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E. Ballico[†]

NON-SIMPLE VECTOR BUNDLES ON CURVES

Abstract. Let A be a finite dimensional unitary algebra over an algebraically closed field $\mathbf{K}$. Here we study the vector bundles on a smooth projective curve which are equipped with a faithful action of A.

1. Introduction

Let $\mathbf{K}$ be an algebraically closed field, A a finite dimensional unitary $\mathbf{K}$-algebra, X a smooth connected complete curve of genus g defined over Spec($\mathbf{K}$), E a vector bundle on X and $h : A \to H^0(X, End(E))$ an injective homomorphism of unitary $\mathbf{K}$-algebras. Hence $Id \in h(A)$. We will say that the pair (E, h) is an A-sheaf or an A-vector bundle. A subsheaf F of A will be called an A-subsheaf of (E, h) (or just an A-subsheaf of E) if it is invariant for the action of $h(E)$ on E. Notice that if $A \neq \mathbf{K}$, then E is not simple and in particular rank$(E) > 1$ and E is not stable. For any vector bundle G on X let $\mu := \deg(G)/\mathrm{rank}(G)$ denote its slope. We will say that (A, h) is A-stable (resp. A-semistable) if for every A-subsheaf F of E with $0 < \mathrm{rank}(F) < \mathrm{rank}(G)$ we have $\mu(F) < \mu(E)$ (resp. $\mu(F) \leq \mu(E)$). In section 2 we will prove the following results which give the connection between semistability and A-stability.

THEOREM 1. *Let (E, h) be an A-vector bundle. E is semistable if and only if (E, h) is A-semistable.*

THEOREM 2. *Let (E, h) be an A-vector bundle. Assume that E is polystable as an abstract bundle, i.e. assume that E is a direct sum of stable vector bundles with the same slope. (E, h) is A-stable if and only if there is an integer $r \geq 1$ and a stable vector bundle F such that $E \cong F^{\oplus r}$ and A is a unitary $\mathbf{K}$-subalgebra of the unitary $\mathbf{K}$-algebra $M_{r\times r}(\mathbf{K})$ of $r \times r$ matrices whose action on $\mathbf{K}^{\oplus r}$ is irreducible.*

THEOREM 3. *Let (E, h) be an A-sheaf. Assume that E is semistable but not polystable. Then E is not A-stable.*

DEFINITION 1. *Let (E, h) be an A-sheaf. For any A-subsheaf F of E let $h(A, F)$ be the image of $h(A)$ into $H^0(X, End(F))$. Set $c(h, F) := dim_{\mathbf{K}} h(A, F)$, $\lambda_A(F) := \mu(F)/c(h, F)$ and $\epsilon_A(F) = \mu(F)c(h, F)$. We will say that (E, h) (or just E) is λ_A-stable (resp. λ_A-semistable) if for every proper A-subsheaf F of E we have $\lambda_A(F) < \lambda_A(E)$ (resp. $\lambda_A(F) \leq \lambda_A(E)$). We will say that (E, h) (or just E) is ϵ_A-stable (resp. ϵ_A-semistable) if for every proper A-subsheaf F of E we have $\epsilon_A(F) < \epsilon_A(E)$ (resp. $\epsilon_A(F) \leq \epsilon_A(E)$).*

For any subsheaf F of the vector bundle E on X the saturation G of F in E is the only subsheaf G of E such that $F \subseteq G$, rank$(G) =$ rank(F) and E/G has no torsion, i.e. E/G is

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locally free if $\text{rank}(F) < \text{rank}(E)$, while $G = E$ if $\text{rank}(F) = \text{rank}(E)$.

REMARK 1. Let (E, h) be an A-sheaf, F an A-subsheaf of E and G the saturation of F in E. G is $h(A)$-invariant and hence it is an A-sheaf. Since $h(A, F) = h(A, G)$, we have $\lambda_A(F) \leq \lambda_A(G)$, $\epsilon_A(F) \leq \epsilon_A(G)$, $\lambda_A(F) = \lambda_A(G)$ if and only if $G = F$ and $\epsilon_A(F) = \epsilon_A(G)$ if and only if $G = F$

For any vector bundle F and any line bundle L we have $End(F) \cong End(F \otimes L)$ and $\mu(F \otimes L) = \mu(F) + \deg(L)$. This shows that in general the notions of λ_A-stability, λ_A-semistability, ϵ_A-stability and ϵ_A-semistability are NOT invariant for the twist by a line bundle (see Example 1). We believe that ϵ_A-stability is the correct notion for the Brill - Noether theory of non-simple vector bundles. In section 3 we will describe all the **K**-algebras arising for rank two vector bundles.

2. Proofs of Theorems 1, 2 and 3

Let (E, h) be an A-sheaf on X. Since the saturation of an A-subsheaf of E is an A-subsheaf of E, the usual proof of the existence of an Harder - Narasimhan filtration of any vector bundle on X (see for instance [2], pp. 15–16) gives the following result.

PROPOSITION 1. *Let (E, h) be an A-sheaf. There is an increasing filtration $\{E_i\}_{0 \leq i \leq r}$ of E by saturated A-subsheaves such that $E_0 = \{0\}$, $E_r = E$, E_i is saturated in E_{i+1} for $0 \leq i < r$ and E_{i+1}/E_i is A_i-semistable, where $A_i \subseteq H^0(X, End(E_{i+1}/E_i))$ is the image of $h(E)$ in $H^0(X, End(E_{i+1}/E_i))$ and $\mu(E_{i+1}/E_i) > \mu(B)$ for every other A_i-subsheaf of E/E_i.*

Proof of Theorem 1. If E is semistable, then obviously it is A-semistable. Assume that E is not semistable and let F be the first step of the Harder - Narasimhan filtration of E. Thus $\{0\} \neq F$ and $\mu(F) > \mu(E)$. By the uniqueness of the Harder - Narasimhan filtration of E the subsheaf F of E is invariant for the action of $\text{Aut}(E)$. Since $\text{Aut}(E)$ is a non-empty open subset of $H^0(X, End(E))$, F is invariant for the action of the **K**-algebra $H^0(X, End(E))$. Since $h(A) \subseteq H^0(X, End(E))$, F is an A-subsheaf of E. Thus E is not A-semistable.

□

Proof of Theorem 2. The if part is easy (see Example 2). Here we will check the other implication. Since E is polystable, there is an integer $s \geq 1$, stable bundles $F_1, \dots, F_s$ (uniquely determined up to a permutation of their indices) with $F_i \ncong F_j$ if $i \neq j$ and positive integers $r_1, \dots, r_s$ such that $E \cong \oplus_{1 \leq i \leq s} F_i^{\bigoplus r_i}$. Since E is polystable, $\mu(F_i) = \mu(F_j)$ for all i, j. Since F_i and F_j are stable, with the same slope and not isomorphic, $h^0(X, Hom(F_i, F_j)) = 0$ if $i \neq j$. Hence $H^0(X, End(E)) \cong \bigoplus_{1 \leq i \leq s} M_{r_i \times r_i}(\mathbf{K})$. Since each factor $F_i^{\oplus r_i}$ is invariant for the action of the group $\text{Aut}(E)$, it is $H^0(X, End(E))$-invariant and hence $h(A)$-invariant, i.e. it is an A-sheaf. Since $\mu(F_i) = \mu(F_j)$ for any i, j, E is A-stable only if $s = 1$. Obviously, A is a unitary **K**-subalgebra of the unitary **K**-algebra $M_{r_1 \times r_1}(\mathbf{K})$ of $r_1 \times r_1$ matrices and the induced action of A is irreducible because no proper direct factor of $F_1^{\oplus r_1}$ is A-invariant.

□

Proof of Theorem 3. Since E is semistable but not polystable, the existence of a Jordan - Hölder filtration of E shows the existence of a maximal proper subsheaf F of E with $0 \neq F \neq E$ and $\mu(F) = \mu(E)$. Indeed, F contains all proper subsheaves of E with slope $\mu(E)$. Thus F is invariant for the action of the group $\mathrm{Aut}(E)$. Hence F is $H^0(X, End(E))$-invariant and hence an A-sheaf. Thus E is not A-stable.

□

EXAMPLE 1. Take (E, h) with $A \neq \mathbf{K}$, $\mathrm{rank}(E) = 2$ and E non-split extension of a line bundle M by a line bundle L. Set $a = \dim_{\mathbf{K}}(A)$. Assume that L is A-invariant and that E has no A-invariant line subbundle of degree $> \deg(L)$; the last condition is always satisfied if $\deg(L) \geq \deg(M)$; both conditions are satisfied if $\deg(L) \geq \deg(M)$ and $E \ncong L \oplus M$. Hence, with the notation of Example 3, $A \cong A(V)$ for some vector subspace V of $H^0(X, Hom(M, L))$. Hence $\deg(M) \geq \deg(L)$. We have $\lambda_A(L) = \deg(L)$ and $\lambda_A(E) = \deg(E)/2a = (\deg(L) + \deg(M))/2a$. Since $h^0(X, Hom(M, L)) > 0$, E is not λ_A-stable if $\deg(M) \geq 0$. If $\deg(M) \geq 0$, then E is λ_A-semistable if and only if $L \cong M$ (i.e. equivalently by the condition $h^0(X, Hom(M, L)) > 0$ if and only if $\deg(M) \geq \deg(L)$) and $a = 2$. If $2(\deg(L)) < a(\deg(L) + \deg(M))$ (resp. $2(\deg(L)) \leq a(\deg(L) + \deg(M))$, then E is ϵ_A-stable (resp. ϵ_A-semistable). Hence if $\deg(M) \geq 0$, E is always ϵ_A-semistable and it is ϵ_A-stable if and only if either $\deg(M) > 0$ or $a \geq 3$.

REMARK 2. If (E, h) is λ_A-semistable (resp. λ_A-stable) then it is A-semistable (resp. A-stable) because $c(h, F) \leq c(h, E)$ for every A-subsheaf F of E.

PROPOSITION 2. *Fix integers a, r, d with $a \geq 1$ and $r \geq 2$. Let X be a smooth and connected projective curve. Let $R(r, d, a)$ (resp. $S(r, d, a)$, resp. $T(r, d, a)$) be the set of all vector bundles E on X such that there exists a unitary* **K***-algebra A with $\dim(A) = a$ and an injective homomorphism of* **K***-algebras $h : A \to H^0(X, End(E))$ such that the pair (E, h) is A-semistable (resp. λ_A-semistable, resp. ϵ_A-semistable). Then $R(r, d, a)$, $S(r, d, a)$ and $T(r, d, a)$ are bounded.*

Proof. The boundedness of $R(r, d, a)$ follows from Theorem 1 and the boundedness of the set of all isomorphism classes of semistable bundles with rank r and degree d. The boundedness of $S(r, d, a)$ follows from the boundedness of $R(r, d, a)$ and Remark 2. Now we will check the boundedness of $T(r, d, a)$ proving that it is a finite union of bounded sets. The intersection of $T(r, d, a)$ with the set of all semistable bundles is obviously bounded. Hence we may consider only unstable bundles. Let $T(r, d, a; c_1, \ldots, c_x)$ be the set of all bundles $E \in T(r, d, a)$ formed by the vector bundles whose Harder - Narasimhan filtration is of the form $\{E_i\}_{0 \leq i \leq x+1}$ with $E_0 = \{0\}$, $\mathrm{rank}(E_i) = c_i$ for $1 \leq i \leq x$ and $E_{x+1} = E$. Since $E \in T(r, d, a)$ and each E_i is an A-sheaf, we have $\deg(E_i)c(h, E_i)/c_i \leq \deg(E)a/r$ and hence $\deg(E/E_i) = \deg(E) - \deg(E_i) \geq \deg(E)(1 - ac_i/rc(h, E_i))$. The set of all vector bundles on X with rank r, degree d and an $x + 1$ steps Harder - Narasimhan filtration satisfying these x inequalities is bounded ([1]); in this particular case this may be checked in the following way; for $0 \leq i \leq x$ the set of all semistable bundles E_{i+1}/E_i is bounded; in particular the set of all possible E_1 is bounded; the set of all possible E_{i+1} is contained in the set of all extensions of members of two bounded families, the one containing E_{i+1}/E_i and the one containing E_i, and hence it is bounded; inductively, after at most r steps we obtain the result.

□

From now on in this section we consider the case in which X is an integral projective curve. Set $g := p_a(X)$. An A-sheaf is a pair (E, h) where E is a torsion free sheaf on X and $h : A \to H^0(X, End(E))$ is an injective homomorphism of unitary **K**-algebras. A subsheaf F of E is saturated in E if and only if either $F = E$ or E/F is torsion free. Every subsheaf F of

E admits a unique saturation, i.e. it is contained in a unique saturated subsheaf of E with rank $\mathrm{rank}(F)$.

REMARK 3. Proposition 1 is true for a torsion free pair (E, h) on X; obviously in its statement the sheaves E_i, $1 \leq i < r$, are not necessarly locally free but each sheaf E_{i+1}/E_i is torsion free. The proofs of Theorems 1, 2, 3 and of Proposition 2 work verbatim.

3. Nilpotent algebras

DEFINITION 2. *We will say that A is pointwise nilpotent if for every $f \in A$ there is $\lambda \in \mathbf{K}$ and an integer $t > 0$ such that $(f - \lambda)^t = 0$. In this case λ is called the eigenvalue of f and the minimal such integer t is called the nil-exponent of f. The nil-exponent is a semicontinuos function on the finite-dimensional $\mathbf{K}$-vector space A with respect to the Zariski topology. Hence in the definition of pointwise-nilpotency we may take the same integer t for all $f \in A$.*

REMARK 4. Fix $f \in h(A)$ such that there is $\lambda \in \mathbf{K}$ and $t \geq 2$ such that $(f - \lambda Id)^t = 0$ and $(f - \lambda Id)^{t-1} \neq 0$. For any integer $u \geq 0$ set $E(f, u) := \mathrm{Ker}((f - \lambda Id)^u)$. Since $\mathrm{Im}((f - \lambda Id)^u) \subseteq E$, $\mathrm{Im}((f - \lambda)^u)$ is torsion free and hence $E(f, u)$ is saturated in E and in $E(f, u + 1)$. Looking at the Jordan normal form of the endomorphism of the fiber $E|\{P\}$, P general in X, induced by $f - \lambda Id$, we see that $\mathrm{rank}(E(f, u)) < \mathrm{rank}(E(f, u + 1))$ for every integer u with $0 \leq u < t$. In particular $t \leq \mathrm{rank}(E)$ and we have $t = \mathrm{rank}(E)$ if and only if $E(f, 1)$ is a line subbundle of E.

EXAMPLE 2. Fix an integer $r \geq 2$ and let A be a unitary $\mathbf{K}$-subalgebra of the unitary $\mathbf{K}$-algebra $M_{r \times r}(\mathbf{K})$ of $r \times r$ matrices whose action on $\mathbf{K}^{\oplus r}$ is irreducible. For any $L \in \mathrm{Pic}(X)$ the vector bundle $E := L^{\oplus r}$ is an A-sheaf. E is semistable as an abstract vector bundle and every rank s subbundle F of E with $\mu(F) = \mu(E)$ is isomorphic to $L^{\oplus s}$ and obtained from E fixing an s-dimensional linear subspace of $\mathbf{K}^{\oplus r}$. Thus we easily check that E is A-stable. Similarly, for any stable vector bundle G the vector bundle $G^{\oplus r}$ is A-stable.

EXAMPLE 3. Assume $A \neq \mathbf{K}Id$ and take an A-paier (E, h) with $\mathrm{rank}(E) = 2$. Hence E is not simple but no proper saturated subsheaf L of E may have a faithful representation $A \to H^0(X, End(L))$; more precisely, a saturated proper subsheaf L of E is an A-subsheaf of E if and only if each element of $h(A)$ acts as a multiple of the identity on L. First assume E indecomposable. Since E is not simple but indecomposable, it is easy to check the existence of uniquely determined line bundles L, M on X such that E is a non-split extension of M by L and $\deg(L) \geq \deg(M)$. we have $h^0(X, End(E)) = 1 + h^0(X, Hom(M, L))$ and there is a linear surjective map $H^0(X, End(E)) \to H^0(X, Hom(M, L))$ with $\mathrm{Ker}(u) = \mathbf{K}Id$. For every linear subspace V of $H^0(X, Hom(M, L))$ there is a unique unitary $\mathbf{K}$-subalgebra $A(V)$ of $H^0(X, End(E))$ with $u(A(V)) = V$. We have $\dim(A(V)) = 1 + \dim(V)$ and $A(V)$ is pointwise-nilpotent with nil-esponent two (except the case $V = \{0\}$ because $A(\{0\}) = \mathbf{K}Id$). Each algebra $A(V)$ is commutative. For every unitary $\mathbf{K}$-subalgebra B of $H^0(X, End(E))$ there is a unique linear subspace V of $H^0(X, Hom(M, L))$ such that $B = A(V)$. Now assume E decomposable, say $E = L \oplus M$. $H^0(X, End(E))$ is not pointwise-nilpotent. We have $h^0(X, End(E)) = 2 + h^0(X, Hom(M, L))$. If $L \cong M$, then $H^0(X, End(E)) \cong M_{2\times 2}(\mathbf{K})$. Any commutative subalgebra of $H^0(X, End(E))$ has dimension at most two and it is isomorphic to $\mathbf{K} \oplus \mathbf{K}$ with componentwise multiplication. Any pointwise-nilpotent subalgebra of $H^0(X, End(E))$ has dimension at most two and if it is not trivial it has nil-exponent two. Now assume $L \ncong M$. Hence either $h^0(X, Hom(M, L)) = 0$

or $h^0(X, Hom(L, M))$. Just to fix the notation we assume $h^0(X, Hom(L, M)) = 0$. Every non-trivial pointwise nilpotent subalgebra B of $H^0(X, End(E))$ has nil-exponent two and dimension at most $1 + h^0(X, Hom(M, L))$. For any integer v with $0 \leq v \leq h^0(X, Hom(M, L))$ and for every linesr subspace V of $H^0(X, Hom(M, L))$ with $\dim(V) = v$ there is a pointwise nilpotent subalgebra B of $H^0(X, End(E))$ and the isomorphism class of B as abstract **K**-algebra depends only from v, not the choice of V and are isomorphic to the algebra $A(V)$ just described in the indecomposable case. A byproduct of the discussion just given is that E is A-stable if and only if $A \cong M_{2\times 2}(\mathbf{K})$ and $E \cong L \oplus L$.

EXAMPLE 4. Fix an integer $a \geq 2$ and two vector bundles B, D on X such that $h^0(X, Hom(B, D)) \geq a - 1$. Fix a linear subspace V of $H^0(X, Hom(B, D))$ with $\dim(V) = a - 1$ and let $D(V) := \mathbf{K}Id \oplus V$ be the unitary **K**-algebra obtained taking the trivial multiplication on V, i.e. such that $uw = 0$ for all $u, w \in V$. Notice that $D(V)$ is commutative. Consider an extension

$$0 \to B \to E \to D \to 0 \tag{1}$$

of D by B. There is a unique injection $h : D(V) \to H^0(X, End(E))$ of unitary **K**-algebras obtained sending the element $v \in V \subset D(V)$ into the endomorphism $f_v : E \to E$ obtained as composition of the surjection $E \to D$ given by (1), the map $v : D \to B$ and the inclusion $B \to E$ given by (1).

PROPOSITION 3. *Assume char*$(\mathbf{K}) \neq 2$*. Let A be a commutative pointwise-nilpotent algebra with nil-exponent two and* (E, h) *an A-sheaf. Set* $a := \dim(A)$*. Then there exist vector bundles B, D and a linear subspace V of* $H^0(X, Hom(B, D))$ *with* $\dim(V) = a - 1$ *such that, with the notation of Example 4, E fits in an exact sequence (1),* $A \cong D(V)$ *and h is obtained as in Example 4, up to the identification of A with* $D(V)$.

Proof. Take a general $h \in h(A)$ and let λ be its eigenvalue. Set $u = f - \lambda Id$, $B' = \mathrm{Ker}(u)$ and $D' = E/B'$. Since $a \geq 2$, $f \notin \mathbf{K}Id$ and hence $u \neq 0$. Thus $D' \neq \{0\}$. Since $\mathrm{Im}(u) \subseteq E$, B' is saturated in E. Hence D' is a vector bundle. Since $u^2 = 0$, $B' \neq \{0\}$. There is a non-empty Zariski open subset W of A such that for every $m \in W$, calling λ_m the eigenvalue associated to m, we have $\mathrm{rank}(\mathrm{Ker}(m - \lambda_m Id)) = \mathrm{rank}(B')$ and $\deg(\mathrm{Ker}(m - \lambda_m Id)) = \deg(B')$. Set $w = m - \lambda_m Id$. Since $(u - w)^2 = 0$ and $u^2 = w^2 = 0$, we have $uw + wu = 0$. Since A is commutative and $\mathrm{char}(\mathbf{K}) \neq 2$ we obtain $uw = wu = 0$. Since $u^2 = w^2 = 0$ we obtain $\mathrm{Im}(u) \subseteq \mathrm{Ker}(u) \cap \mathrm{Ker}(w)$ and $\mathrm{Im}(w) \subseteq \mathrm{Ker}(u) \cap \mathrm{Ker}(w)$. Vary m in W and call B the saturation of the union T of all subsheaves $\mathrm{Im}(w_1) + \cdots + \mathrm{Im}(w_x)$, $x \geq 1$, and $w_i \in W$ and nilpotent for every i. T is a coherent subsheaf of $\mathrm{Ker}(u)$ because the set of all such sums $\mathrm{Im}(w_1) + \cdots + \mathrm{Im}(w_x)$ is directed and we may use [3], 0.12. Set $D := E/B$. Thus we have an exact sequence (1). We just proved that B is contained in $\mathrm{Ker}(w)$ for all nilpotent w coming from some $f \in W$. Since W is dense in $h(A)$, we have $B \subseteq \mathrm{Ker}(w)$ for every nilpotent $w \in h(A)$, i.e. every $f \in h(A)$ is obtained composing the surjection $E \to D$ given by (1) with a map $D \in B$ and then with the inclusion of B in E given by (1). Hence $h(A) \cong D(V)$ for some V.

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Edoardo BALLICO
Dipartimento di Matematica
Università of Trento
38050 Povo (TN), ITALIA
e-mail: ballico@science.unitn.it