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# SOME GENERATING FUNCTIONS INVOLVING THE STIRLING NUMBERS OF THE SECOND KIND

**Abstract.** Certain general results on generating functions (associated with the Stirling numbers of the second kind) are applied here to several interesting sequences of special functions and polynomials in one and more variables. Relevant connections of the generating functions, which are derived in this paper, with those given in earlier works on the subject are also indicated.

## 1. Introduction, Definitions and Preliminaries

Following the work of Riordan [11] (p. 90 *et seq.*), we denote by $S(n,k)$ the Stirling numbers of the second kind, defined by

$$S(n,k) := \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n, \tag{1}$$

so that

$$S(n,0) = \begin{cases} 1 & (n=0) \\ 0 & (n \in \mathbb{N} := \{1,2,3,\dots\}) \end{cases} \tag{2}$$

and

$$S(n,1) = S(n,n) = 1 \qquad \text{and} \qquad S(n,n-1) = \binom{n}{2}.$$

Recently, several authors (see, for example, Gabutti and Lyness [3], Mathis and Sismondi [7], and Srivastava [12]) considered various families of generating functions associated with the Stirling numbers $S(n,k)$ defined by (1). We choose to recall here the following general results on these families of generating functions, which were given by Srivastava [12].

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THEOREM 1 (SRIVASTAVA [12], P. 754, THEOREM 1). *Let the sequence* $\{\mathcal{S}_n(x)\}_{n=0}^{\infty}$ *be generated by*

$$\sum_{k=0}^{\infty} \binom{n+k}{k} \mathcal{S}_{n+k}(x)\, t^k = f(x,t)\, \{g(x,t)\}^{-n}\, \mathcal{S}_n(h(x,t)) \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}),$$

*where* $f$, $g$ *and* $h$ *are suitable functions of* $x$ *and* $t$.

*Then, in terms of the Stirling numbers* $S(n,k)$ *defined by* (1)*, the following family of generating functions holds true:*

$$\sum_{k=0}^{\infty} k^n\, \mathcal{S}_k(h(x,-z)) \left(\frac{z}{g(x,-z)}\right)^k = \{f(x,-z)\}^{-1} \sum_{k=0}^{n} k!\, S(n,k)\, \mathcal{S}_k(x)\, z^k \quad (n \in \mathbb{N}_0), \tag{3}$$

*provided that each member of* (3) *exists.*

THEOREM 2 (SRIVASTAVA [12], P.765, THEOREM 2). *Suppose that the multivariable sequence*

$$\{\Xi_n(x_1,\ldots,x_s)\}_{n=0}^{\infty}$$

*is generated by*

$$\begin{aligned}\sum_{k=0}^{\infty} \binom{n+k}{k} \Xi_{n+k}(x_1,\ldots,x_s)\, t^k &= \theta(x_1,\ldots,x_s;t)\,\{\phi(x_1,\ldots,x_s;t)\}^{-n} \\ &\quad \cdot \Xi_n(\psi_1(x_1,\ldots,x_s;t),\ldots,\psi_s(x_1,\ldots,x_s;t)) \quad (n \in \mathbb{N}_0;\ s \in \mathbb{N}),\end{aligned} \tag{4}$$

*where* $\theta$, $\phi$, $\psi_1,\ldots,\psi_s$ *are suitable functions of* $x_1,\ldots,x_s$ *and* $t$. *Also let* $S(n,k)$ *denote the Stirling numbers of the second kind, defined by* (1).

*Then the following family of multivariable generating functions holds true:*

$$\begin{aligned}&\sum_{k=0}^{\infty} k^n\, \Xi_k(\psi_1(x_1,\ldots,x_s;-z),\ldots,\psi_s(x_1,\ldots,x_s;-z)) \\ &\quad \cdot \left(\frac{z}{\phi(x_1,\ldots,x_s;-z)}\right)^k \\ &= \{\theta(x_1,\ldots,x_s;-z)\}^{-1} \sum_{k=0}^{n} k!\, S(n,k) \\ &\quad \cdot \Xi_k(x_1,\ldots,x_s)\, z^k \quad (n \in \mathbb{N}_0;\ s \in \mathbb{N}),\end{aligned} \tag{5}$$

*provided that each member of* (5) *exists.*

Srivastava [12] also applied his general result (Theorem 1 above) *as well as* its multivariable extension (Theorem 2 above) with a view to obtaining generating functions (associated with the Stirling numbers of the second kind) for a fairly wide variety of special functions and polynomials in one, two, and more variables, thereby extending the corresponding results given earlier by Gabutti and Lyness [3] (and, subsequently, by Mathis and Sismondi [7]). The main object of this sequel to the work of Srivastava [12] is to derive several *further* applications of Theorem 1 and Theorem 2.

For the sake of convenience in our present investigation, we first make use of the following notational changes:

$$\mathcal{S}_n(x) = \frac{1}{n!}\,\mathcal{T}_n(x) \qquad \text{and} \qquad \Xi_n(x_1,\ldots,x_s) = \frac{1}{n!}\,\Lambda_n(x_1,\ldots,x_s)$$

in order to restate Theorem 1 and Theorem 2 in their *equivalent* forms given by Theorem 3 and Theorem 4, respectively.

THEOREM 3. *Suppose that the sequence* $\{\mathcal{T}_n(x)\}_{n=0}^{\infty}$ *is generated by*

$$\sum_{k=0}^{\infty} \mathcal{T}_{n+k}(x)\,\frac{t^k}{k!} = f(x,t)\,\{g(x,t)\}^{-n}\,\mathcal{T}_n(h(x,t)) \quad (n \in \mathbb{N}_0), \tag{6}$$

*where* $f$, $g$ *and* $h$ *are suitable functions of* $x$ *and* $t$. *Also let* $S(n,k)$ *denote the Stirling numbers defined by* (1).

*Then the following family of generating functions holds true:*

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{k^n}{k!}\,\mathcal{T}_k(h(x,-z))\left(\frac{z}{g(x,-z)}\right)^k \\ = \{f(x,-z)\}^{-1}\sum_{k=0}^{n} S(n,k)\,\mathcal{T}_k(x)\,z^k \quad (n \in \mathbb{N}_0),\end{aligned} \tag{7}$$

*provided that each member of* (7) *exists.*

THEOREM 4. *Let the multivariable sequence*

$$\{\Lambda_n(x_1,\ldots,x_s)\}_{n=0}^{\infty}$$

*be generated by*

$$\begin{aligned}\sum_{k=0}^{\infty} \Lambda_{n+k}(x_1,\ldots,x_s)\,\frac{t^k}{k!} = \theta(x_1,\ldots,x_s;t)\,\{\phi(x_1,\ldots,x_s;t)\}^{-n} \\ \cdot\Lambda_n(\psi_1(x_1,\ldots,x_s;t),\ldots,\psi_s(x_1,\ldots,x_s;t)) \quad (n \in \mathbb{N}_0;\ s \in \mathbb{N}),\end{aligned} \tag{8}$$

*where* $\theta$, $\phi$, $\psi_1,\ldots,\psi_s$ *are suitable functions of* $x_1,\ldots,x_s$ *and* $t$. *Suppose also that* $S(n,k)$ *denotes the Stirling numbers defined by* (1).

*Then the following family of multivariable generating functions holds true:*

$$
\begin{aligned}
(9)\qquad & \sum_{k=0}^{\infty} \frac{k^n}{k!}\ \Lambda_k\left(\psi_1\left(x_1,\ldots,x_s;-z\right),\ldots,\psi_s\left(x_1,\ldots,x_s;-z\right)\right) \\
& \cdot\left(\frac{z}{\phi\left(x_1,\ldots,x_s;-z\right)}\right)^k \\
& = \left\{\theta\left(x_1,\ldots,x_s;-z\right)\right\}^{-1} \sum_{k=0}^{n} S(n,k) \\
& \cdot \Lambda_k\left(x_1,\ldots,x_s\right) z^k \quad \left(n \in \mathbb{N}_0;\ s \in \mathbb{N}\right),
\end{aligned}
$$

*provided that each member of* (9) *exists.*

## 2. Applications of Theorems 1 and 3

### 2.1. Hermite Polynomials

For the classical Hermite polynomials defined by (cf., e.g., [10], Chapter 11)

$$H_n(x) := \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \frac{(2k)!}{k!} (2x)^{n-2k}$$

or, equivalently, by

$$H_n(x) = (2x)^n\ {}_2F_0\left(-\frac{1}{2}n, -\frac{1}{2}n+\frac{1}{2};\ -;\ -\frac{1}{x^2}\right)$$

in terms of hypergeometric functions, it is known that (cf. [10], p. 197, Equation (1); see also [14], p. 419, Equation 8.4 (13))

$$(10)\qquad \sum_{k=0}^{\infty} H_{n+k}(x) \frac{t^k}{k!} = \exp\left(2xt - t^2\right) H_n(x-t) \qquad \left(n \in \mathbb{N}_0\right),$$

which obviously belongs to the family given by (6). Indeed, by comparing (10) with (6), it is readily observed that

$$f(x,t) = \exp\left(2xt - t^2\right), \qquad g(x,t) = 1, \qquad h(x,t) = x - t,$$

and

$$\mathcal{T}_k(x) \longmapsto H_k(x) \qquad \left(k \in \mathbb{N}_0\right).$$

Thus the assertion (7) of Theorem 3 leads us to the following (*presumably new*) generating function for the classical Hermite polynomials:

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} H_k(x+z) z^k = \exp\left(2xz + z^2\right) \sum_{k=0}^{n} S(n,k) H_k(x) z^k \quad (n \in \mathbb{N}_0),$$

which, for $x \longmapsto x - z$, assumes the form:

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} H_k(x) z^k = \exp\left(2xz - z^2\right) \cdot \sum_{k=0}^{n} S(n,k) H_k(x-z) z^k \quad (n \in \mathbb{N}_0). \tag{11}$$

In view of the evaluation (2), a special case of (11) when $n = 0$ would immediately yield the classical generating function for the Hermite polynomials (cf., e.g., [15], p. 106, Equation(5.5.7)).

### 2.2. Bessel Functions

For the Bessel function $J_\nu(z)$ of the first kind (and of order $\nu \in \mathbb{C}$), defined by

$$J_\nu(z) := \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}z\right)^{\nu+2k}}{k!\, \Gamma(\nu+k+1)} \qquad (z \in \mathbb{C} \setminus (-\infty, 0]),$$

the following generating function is well-known [18], p. 141, Equation 5.22 (5):

$$\sum_{k=0}^{\infty} J_{\nu+k}(x) \frac{t^k}{k!} = \left(1 - \frac{2t}{x}\right)^{-\frac{1}{2}\nu} J_\nu\left(\sqrt{x^2 - 2xt}\right) \qquad \left(\nu \in \mathbb{C};\ |t| < \frac{1}{2}|x|\right), \tag{12}$$

which is in the family given by (6) with, of course, $\nu \longmapsto \nu + n$ $(n \in \mathbb{N}_0)$,

$$f(x,t) = \left(1 - \frac{2t}{x}\right)^{-\frac{1}{2}\nu}, \qquad g(x,t) = \sqrt{1 - \frac{2t}{x}},$$
$$h(x,t) = \sqrt{x^2 - 2xt}, \qquad \text{and} \qquad \mathcal{T}_k(x) \longmapsto J_{\nu+k}(x) \qquad (\nu \in \mathbb{C};\ k \in \mathbb{N}_0).$$

Thus, by applying Theorem 3, we obtain the following class of generating functions for the Bessel function $J_\nu(z)$:

$$\begin{aligned}(13)\qquad &\sum_{k=0}^{\infty}\frac{k^n}{k!}\,J_{\nu+k}\left(\sqrt{x^2+2xz}\right)\left(\frac{z}{\sqrt{1+2\,(z/x)}}\right)^k\\ &=\left(1+\frac{2z}{x}\right)^{\frac{1}{2}\nu}\sum_{k=0}^{n}S\,(n,k)\,J_{\nu+k}\,(x)\,z^k\quad\left(\nu\in\mathbb{C};\ |z|<\frac{1}{2}\,|x|\,;\ n\in\mathbb{N}_0\right).\end{aligned}$$

In the generating function (13), we first set $z = XZ/x$ and then let

$$x=\sqrt{X^2-2XZ}.$$

Upon replacing $X$ and $Z$ by $x$ and $z$, respectively, we finally obtain the generating function:

$$\begin{aligned}(14)\qquad &\sum_{k=0}^{\infty}\frac{k^n}{k!}\,J_{\nu+k}\,(x)\,z^k=\left(1-\frac{2z}{x}\right)^{-\frac{1}{2}\nu}\\ &\cdot\sum_{k=0}^{n}S\,(n,k)\,J_{\nu+k}\left(\sqrt{x^2-2xz}\right)\left(\frac{z}{\sqrt{1-2\,(z/x)}}\right)^k\\ &\qquad\left(\nu\in\mathbb{C};\ \ |z|<\frac{1}{2}\,|x|\,;\ n\in\mathbb{N}_0\right),\end{aligned}$$

which, for $n = 0$, corresponds to the classical result (12).

### 2.3. Gottlieb Polynomials

For the Gottlieb polynomials $\mathcal{L}_n\,(x;\lambda)$ defined by (cf., e.g., [14], p. 185, Problem 47)

$$\mathcal{L}_n\,(x;\lambda):=e^{-n\lambda}\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}\left(1-e^{\lambda}\right)^k=e^{-n\lambda}\,{}_2F_1\left(-n,-x;1;1-e^{\lambda}\right)$$

in terms of the Gauss hypergeometric function, it is known that [14], p. 449, Problem 20(i)

$$\begin{aligned}(15)\qquad &\sum_{k=0}^{\infty}\binom{n+k}{k}\mathcal{L}_{n+k}\,(\alpha;x)\,t^k\\ &=(1-t)^{\alpha-n}\left(1-te^{-x}\right)^{-\alpha-1}\mathcal{L}_n\left(\alpha;\log_e\left(\frac{e^x-t}{1-t}\right)\right)\\ &\qquad(n\in\mathbb{N}_0;\ \ |t|<1)\,.\end{aligned}$$

Thus Theorem 1 (or Theorem 3), when applied to (15), yields the following (*presumably new*) generating function for the Gottlieb polynomials:

$$\sum_{k=0}^{\infty} k^n \mathcal{L}_k\left(\alpha; \log_e\left(\frac{e^x+z}{1+z}\right)\right)\left(\frac{z}{1+z}\right)^k$$
$$= (1+z)^{-\alpha}\left(1+ze^{-x}\right)^{\alpha+1} \sum_{k=0}^{n} k!\, S(n,k)\, \mathcal{L}_k(\alpha; x)\, z^k$$
$$(n \in \mathbb{N}_0;\ \ |z|<1),$$

which, for $z \longmapsto z/(1-z)$, assumes the form:

$$\sum_{k=0}^{\infty} k^n \mathcal{L}_k\Big(\alpha; \log_e\big(z+(1-z)\,e^x\big)\Big) z^k$$
$$= (1-z)^{-1}\left(1-z+ze^{-x}\right)^{\alpha+1} \sum_{k=0}^{n} k!\, S(n,k)\, \mathcal{L}_k(\alpha; x)\left(\frac{z}{1-z}\right)^k$$
$$(n \in \mathbb{N}_0;\ \ |z|<1).$$

### 2.4. Meixner Polynomials

The Meixner polynomials $\mathcal{M}_n(x; \beta, c)$ are defined by (cf., e.g., [14], p. 75, Equation 1.9 (3); p. 443, Problem 5)

$$\begin{aligned} \mathcal{M}_n(x; \beta, c) &:= \binom{\beta+n-1}{n} n!\, {}_2F_1\left(-n, -x; \beta; 1-c^{-1}\right) \\ &= n!\, P_n^{(\beta-1, -\beta-x-n)}\left(\tfrac{2}{c}-1\right), \\ &\quad (\beta>0;\ 0<c<1;\ x \in \mathbb{N}_0) \end{aligned} \tag{16}$$

in terms of the classical Jacobi polynomials [15], Chapter 4; in fact, these polynomials are known to satisfy the generating-function relationship [14], p. 449, Problem 20 (ii):

$$\sum_{k=0}^{\infty} \mathcal{M}_{n+k}(\alpha; \beta, x)\frac{t^k}{k!} = (1-t)^{-\alpha-\beta-n}\left(1-\frac{t}{x}\right)^{\alpha} \mathcal{M}_n\left(\alpha; \beta, \frac{x-t}{1-t}\right)$$
$$(n \in \mathbb{N}_0;\ \ |t| < \min\{1, |x|\}),$$

which obviously belongs to the family (6) involved in Theorem 3. Thus the following (*presumably new*) generating function holds true for the Meixner polynomials defined by (16):

$$\sum_{k=0}^{\infty} \frac{k^n}{k!}\ \ \mathcal{M}_k\left(\alpha; \beta, \frac{x+z}{1+z}\right)\left(\frac{z}{1+z}\right)^k$$
$$= (1+z)^{\alpha+\beta}\left(1+\frac{z}{x}\right)^{-\alpha} \sum_{k=0}^{n} S(n,k)\, \mathcal{M}_k(\alpha; \beta, x)\, z^k$$
$$(n \in \mathbb{N}_0;\ \ |z| < \min\{1, |x|\}),$$

which, for $z \longmapsto z/(1-z)$, assumes the form:

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} \quad \mathcal{M}_k(\alpha; \beta, z+(1-z)x) z^k$$
$$= (1-z)^{-\beta} \left(1 - z + \frac{z}{x}\right)^{-\alpha} \sum_{k=0}^{n} S(n,k) \mathcal{M}_k(\alpha; \beta, x) \left(\frac{z}{1-z}\right)^k$$
$$\left(n \in \mathbb{N}_0; \quad |z| < \min\{1, |x/(1-x)|\}\right).$$

### 2.5. Cesàro Polynomials

For the Cesàro polynomials $\mathcal{G}_n^{(s)}(x)$ defined by (cf. [14], p. 449, Problem 20)

$$\text{(17)} \qquad \begin{aligned} \mathcal{G}_n^{(s)}(x) &:= \textstyle\sum_{k=0}^{n} \binom{s+n-k}{n-k} x^k = \binom{s+n}{n} {}_2F_1(-n, 1; -s-n; x) \\ &= P_n^{(s+1,-s-n-1)}(2x-1), \end{aligned}$$

it is known that [14], p. 449, Problem 20 (iii)

$$\sum_{k=0}^{\infty} \binom{n+k}{k} \mathcal{G}_{n+k}^{(s)}(x) t^k = (1-t)^{-s-n-1} (1-xt)^{-1} \mathcal{G}_n^{(s)}\left(\frac{x(1-t)}{1-xt}\right)$$
$$\left(n \in \mathbb{N}_0; \quad |t| < \min\left\{1, |x|^{-1}\right\}\right).$$

By applying Theorem 1 (or Theorem 3), we immediately obtain the following (*presumably new*) generating function for the Cesàro polynomials defined by (17):

$$\sum_{k=0}^{\infty} k^n \mathcal{G}_k^{(s)}\left(\frac{x(1+z)}{1+xz}\right)\left(\frac{z}{1+z}\right)^k$$
$$= (1+z)^{s+1} (1+xz) \sum_{k=0}^{n} k!\, S(n,k)\, \mathcal{G}_k^{(s)}(x) z^k$$
$$\left(n \in \mathbb{N}_0; \quad |z| < 1\right),$$

which, for $z \longmapsto z/(1-z)$ and $x \longmapsto x(1-z)/(1-xz)$, assumes the form:

$$\sum_{k=0}^{\infty} k^n \mathcal{G}_k^{(s)}(x) z^k = (1-z)^{-s-1} (1-xz)^{-1}$$
$$\cdot \sum_{k=0}^{n} k!\, S(n,k)\, \mathcal{G}_k^{(s)}\left(\frac{x(1-z)}{1-xz}\right)\left(\frac{z}{1-z}\right)^k$$
$$\left(n \in \mathbb{N}_0; \quad |z| < 1\right).$$

### 2.6. Generalized Sylvester Polynomials

For the generalized Sylvester polynomials $\varphi_n(x; c)$ defined by [14], p. 450, Problem 20 (iv)

$$\begin{aligned}\varphi_n(x; c) :&= \frac{(cx)^n}{n!}\, {}_2F_0\left(-n, x;\ -\ ; -\frac{1}{cx}\right) \\ &= (-1)^n\, L_n^{(-x-n)}(cx)\end{aligned}$$

in terms of the classical Laguerre polynomials [15], Chapter 5, it is known that [14], p. 450, Problem 20 (v)

$$\sum_{k=0}^{\infty} \binom{n+k}{k} \varphi_{n+k}(\alpha; x)\, t^k = (1-t)^{-\alpha-n}\, e^{\alpha x t}\, \varphi_n(\alpha; x(1-t)) \qquad (n \in \mathbb{N}_0;\ \ |t| < 1),$$

so that Theorem 1 immediately yields the generating function:

$$\begin{aligned}&\sum_{k=0}^{\infty} k^n\, \varphi_k(\alpha; x(1+z)) \left(\frac{z}{1+z}\right)^k \\ &= (1+z)^{\alpha}\, e^{\alpha x z} \sum_{k=0}^{n} k!\, S(n,k)\, \varphi_k(\alpha; x)\, z^k \\ &\qquad\qquad (n \in \mathbb{N}_0;\ \ |z| < 1),\end{aligned}$$

which, for $z \longmapsto z/(1-z)$ and $x \longmapsto x(1-z)$, assumes the form:

$$\begin{aligned}&\sum_{k=0}^{\infty} k^n\, \varphi_k(\alpha; x)\, z^k = (1-z)^{-\alpha}\, e^{\alpha x z} \\ &\cdot \sum_{k=0}^{n} k!\, S(n,k)\, \varphi_k(\alpha; x(1-z)) \left(\frac{z}{1-z}\right)^k \\ &\qquad\qquad (n \in \mathbb{N}_0;\ \ |z| < 1).\end{aligned}$$

### 2.7. Bessel Polynomials

The Bessel polynomials $y_n(x, \alpha, \beta)$ are defined by [14], p. 75, Equation 1.9 (1)

$$\begin{aligned}y_n(x, \alpha, \beta) :&= \sum_{k=0}^{n} \binom{n}{k} \binom{\alpha+n+k-2}{k} k! \left(\frac{x}{\beta}\right)^k \\ &= {}_2F_0\left(-n, \alpha+n-1;\ -\ ; -\frac{x}{\beta}\right) \\ &= \left(-\frac{x}{\beta}\right)^n n!\, L_n^{(1-\alpha-2n)}\left(\frac{\beta}{x}\right)\end{aligned}$$

and satisfy the generating-function relationship [14], p. 419, Equation 8.4 (8):

$$\sum_{k=0}^{\infty} y_{n+k}(x, \alpha - k, \beta) \frac{t^k}{k!} = \left(1 - \frac{xt}{\beta}\right)^{1-\alpha-n} e^t y_n\left(x\left(1 - \frac{xt}{\beta}\right)^{-1}, \alpha, \beta\right) \qquad (n \in \mathbb{N}_0;\ |t| < |\beta/x|)\,. \tag{18}$$

On the other hand, for the *simple* Bessel polynomials $y_n(x)$ defined by

$$y_n(x) := y_n(x, 2, 2)\,,$$

it is known that [14], p. 419, Equation 8.4 (10)

$$\sum_{k=0}^{\infty} y_{n+k}(x) \frac{t^k}{k!} = (1 - 2xt)^{-\frac{1}{2}(n+1)} \cdot \exp\left(x^{-1}\left[1 - \sqrt{1 - 2xt}\right]\right) y_n\left(\frac{x}{\sqrt{1 - 2xt}}\right) \qquad \left(n \in \mathbb{N}_0;\ |t| < \frac{1}{2}|x|^{-1}\right). \tag{19}$$

Thus, in view of the obviously independent results (18) and (19), Theorem 3 yields the following (*presumably new*) generating functions for the Bessel polynomials:

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} y_k\left(x\left(1 + \frac{xz}{\beta}\right)^{-1}, \alpha - k, \beta\right) z^k = \left(1 + \frac{xz}{\beta}\right)^{\alpha-1} e^z \sum_{k=0}^{n} S(n,k)\, y_k(x, \alpha - k, \beta)\, z^k \qquad (n \in \mathbb{N}_0;\ |z| < |\beta/x|)\,,$$

which, for

$$x \longmapsto x\left(1 - \frac{xz}{\beta}\right)^{-1},$$

assumes the form:

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} y_k(x, \alpha - k, \beta)\, z^k = \left(1 - \frac{xz}{\beta}\right)^{1-\alpha} e^z \cdot \sum_{k=0}^{n} S(n,k)\, y_k\left(x\left(1 - \frac{xz}{\beta}\right)^{-1}, \alpha - k, \beta\right) z^k \qquad (n \in \mathbb{N}_0;\ |z| < |\beta/x|)\,;$$

$$\sum_{k=0}^{\infty} \frac{k^n}{k!}\, y_k\left(\frac{x}{\sqrt{1+2xz}}\right)\left(\frac{z}{\sqrt{1+2xz}}\right)^k$$
$$= \sqrt{1+2xz}\; \exp\left(-x^{-1}\left[1-\sqrt{1+2xz}\right]\right)$$
$$\cdot \sum_{k=0}^{n} S\,(n,k)\, y_k\,(x)\, z^k \quad \left(n \in \mathbb{N}_0;\;\; |z| < \frac{1}{2}\,|x|^{-1}\right).$$

### 2.8. Generalized Heat Polynomials

For the generalized heat polynomials $\mathcal{P}_{n,\nu}\,(x,u)$ defined by [14], p. 426, Equation 8.4 (52)

$$\mathcal{P}_{n,\nu}\,(x,u) := \sum_{k=0}^{n} 2^{2k}\binom{n}{k}\binom{\nu+n-\frac{1}{2}}{k} k!\; x^{2n-2k}u^k$$
$$= (4u)^n\, n!\; L_n^{\left(\nu-\frac{1}{2}\right)}\left(-\frac{x^2}{4u}\right),$$

it is easily observed that

$$\sum_{k=0}^{\infty} \mathcal{P}_{n+k,\nu}\,(x,u)\,\frac{t^k}{k!} = (1-4ut)^{-\nu-n-\frac{1}{2}}$$
$$\cdot \exp\left(\frac{x^2 t}{1-4ut}\right)\mathcal{P}_{n,\nu}\left(\frac{x}{\sqrt{1-4ut}}, u\right)$$
$$\left(n \in \mathbb{N}_0;\;\; |t| < \frac{1}{4}\,|u|^{-1}\right),$$

so that Theorem 3 yields the generating function:

$$\sum_{k=0}^{\infty} \frac{k^n}{k!}\, \mathcal{P}_{k,\nu}\left(\frac{x}{\sqrt{1+4uz}}, u\right)\left(\frac{z}{1+4uz}\right)^k$$
$$= (1+4uz)^{\nu+\frac{1}{2}} \exp\left(\frac{x^2 z}{1+4uz}\right)\sum_{k=0}^{n} S\,(n,k)\,\mathcal{P}_{k,\nu}\,(x,u)\, z^k$$
$$\left(n \in \mathbb{N}_0;\;\; |z| < \frac{1}{4}\,|u|^{-1}\right).$$

### 2.9. Modified Laguerre Polynomials

For the *modified* Laguerre polynomials $f_n^{\alpha}\,(x)$ defined by (cf. [8], p. 68; see also [14], p. 425, Equation 8.4 (45))

$$f_n^{\alpha}\,(x) := \sum_{k=0}^{n} (-1)^{n-k}\binom{-\alpha}{n-k}\frac{x^k}{k!} = (-1)^n\, L_n^{(-\alpha-n)}\,(x)\,, \tag{20}$$

it is readily seen that (cf. [8], p. 70, Equation (4))

$$\sum_{k=0}^{\infty} \binom{n+k}{k} f_{n+k}^{\alpha}(x)\, t^k = (1-t)^{-\alpha-n} \exp(xt)\, f_n^{\alpha}(x(1-t))$$

$$(n \in \mathbb{N}_0;\ |t| < 1).$$

Thus Theorem 1 immediately yields the generating function:

$$\sum_{k=0}^{\infty} k^n f_k^{\alpha}(x(1+z)) \left(\frac{z}{1+z}\right)^k$$
$$= (1+z)^{\alpha} \exp(xz) \sum_{k=0}^{n} k!\, S(n,k)\, f_k^{\alpha}(x)\, z^k$$
$$(n \in \mathbb{N}_0;\ |z| < 1),$$

which, for $z \longmapsto z/(1-z)$ and $x \longmapsto x(1-z)$, assumes the form:

$$\sum_{k=0}^{\infty} k^n f_k^{\alpha}(x)\, z^k = (1-z)^{-\alpha} \exp(xz) \tag{21}$$
$$\cdot \sum_{k=0}^{n} k!\, S(n,k)\, f_k^{\alpha}(x(1-z)) \left(\frac{z}{1-z}\right)^k$$
$$(n \in \mathbb{N}_0;\ |z| < 1).$$

### 2.10. Poisson-Charlier Polynomials

For the Poisson-Charlier polynomials $c_n(x;\alpha)$ defined by (cf., e.g., [14], p. 425, Equation 8.4 (47); see also [15], p. 35, Equation (2.81.2))

$$c_n(x;\alpha) := \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{x}{k} k!\, \alpha^{-k}$$
$$= (-\alpha)^{-n}\, n!\, L_n^{(x-n)}(\alpha) \quad (\alpha > 0;\ x \in \mathbb{N}_0),$$

it is not difficult to observe that (*cf.* [8, p. 71])

$$\sum_{k=0}^{\infty} c_{n+k}(\alpha;x) \frac{t^k}{k!} = \left(1-\frac{t}{x}\right)^{\alpha} e^t\, c_n(\alpha;x-t) \tag{22}$$
$$(n \in \mathbb{N}_0;\ |t| < |x|).$$

Thus, by means of (22), Theorem 3 would yield the following (*presumably new*) generating function for the Poisson-Charlier polynomials:

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} c_k(\alpha;x+z)\, z^k = \left(1+\frac{z}{x}\right)^{-\alpha} e^z \sum_{k=0}^{n} S(n,k)\, c_k(\alpha;x)\, z^k$$
$$(n \in \mathbb{N}_0;\ |z| < |x|),$$

which, for $x \longmapsto x - z$, assumes the form:

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} c_k(\alpha; x) z^k = \left(1 - \frac{z}{x}\right)^{\alpha} e^z \sum_{k=0}^{n} S(n,k) c_k(\alpha; x - z) z^k$$
$$(n \in \mathbb{N}_0;\ \ |z| < |x|).$$

### 2.11. Sequences of Generalized Hypergeometric Functions

We first consider the sequence of generalized hypergeometric functions:

$$\left\{\omega_{n,N}^{(\lambda)}[\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : x]\right\}_{n=0}^{\infty}$$

defined by (cf., e.g., [13], p. 18, Equation (4.4); see also [14], p. 428, Equation 8.4 (59))

(23)
$$\omega_{n,N}^{(\lambda)}[\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : x]$$
$$:= {}_{N+u}F_v[\Delta(N; \lambda + n), \alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : x]$$
$$(n \in \mathbb{N}_0;\ N \in \mathbb{N}),$$

where, for convenience, $\Delta(N; \lambda)$ abbreviates the array of $N$ parameters

$$\frac{\lambda}{N}, \frac{\lambda+1}{N}, \ldots, \frac{\lambda+N-1}{N} \qquad (N \in \mathbb{N}).$$

For the sequence defined by (23), it is known that [14], p. 429, Equation 8.4 (60)

(24)
$$\sum_{k=0}^{\infty} \binom{\lambda+n+k-1}{k} \omega_{n+k,N}^{(\lambda)}[\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : x] t^k$$
$$= (1-t)^{-\lambda-n} \omega_{n,N}^{(\lambda)}\left[\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : \frac{x}{(1-t)^N}\right]$$
$$(n \in \mathbb{N}_0;\ N \in \mathbb{N};\ \ |t| < 1),$$

which is of the form (6) with

$$f(x,t) = (1-t)^{-\lambda}, \qquad g(x,t) = 1 - t, \qquad h(x,t) = \frac{x}{(1-t)^N},$$

and

$$\mathcal{T}_k(x) \longmapsto \binom{\lambda+k-1}{k} k!\, \omega_{k,N}^{(\lambda)}[\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : x] \qquad (k \in \mathbb{N}_0).$$

Thus, by appealing to Theorem 3 once again, we obtain

$$\sum_{k=0}^{\infty} \binom{\lambda+k-1}{k} k^n \, \omega_{k,N}^{(\lambda)} \left[\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : \frac{x}{(1+z)^N}\right] \left(\frac{z}{1+z}\right)^k$$
$$= (1+z)^{\lambda} \sum_{k=0}^{n} \binom{\lambda+k-1}{k} k! \, S(n,k) \, \omega_{k,N}^{(\lambda)} [\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : x] \, z^k$$
$$(n \in \mathbb{N}_0; \; N \in \mathbb{N}; \; |z| < 1),$$

which, upon letting

$$z \longmapsto \frac{z}{1-z} \qquad \text{and} \qquad x \longmapsto \frac{x}{(1-z)^N},$$

yields the following generating function:

$$\sum_{k=0}^{\infty} \binom{\lambda+k-1}{k} k^n \, \omega_{k,N}^{(\lambda)} [\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : x] \, z^k \tag{25}$$
$$= (1-z)^{-\lambda} \sum_{k=0}^{n} \binom{\lambda+k-1}{k} k! \, S(n,k)$$
$$\cdot \, \omega_{k,N}^{(\lambda)} \left[\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : \frac{x}{(1-z)^N}\right] \left(\frac{z}{1-z}\right)^k$$
$$(n \in \mathbb{N}_0; \; N \in \mathbb{N}; \; |z| < 1),$$

which, in the *special* case when $\lambda = 1$, reduces immediately to Srivastava's result [12], p. 765, Equation (4.15).

For another sequence of generalized hypergeometric functions:

$$\left\{\zeta_{n,N}^{(\lambda)} [\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : x]\right\}_{n=0}^{\infty}$$

defined by [1], p. 171, Equation (5.14)

$$\zeta_{n,N}^{(\lambda)} [\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : x] \tag{26}$$
$$:= {}_uF_{N+v} [\alpha_1, \ldots, \alpha_u; \Delta(N; 1-\lambda-n), \beta_1, \ldots, \beta_v; x]$$
$$(n \in \mathbb{N}_0; \; N \in \mathbb{N}),$$

it is known that [1], p. 171, Equation (5.15)

$$\sum_{k=0}^{\infty} \binom{\lambda+n+k-1}{k} \zeta_{n+k,N}^{(\lambda)} [\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : x] \, t^k \tag{27}$$
$$= (1-t)^{-\lambda-n} \, \zeta_{n,N}^{(\lambda)} \left[\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v : x(1-t)^N\right]$$
$$(n \in \mathbb{N}_0; \; N \in \mathbb{N}; \; |t| < 1),$$

which also belongs to the family (6) with

$$f(x,t) = (1-t)^{-\lambda}, \qquad g(x,t) = 1-t, \qquad h(x,t) = x(1-t)^N,$$

and

$$\mathcal{T}_k(x) \longmapsto \binom{\lambda+k-1}{k} k!\, \zeta_{k,N}^{(\lambda)}[\alpha_1,\ldots,\alpha_u;\beta_1,\ldots,\beta_v : x] \qquad (k \in \mathbb{N}_0).$$

Theorem 3, when applied to the generating function (27), yields

$$\sum_{k=0}^{\infty}\binom{\lambda+k-1}{k} k^n\, \zeta_{k,N}^{(\lambda)}\left[\alpha_1,\ldots,\alpha_u;\beta_1,\ldots,\beta_v : x(1+z)^N\right]\left(\frac{z}{1+z}\right)^k$$

$$(28) \qquad = (1+z)^{\lambda}\sum_{k=0}^{n}\binom{\lambda+k-1}{k} k!\, S(n,k)$$

$$\cdot\, \zeta_{k,N}^{(\lambda)}[\alpha_1,\ldots,\alpha_u;\beta_1,\ldots,\beta_v : x]\, z^k \quad (n \in \mathbb{N}_0;\ N \in \mathbb{N};\ |z|<1),$$

which, upon letting

$$z \longmapsto \frac{z}{1-z} \qquad \text{and} \qquad x \longmapsto x(1-z)^N,$$

assumes the form:

$$(29) \qquad \sum_{k=0}^{\infty}\binom{\lambda+k-1}{k} k^n\, \zeta_{k,N}^{(\lambda)}[\alpha_1,\ldots,\alpha_u;\beta_1,\ldots,\beta_v : x]\, z^k$$

$$= (1-z)^{-\lambda}\sum_{k=0}^{n}\binom{\lambda+k-1}{k} k!\, S(n,k)$$

$$\cdot\, \zeta_{k,N}^{(\lambda)}\left[\alpha_1,\ldots,\alpha_u;\beta_1,\ldots,\beta_v : x(1-z)^N\right]\left(\frac{z}{1-z}\right)^k$$

$$(n \in \mathbb{N}_0;\ N \in \mathbb{N};\ |z|<1).$$

For the Konhauser *biorthogonal* polynomials $Z_n^{\alpha}(x;\kappa)$ $(\kappa \in \mathbb{N})$ of the *second* kind, defined by (cf. [5], p. 304, Equation (5); see also [14], p. 197, Problem 65)

$$(30) \qquad Z_n^{\alpha}(x;\kappa) := \binom{\alpha+\kappa n}{\kappa n}\frac{(\kappa n)!}{n!}\, {}_1F_{\kappa}\left[-n;\Delta(\kappa;\alpha+1);\left(\frac{x}{\kappa}\right)^{\kappa}\right]$$

$$(\kappa \in \mathbb{N}),$$

which incidentally were considered *earlier* by Toscano [16] *without* their *biorthogonality* property (emphasized upon in Konhauser's work [5]), it is easily seen by comparing (26) and (30) that

$$(31) \qquad Z_n^{\alpha-k}(x;\kappa) = \binom{\alpha-k+\kappa n}{\kappa n}\frac{(\kappa n)!}{n!}\zeta_{k,\kappa}^{(-\alpha)}\left[-n;\ -:\left(\frac{x}{\kappa}\right)^{\kappa}\right]$$

$$(k \in \mathbb{N}_0;\ \kappa \in \mathbb{N}).$$

In view of the relationship (31), the generating function (27) can easily be specialized to the form [4], p. 157, Equation (5.29):

$$\sum_{k=0}^{\infty} \binom{k-\alpha-\kappa n-1}{k} Z_n^{\alpha-k}(x;\kappa)\, t^k = (1-t)^{\alpha} Z_n^{\alpha}(x(1-t);\kappa) \tag{32}$$

$$(n \in \mathbb{N}_0;\ \kappa \in \mathbb{N};\ |t|<1).$$

Upon replacing $n$ and $\alpha$ in (32) by $N$ and $\alpha - n$, respectively, if we apply Theorem 3 to the resulting generating function, we obtain

$$\sum_{k=0}^{\infty} \binom{k-\alpha-\kappa N-1}{k} k^n Z_N^{\alpha-k}(x(1+z);\kappa) \left(\frac{z}{1+z}\right)^k \tag{33}$$

$$= (1+z)^{-\alpha} \sum_{k=0}^{n} \binom{k-\alpha-\kappa N-1}{k} k!\, S(n,k)\, Z_N^{\alpha-k}(x;\kappa)\, z^k$$

$$(n \in \mathbb{N}_0;\ \kappa \in \mathbb{N};\ |z|<1),$$

which, for

$$z \longmapsto \frac{z}{1-z} \qquad \text{and} \qquad x \longmapsto x(1-z),$$

assumes the form:

$$\sum_{k=0}^{\infty} \binom{k-\alpha-\kappa N-1}{k} k^n Z_N^{\alpha-k}(x;\kappa)\, z^k = (1-z)^{\alpha} \tag{34}$$

$$\cdot \sum_{k=0}^{n} \binom{k-\alpha-\kappa N-1}{k} k!\, S(n,k)\, Z_N^{\alpha-k}(x(1-z);\kappa) \left(\frac{z}{1-z}\right)^k$$

$$(n \in \mathbb{N}_0;\ \kappa \in \mathbb{N};\ |z|<1).$$

The generating functions (33) and (34) can *alternatively* be deduced from the corresponding general results (28) and (29), respectively, by appealing to the relationship (31).

### 2.12. Jacobi and Laguerre Polynomials

Srivastava [12] did not consider several *unusual* generating functions for the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and the classical Laguerre polynomials $L_n^{(\alpha)}(x)$ in which the summation index appears *only* in these polynomials' indices $\alpha$ and $\beta$, just as in the generating function (32) for the Konhauser *biorthogonal* polynomials $Z_n^{\alpha}(x;\kappa)$ ($\kappa \in \mathbb{N}$) defined by (30). However, as pointed out earlier by Chen and Srivastava [1], p. 180, each of these *unusual* generating functions is actually a special case of the hypergeometric generating function (27) *with* $N = 1$. Consequently, instead of applying Theorem 3 to each of the following known generating functions *individually*, we can deduce the corresponding result by suitably specializing the generating functions (28) and (29) (cf. [1], pp. 168, 171 and 177):

(35)
$$\sum_{k=0}^{\infty} \binom{k-\alpha-n-1}{k} P_n^{(\alpha-k,\beta+k)}(x)\, t^k = (1-t)^{\alpha}\, P_n^{(\alpha,\beta)}(x-(x-1)\,t) \qquad (n \in \mathbb{N}_0;\ \ |t|<1)\,;$$

(36)
$$\sum_{k=0}^{\infty} \binom{k-\beta-n-1}{k} P_n^{(\alpha+k,\beta-k)}(x)\, t^k = (1-t)^{\beta}\, P_n^{(\alpha,\beta)}(x-(x+1)\,t) \qquad (n \in \mathbb{N}_0;\ \ |t|<1)\,;$$

(37)
$$\sum_{k=0}^{\infty} \binom{k-\alpha-n-1}{k} P_n^{(\alpha-k,\beta)}(x)\, t^k = (1-t)^{\alpha} \left\{1+\frac{1}{2}(x-1)\,t\right\}^n P_n^{(\alpha,\beta)}\left(\frac{x-\frac{1}{2}(x-1)\,t}{1+\frac{1}{2}(x-1)\,t}\right) \qquad (n \in \mathbb{N}_0;\ \ |t|<1)\,;$$

(38)
$$\sum_{k=0}^{\infty} \binom{k-\beta-n-1}{k} P_n^{(\alpha,\beta-k)}(x)\, t^k = (1-t)^{\beta} \left\{1-\frac{1}{2}(x+1)\,t\right\}^n P_n^{(\alpha,\beta)}\left(\frac{x-\frac{1}{2}(x+1)\,t}{1-\frac{1}{2}(x+1)\,t}\right) \qquad (n \in \mathbb{N}_0;\ \ |t|<1)\,;$$

(39)
$$\sum_{k=0}^{\infty} \binom{\alpha+\beta+n+k}{k} P_n^{(\alpha+k,\beta)}(x)\, t^k = (1-t)^{-\alpha-\beta-n-1}\, P_n^{(\alpha,\beta)}\left(\frac{x+t}{1-t}\right) \qquad (n \in \mathbb{N}_0;\ \ |t|<1)\,;$$

(40)
$$\sum_{k=0}^{\infty} \binom{\alpha+\beta+n+k}{k} P_n^{(\alpha,\beta+k)}(x)\, t^k = (1-t)^{-\alpha-\beta-n-1}\, P_n^{(\alpha,\beta)}\left(\frac{x-t}{1-t}\right) \qquad (n \in \mathbb{N}_0;\ \ |t|<1)\,;$$

(41)
$$\sum_{k=0}^{\infty} \binom{k-\alpha-n-1}{k} L_n^{(\alpha-k)}(x)\, t^k = (1-t)^{\alpha}\, L_n^{(\alpha)}(x\,(1-t)) \qquad (n \in \mathbb{N}_0;\ \ |t|<1)\,,$$

which indcidentally is an obvious special case of (32) when $\kappa = 1$, since

$$Z_n^{\alpha}(x;1) = L_n^{(\alpha)}(x) \qquad (n \in \mathbb{N}_0).$$

Thus, by setting $\kappa = 1$ in (34), we immediately obtain the generating function:

$$\begin{aligned}(42)\qquad \sum_{k=0}^{\infty} \binom{k-\alpha-N-1}{k} k^n\, L_N^{(\alpha-k)}(x)\, z^k &= (1-z)^{\alpha} \\ \cdot \sum_{k=0}^{n} \binom{k-\alpha-N-1}{k} k!\, S(n,k)\, L_N^{(\alpha-k)}&(x(1-z)) \left(\frac{z}{1-z}\right)^k \\ &(n \in \mathbb{N}_0;\ |z| < 1)\end{aligned}$$

for the classical Laguerre polynomials. Furthermore, corresponding to each of the generating functions (35) to (40), we similarly find from the general result (29) that

$$\begin{aligned}(43)\qquad \sum_{k=0}^{\infty} \binom{k-\alpha-N-1}{k} k!\, P_N^{(\alpha-k,\beta+k)}(x)\, z^k &= (1-z)^{\alpha} \\ \cdot \sum_{k=0}^{n} \binom{k-\alpha-N-1}{k} k!\, S(n,k)\, P_N^{(\alpha-k,\beta+k)}&(x-(x-1)z) \left(\frac{z}{1-z}\right)^k \\ &(n \in \mathbb{N}_0;\ N \in \mathbb{N}_0;\ |z| < 1),\end{aligned}$$

$$\begin{aligned}(44)\qquad \sum_{k=0}^{\infty} \binom{k-\beta-N-1}{k} k^n\, P_N^{(\alpha+k,\beta-k)}(x)\, z^k &= (1-z)^{\beta} \\ \cdot \sum_{k=0}^{n} \binom{k-\beta-N-1}{k} k!\, S(n,k)\, P_N^{(\alpha+k,\beta-k)}&(x-(x+1)z) \left(\frac{z}{1-z}\right)^k \\ &(n \in \mathbb{N}_0;\ N \in \mathbb{N}_0;\ |z| < 1),\end{aligned}$$

$$\begin{aligned}(45)\qquad &\sum_{k=0}^{\infty} \binom{k-\alpha-N-1}{k} k^n\, P_N^{(\alpha-k,\beta)}(x)\, z^k \\ &= (1-z)^{\alpha} \left\{1 + \frac{1}{2}(x-1)z\right\}^N \sum_{k=0}^{n} \binom{k-\alpha-N-1}{k} k!\, S(n,k) \\ &\cdot P_N^{(\alpha-k,\beta)} \left(\frac{x-\frac{1}{2}(x-1)z}{1+\frac{1}{2}(x-1)z}\right) \left(\frac{z}{1-z}\right)^k \\ &\qquad (n \in \mathbb{N}_0;\ N \in \mathbb{N}_0;\ |z| < 1),\end{aligned}$$

$$\text{(46)} \quad \sum_{k=0}^{\infty} \binom{k-\beta-N-1}{k} k^n \, P_N^{(\alpha,\beta-k)}(x)\, z^k$$
$$= (1-z)^{\beta} \left\{1 - \frac{1}{2}(x+1)\,z\right\}^N \sum_{k=0}^{n} \binom{k-\alpha-N-1}{k} k!\, S(n,k)$$
$$\cdot\, P_N^{(\alpha,\beta-k)} \left( \frac{x - \frac{1}{2}(x+1)\,z}{1 - \frac{1}{2}(x+1)\,z} \right) \left( \frac{z}{1-z} \right)^k$$
$$(n \in \mathbb{N}_0;\ N \in \mathbb{N}_0;\ |z| < 1)\,,$$

$$\text{(47)} \quad \sum_{k=0}^{\infty} \binom{\alpha+\beta+N+k}{k} k^n \, P_N^{(\alpha+k,\beta)}(x)\, z^k = (1-z)^{-\alpha-\beta-N-1}$$
$$\cdot \sum_{k=0}^{n} \binom{\alpha+\beta+N+k}{k} k!\, S(n,k)\, P_N^{(\alpha+k,\beta)} \left( \frac{x+z}{1-z} \right) \left( \frac{z}{1-z} \right)^k$$
$$(n \in \mathbb{N}_0;\ N \in \mathbb{N}_0;\ |z| < 1)\,,$$

and

$$\text{(48)} \quad \sum_{k=0}^{\infty} \binom{\alpha+\beta+N+k}{k} k^n \, P_N^{(\alpha,\beta+k)}(x)\, z^k = (1-z)^{-\alpha-\beta-N-1}$$
$$\sum_{k=0}^{n} \binom{\alpha+\beta+N+k}{k} k!S(n,k)\, P_N^{(\alpha,\beta+k)} \left( \frac{x-z}{1-z} \right) \left( \frac{z}{1-z} \right)^k$$
$$(n \in \mathbb{N}_0;\ N \in \mathbb{N}_0;\ |z| < 1)\,,$$

respectively.

In view of some well-known *indicial* relationships between Jacobi polynomials themselves (cf. [14] and [15]), the generating functions (35) to (40) (and hence also their consequences (43) to (48)) are all *equivalent* to one another (see, for details, [1]). Furthermore, since [15], p. 103, Equation (5.3.4)

$$\text{(49)} \qquad L_n^{(\alpha)}(x) = \lim_{|\beta|\to\infty} \left\{ P_n^{(\alpha,\beta)} \left( 1 - \frac{2x}{\beta} \right) \right\},$$

the generating function (41) can also be deduced as a limit case of (for example) (35) and (37). On the other hand, by appealing to the limit relationship (49), each of the generating functions (36) and (39) *with*

$$x \longmapsto 1 - \frac{2x}{\beta} \qquad \text{and} \qquad t \longmapsto \pm \frac{t}{\beta}$$

would yield the following well-known (*rather classical*) result (cf., e.g., [14], 172, Problem 22 (ii)]):

$$\sum_{k=0}^{\infty} L_n^{(\alpha+k)}(x)\, \frac{t^k}{k!} = e^t L_n^{(\alpha)}(x-t)\,,$$

which, by means of Theorem 3, leads us *eventually* to the generating function (cf. Equation (42)):

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} L_N^{(\alpha+k)}(x)\, z^k = e^z \sum_{k=0}^{n} S(n,k)\, L_N^{(\alpha+k)}(x-z)\, z^k \qquad (n \in \mathbb{N}_0;\; N \in \mathbb{N}_0)$$

associated with the Stirling numbers $S(n,k)$ defined by (1).

In the cases of the polynomials which are related rather closely to the classical Jacobi or classical Laguerre polynomials, some of the generating functions presented in this section can *alternatively* be derived from the corresponding results considered by Srivastava [12], Section 3, by suitably exploiting these relationships. For example, in view of the relationship (20) with the classical Laguerre polynomials $L_n^{(\alpha)}(x)$, the generating function (21) for the *modified* Laguerre polynomials $f_n^{\alpha}(x)$ can *alternatively* be deduced from the following result of Srivastava [12], p. 760, Equation (3.27):

$$\begin{aligned}\sum_{k=0}^{\infty} k^n L_k^{(\alpha-k)}(x)\, z^k &= (1+z)^{\alpha} \exp(-xz) \\ &\cdot \sum_{k=0}^{n} k!\, S(n,k)\, L_k^{(\alpha-k)}(x(1+z)) \left(\frac{z}{1+z}\right)^k \qquad (n \in \mathbb{N}_0;\; |z| < 1)\end{aligned} \tag{50}$$

by first letting $\alpha \longmapsto -\alpha$ *and* $z \longmapsto -z$, and then applying the relationship (20) on both sides of the resulting equation.

## 3. Applications of Theorems 2 and 4

### 3.1. Generalized Bessel Functions

In terms of the generalized Bessel function $J_\nu^{(\mu)}(x,y;\tau)$ defined by (cf., e.g., Dattoli *et al.* [2])

$$J_\nu^{(\mu)}(x,y;\tau) := \sum_{l=-\infty}^{\infty} J_l(x)\, J_{\nu+\mu l}(y)\, \tau^l,$$

so that

$$\lim_{x\to 0} \left\{ J_\nu^{(\mu)}(x,y;\tau) \right\} = J_\nu(y)$$

and

$$J_\nu^{(-1)}(x,y;1) = \sum_{l=-\infty}^{\infty} J_l(x)\, J_{\nu-l}(y) = J_\nu(x+y),$$

a generalization of the generating function (12) in the form:

$$\sum_{k=0}^{\infty} J_{\nu+k}^{(\mu)}(x, y; \tau) \frac{t^k}{k!} = \left(1 - \frac{2t}{y}\right)^{-\frac{1}{2}\nu} \cdot J_{\nu}^{(\mu)}\left(x, \sqrt{y^2 - 2yt}; \tau\,[1 - 2\,(t/y)]^{-\frac{1}{2}\mu}\right) \tag{51}$$

$$\left(\mu, \nu \in \mathbb{C};\ \ |t| < \frac{1}{2}\,|y|\right)$$

was given recently by Pathan *et al.* [9, p. 179, Equation (3.4)]. Thus, by applying Theorem 4 to the generating function (51), we obtain

$$\sum_{k=0}^{\infty} \frac{k^n}{k!}\, J_{\nu+k}^{(\mu)}\left(x, \sqrt{y^2 + 2yz}; \tau\,[1 + 2\,(z/y)]^{-\frac{1}{2}\mu}\right) \left(\frac{z}{\sqrt{1 + 2\,(z/y)}}\right)^k$$
$$= \left(1 + \frac{2z}{y}\right)^{\frac{1}{2}\nu} \sum_{k=0}^{n} S\,(n, k)\, J_{\nu+k}^{(\mu)}(x, y; \tau)\, z^k$$
$$\left(\mu, \nu \in \mathbb{C};\ \ |t| < \frac{1}{2}\,|y|\,;\ n \in \mathbb{N}_0\right),$$

which, under such variable and notational changes as in the transition from (13) to (14), but involving $z$ and $y$ (instead of $z$ and $x$, respectively), yields

$$\sum_{k=0}^{\infty} \frac{k^n}{k!}\, J_{\nu+k}^{(\mu)}(x, y; \tau)\, z^k = \left(1 - \frac{2z}{y}\right)^{-\frac{1}{2}\nu} \cdot \sum_{k=0}^{n} S\,(n, k)\, J_{\nu+k}^{(\mu)}\left(x, \sqrt{y^2 - 2yz}; \tau\,[1 - 2\,(z/y)]^{-\frac{1}{2}\mu}\right) \left(\frac{z}{\sqrt{1 - 2\,(z/y)}}\right)^k \tag{52}$$

$$\left(\mu, \nu \in \mathbb{C};\ \ |t| < \frac{1}{2}\,|y|\,;\ n \in \mathbb{N}_0\right),$$

where we have also set

$$\tau \longmapsto \tau \left(1 - \frac{2z}{y}\right)^{-\frac{1}{2}\mu}$$

*after* the aforementioned transition.

The generating function (52) unifies (as well as extends) each of the results (12), (14), and (51) above.

### 3.2. Lauricella Polynomials

Another application of Theorem 4 involves the so-called Lauricella polynomials in several variables (cf. [6]):

$$
\begin{aligned}
&F_D^{(s)}\left[-n, b_1, \ldots, b_s; c; x_1, \ldots, x_s\right] \\
&\quad := \sum_{k_1, \ldots, k_s=0}^{k_1+\cdots+k_s \leqq n} \frac{(-n)_{k_1+\cdots+k_s}\,(b_1)_{k_1} \cdots (b_s)_{k_s}}{(c)_{k_1+\cdots+k_s}} \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_s^{k_s}}{k_s!},
\end{aligned}
$$

where, as usual in the theory of hypergeometric series,

$$
(\lambda)_0 := 1 \quad \text{and} \quad (\lambda)_k := \lambda\,(\lambda+1) \cdots (\lambda+k-1) \quad (k \in \mathbb{N}).
$$

These Lauricella polynomials are known to satisfy a generating-function relationship in the form (cf. [17], p. 240; see also [14], p. 439, Equation 8.5 (8)):

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(c+n)_k}{k!}\, F_D^{(s)}\left[-n-k, b_1, \ldots, b_s; c; x_1, \ldots, x_s\right] t^k \\
&\quad = (1-t)^{-c-n} \prod_{j=1}^{s} \left\{\left(1+\frac{x_j t}{1-t}\right)^{-b_j}\right\} \\
&\quad \cdot F_D^{(s)}\left[-n, b_1, \ldots, b_s; c; \frac{x_1}{1-t+x_1 t}, \ldots, \frac{x_s}{1-t+x_s t}\right] \\
&\qquad \left(n \in \mathbb{N}_0;\ \ |t| < \min_{1 \leqq j \leqq s}\left\{1, \left|x_j - 1\right|^{-1}\right\}\right),
\end{aligned}
\tag{53}
$$

which fits easily into the pattern (8) with, of course,

$$
\begin{aligned}
&\theta\,(x_1, \ldots, x_s; t) = (1-t)^{-c} \prod_{j=1}^{s} \left\{\left(1+\frac{x_j t}{1-t}\right)^{-b_j}\right\}, \\
&\phi\,(x_1, \ldots, x_s; t) = 1-t, \qquad \psi_j\,(x_1, \ldots, x_s; t) = \frac{x_j}{1-t+x_j t} \\
&\qquad (j = 1, \ldots, s),
\end{aligned}
$$

and

$$
\Lambda_k\,(x_1, \ldots, x_s) \longmapsto (c)_k\, F_D^{(s)}\left[-k, b_1, \ldots, b_s; c; x_1, \ldots, x_s\right] \quad (k \in \mathbb{N}_0).
$$

Theorem 4, when applied to the generating function (53), yields

$$\sum_{k=0}^{\infty} \frac{(c)_k\, k^n}{k!}\, F_D^{(s)}\left[-k, b_1, \ldots, b_s; c; \frac{x_1}{1+z-x_1 z}, \ldots, \frac{x_s}{1+z-x_s z}\right] \cdot \left(\frac{z}{1+z}\right)^k = (1+z)^c \prod_{j=1}^{s}\left\{\left(1-\frac{x_j z}{1+z}\right)^{b_j}\right\} \cdot \sum_{k=0}^{n} (c)_k\, S(n,k)\, F_D^{(s)}[-k, b_1, \ldots, b_s; c; x_1, \ldots, x_s]\, z^k$$
$$\left(n \in \mathbb{N}_0;\ \ |z| < \min_{1 \leqq j \leqq s}\left\{1, \left|x_j - 1\right|^{-1}\right\}\right), \tag{54}$$

which, for

$$z \longmapsto \frac{z}{1-z} \quad \text{and} \quad x_j \longmapsto \frac{x_j}{1-z+x_j z} \qquad (j = 1, \ldots, s),$$

assumes the form:

$$\sum_{k=0}^{\infty} \frac{(c)_k\, k^n}{k!}\, F_D^{(s)}[-k, b_1, \ldots, b_s; c; x_1, \ldots, x_s]\, z^k = (1-z)^{-c} \prod_{j=1}^{s}\left\{\left(1+\frac{x_j z}{1-z}\right)^{-b_j}\right\} \sum_{k=0}^{n} (c)_k\, S(n,k) \cdot F_D^{(s)}\left[-k, b_1, \ldots, b_s; c; \frac{x_1}{1-z+x_1 z}, \ldots, \frac{x_s}{1-z+x_s z}\right]\left(\frac{z}{1-z}\right)^k$$
$$\left(n \in \mathbb{N}_0;\ \ |z| < \min_{1 \leqq j \leqq s}\left\{1, \left|x_j - 1\right|^{-1}\right\}\right). \tag{55}$$

For $c = 1$, these last generating functions (54) and (54) were derived earlier by Srivastava [12], p. 768, Equations (5.11) and (5.12).

### 3.3. Multivariable Sequences

Some general multivariable extensions of the hypergeometric generating functions (24) and (27) were considered by Chen and Srivastava [1], who investigated the multivariable generating functions [1], p. 172, Equation (5.19):

$$\sum_{k=0}^{\infty} \binom{\lambda+n+k-1}{k} \Omega_{n+k}^{(\lambda)}(\sigma_1, \ldots, \sigma_s; x_1, \ldots, x_s)\, t^k = (1-t)^{-\lambda-n}\, \Omega_n^{(\lambda)}\left(\sigma_1, \ldots, \sigma_s; \frac{x_1}{(1-t)^{\sigma_1}}, \ldots, \frac{x_s}{(1-t)^{\sigma_s}}\right)$$
$$(n \in \mathbb{N}_0;\ \ |t| < 1), \tag{56}$$

where

$$\Omega_n^{(\lambda)}(\sigma_1,\ldots,\sigma_s;x_1,\ldots,x_s) := \sum_{m_1,\ldots,m_s=0}^{\infty} (\lambda+n)_M\, \mathcal{A}(m_1,\ldots,m_s)\, x_1^{m_1}\cdots x_s^{m_s}$$
$$\left(M := \sigma_1 m_1 + \cdots + \sigma_s m_s;\ m_j \in \mathbb{N}_0;\ \lambda, \sigma_j \in \mathbb{C};\ j = 1,\ldots,s\right),$$

and [1], p. 172, Equation (5.20)

$$\text{(57)}\qquad \sum_{k=0}^{\infty} \binom{\lambda+n+k-1}{k} \mathcal{Z}_{n+k}^{(\lambda)}(\sigma_1,\ldots,\sigma_s;x_1,\ldots,x_s)\, t^k = (1-t)^{-\lambda-n}\, \mathcal{Z}_n^{(\lambda)}\big(\sigma_1,\ldots,\sigma_s;x_1(1-t)^{\sigma_1},\ldots,x_s(1-t)^{\sigma_s}\big)$$
$$(n \in \mathbb{N}_0;\ |t| < 1),$$

where

$$\mathcal{Z}_n^{(\lambda)}(\sigma_1,\ldots,\sigma_s;x_1,\ldots,x_s) := \sum_{m_1,\ldots,m_s=0}^{\infty} \frac{\mathcal{A}(m_1,\ldots,m_s)}{(1-\lambda-n)_M} x_1^{m_1}\cdots x_s^{m_s}$$
$$\left(M := \sigma_1 m_1 + \cdots + \sigma_s m_s;\ m_j \in \mathbb{N}_0;\ \lambda, \sigma_j \in \mathbb{C};\ j = 1,\ldots,s\right),$$

$\{\mathcal{A}(m_1,\ldots,m_s)\}$ being a suitably bounded multiple sequence of complex numbers. These multivariable generating functions (56) and (57) are, in fact, very specialized cases of much more general multivariable generating functions given earlier by Srivastava (cf., e.g., [14], p. 491, Problem 3; see also [1], p. 173).

By applying Theorem 4 to each of the generating functions (56) and (57), we *finally* obtain the following multivariable generalizations of our results (25) and (29) above:

$$\sum_{k=0}^{\infty} \binom{\lambda+k-1}{k} k^n\, \Omega_k^{(\lambda)}(\sigma_1,\ldots,\sigma_s;x_1,\ldots,x_s)\, z^k = (1-z)^{-\lambda} \sum_{k=0}^{n} \binom{\lambda+k-1}{k} k!\, S(n,k) \cdot \Omega_k^{(\lambda)}\left(\sigma_1,\ldots,\sigma_s;\frac{x_1}{(1-z)^{\sigma_1}},\ldots,\frac{x_s}{(1-z)^{\sigma_s}}\right)\left(\frac{z}{1-z}\right)^k$$
$$(n \in \mathbb{N}_0;\ |z| < 1)$$

and

$$\sum_{k=0}^{\infty} \binom{\lambda+k-1}{k} k^n \mathcal{Z}_k^{(\lambda)} (\sigma_1, \ldots, \sigma_s; x_1, \ldots, x_s) z^k$$
$$= (1-z)^{-\lambda} \sum_{k=0}^{n} \binom{\lambda+k-1}{k} k!\, S(n,k)$$
$$\cdot \mathcal{Z}_k^{(\lambda)} \big(\sigma_1, \ldots, \sigma_s; x_1 (1-z)^{\sigma_1}, \ldots, x_s (1-z)^{\sigma_s}\big)$$
$$(n \in \mathbb{N}_0;\ \ |z| < 1)\,.$$

Many other applications of *each* of the general results (Theorems 1 to 4 above) can indeed be presented in an analogous manner.

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