\{f_1, .., f_k\}$ is an orthonormal basis of the finite dimensional vector space $\mathrm{Ker}AA^* = AA^*(H^s_{\mathcal{P}}(\mathbb{R}^n))^{\perp}$, viewed now as imbedded in both $H^s_{\mathcal{P}}(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$. We consider the operator $B = AA^* + P$ where P is the continuous extension to $H^s_{\mathcal{P}}(\mathbb{R}^n)$ of $Pf = \sum_{j=1}^{k}(f, f_j)_{L^2} f_j$, $f \in S(\mathbb{R}^n)$. P is compact as it has finite rank and is smoothing since it has an integral kernel in $S(\mathbb{R}^n \times \mathbb{R}^n)$. Then B is a Fredholm operator in $E\mathrm{L}^{s}_{\rho,\mathcal{P}}$ and $\mathrm{index}(B) = 0$. It can be easily checked that $B : H^s_{\mathcal{P}}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is injective so that it is a bijection. The continuity of B^{-1} follows from the open mapping theorem and the continuity of B.

□

COROLLARY 2. *Let $A \in \mathrm{L}^{0}_{\rho,\mathcal{P}}$ be invertible in $\mathcal{B}(H^s_{\mathcal{P}}(\mathbb{R}^n))$ for some $s \in \mathbb{R}$, then $A^{-1} \in \mathrm{L}^{0}_{\rho,\mathcal{P}}$.*

Proof. If $T : H^s_{\mathcal{P}}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the order reduction, we know that $B = TAT^{-1} \in \mathrm{L}^{0}_{\rho,\mathcal{P}}$ is invertible on $L^2(\mathbb{R}^n)$. By Theorem 6 $B^{-1} \in \mathrm{L}^{0}_{\rho,\mathcal{P}}$, so $A^{-1} = TB^{-1}T^{-1} \in \mathrm{L}^{0}_{\rho,\mathcal{P}}$.

□

REMARK 1. A consequence of Corollary 2 is that the spectrum of an operator $A \in \mathrm{L}^{0}_{\rho,\mathcal{P}}$ is independent of the space $H^s_{\mathcal{P}}(\mathbb{R}^n)$. This is particularly relevant in view of the developement of a spectral theory for multi-quasi-elliptic operators.

The fact that multi-quasi-elliptic operators have the Fredholm property was proven in [5]; we show here that the converse holds.

THEOREM 7. *Let $A \in L^{m}_{\rho,\mathcal{P}}$, $m \in \mathbb{R}$. Then the following are equivalent:*

(a) $A \in E\mathrm{L}^{m}_{\rho,\mathcal{P}}$.

(b) $A : H^s_{\mathcal{P}}(\mathbb{R}^n) \to H^{s-m}_{\mathcal{P}}(\mathbb{R}^n)$ *is a Fredholm operator for all* $s \in \mathbb{R}$.

(c) $A : H^{s_0}_{\mathcal{P}}(\mathbb{R}^n) \to H^{s_0-m}_{\mathcal{P}}(\mathbb{R}^n)$ *is a Fredholm operator for some* $s_0 \in \mathbb{R}$.

Proof. By [5], (a) implies (b), (b) trivially implies (c). In order to show that (c) implies (a), we can apply order reduction and assume that $m = s_0 = 0$. Next we observe that, for suitable $\varepsilon > 0$, $C_\varepsilon > 0$,

$$\omega^{\rho}_{\mathcal{P}}(x,\xi) \geq C_\varepsilon \langle x\rangle^{\varepsilon}\langle \xi\rangle^{\varepsilon}.$$

This implies that $\Lambda^0_{\rho,\mathcal{P}}(\mathbb{R}^{2n})$ is embedded in the class $\tilde{S}^0_{\varepsilon,0}(\mathbb{R}^n \times \mathbb{R}^n)$ of slowly varying symbols, cf. Kumano-go [16], Chapter III, Definition 5.11. For these symbols it has been shown in [18], Theorem 1.8 that the Fredholm property on $L^2(\mathbb{R}^n)$ implies uniform ellipticity. This concludes the proof, for $H^0_{\mathcal{P}}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$, and the notion of uniform ellipticity coincides with that of multi-quasi-ellipticity of order zero.

□

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