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MODULI OF CURVES VIA ALGEBRAIC GEOMETRY

Abstract. Here we discuss some open problems about moduli spaces of curves from an algebro-geometric point of view. In particular, we focus on Arbarello stratification and we show that its top dimentional stratum is affine.

The moduli space $\overline{\mathcal{M}}_{g,n}$ of stable n-pointed genus g curves is by now a widely explored subject (see for instance the book [10] and the references therein), but many interesting problems in the field are still unsolved, both from a topological and a geometrical point of view. Even though various methods have been fruitfully applied (e. g. Teichmüller spaces, Hodge theory, G.I.T., ...), a purely algebro-geometric approach seems to be quite powerful and rather promising as well. We wish to mention at least the recent paper [3] by Enrico Arbarello and Maurizio Cornalba: as the authors point out in the introduction, what is really new there is the method of proof, which is based on standard algebro-geometric techniques.

Indeed, the only essential result borrowed from geometric topology is a vanishing theorem due to John Harer. Namely, the fact that $H_k(\mathcal{M}_{g,n})$ vanishes for $k > 4g-4+n$ if $n > 0$ and for $k > 4g-5$ if $n = 0$ was deduced in [9] from the construction of a $(4g-4+n)$-dimensional spine for $\mathcal{M}_{g,n}$ by means of Strebel differentials. On the other hand, it is conceivable that Harer's vanishing is only the tip of an iceberg of deeper geometrical properties. For instance, a conjecture of Eduard Looijenga says that $\mathcal{M}_g$ is a union of $g-1$ open subsets (see [7], Conjecture 11.3), but (as far as we know) there are no advances in this direction. Another strategy (see [8], Problem 6.5) in order to avoid the use of Strebel's differentials in the proof of Harer's theorem is to look for an orbifold stratification of $\mathcal{M}_g$ with $g-1$ affine subvarieties as strata.

A natural candidate for such a stratification is provided by a flag of subvarieties introduced by Enrico Arbarello in his Ph.D. thesis. Namely, for each integer n, $2 \leq n \leq g$, he defined the subvariety $W_{n,g} \subset \mathcal{M}_g$ as the sublocus of $\mathcal{M}_g$ described by those points of $\mathcal{M}_g$ which correspond to curves of genus g which can be realized as n-sheeted coverings of $\mathbb{P}^1$ with a point of total ramification (see [2] p. 1). The natural expectation (see [1] p. 326 but also [12] p. 310) was that $W_{n,g} \setminus W_{n-1,g}$ does not contain any complete curve. About ten years later, Steven Diaz was able to prove that a slightly different flag of subvarieties enjoys such a property and he deduced from this fact his celebrated bound on the dimension of complete subvarieties in $\mathcal{M}_g$ (see [5]). It remains instead an open question whether or not the open strata of the Arbarello flag admit complete curves (see [10] p. 291).

Perhaps an even stronger conjecture could be true: since $W_{2,g}$ is the hyperelliptic locus, which is well-known to be affine (see for instance [11] p. 320), one may wonder whether all the open strata $W_{n,g} \setminus W_{n-1,g}$ are affine. We were not able to prove this statement in full generality; however, we found an elementary proof that the top dimensional stratum is indeed affine.

THEOREM 1. *If $g \geq 3$ then $\mathcal{M}_g \setminus W_{g-1,g}$ is affine.*

Proof. Since $\mathcal{M}_g \setminus W_{g-1,g} = \overline{\mathcal{M}}_g \setminus \big(\mathrm{supp}(\overline{W}_{g-1,g}) \cup \partial\overline{\mathcal{M}}_g\big)$, it is sufficient to prove that $\mathrm{supp}(\overline{W}_{g-1,g}) \cup \partial\overline{\mathcal{M}}_g$ is the support of an effective ample divisor on $\overline{\mathcal{M}}_g$. The class of $\overline{W}_{g-1,g}$ in the Picard group of $\overline{\mathcal{M}}_g$ was computed by Steven Diaz in his Ph.D. thesis (see [6]), so we know that

$$\big[\overline{W}_{g-1,g}\big] = a\lambda - \sum_i b_i \delta_i$$

where

$$\begin{aligned} a &:= \frac{g^2(g-1)(3g-1)}{2} \\ b_0 &:= \frac{(g-1)^2 g(g+1)}{6} \\ b_i &:= \frac{i(g-i)g(g^2+g-4)}{2} \qquad (i > 0). \end{aligned}$$

In particular, notice that if $g \geq 3$ then $a > 11$ and $b_i > 1$ for every i. Consider now the following divisor on $\overline{\mathcal{M}}_g$:

$$D := \overline{W}_{g-1,g} + \sum_i (b_i - 1)\Delta_i.$$

Since $b_i > 1$ we see that D is effective; moreover, we have $\mathrm{supp}(D) = \mathrm{supp}(\overline{W}_{g-1,g}) \cup \partial\overline{\mathcal{M}}_g$. We claim that D is ample. Indeed,

$$\begin{aligned} [D] &= \big[\overline{W}_{g-1,g}\big] + \sum_i (b_i - 1)\delta_i \\ &= a\lambda - \sum_i b_i\delta_i + \sum_i b_i\delta_i - \sum_i \delta_i \\ &= a\lambda - \delta. \end{aligned}$$

Since $a > 11$ we may deduce that D is ample from the Cornalba-Harris criterion (see [4], Theorem 1.3), so the proof is over. $\square$

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