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LOCALLY LAGRANGIAN SYMPLECTIC AND POISSON MANIFOLDS

Abstract. We discuss symplectic manifolds where, locally, the structure is that encountered in Lagrangian dynamics. Examples and characteristic properties are given. Then, we refer to the computation of the Maslov classes of a Lagrangian submanifold. Finally, we indicate the generalization of this type of symplectic structures to Poisson manifolds. The paper is the text of a lecture presented at the Conference "Poisson 2000" held at CIRM, Luminy, France, between June 26 and June 30, 2000. It reviews results contained in the author's papers [9, 12, 13] as well as in papers by other authors.

1. Locally Lagrangian Symplectic Manifolds

The present paper is the text of a lecture presented at the Conference "Poisson 2000" held at CIRM, Luminy, France, between June 26 and June 30, 2000, and it reviews results contained in the author's papers [9, 12, 13] as well as in papers by other authors. The notion of a locally Lagrangian Poisson manifold is defined for the first time here.

The symplectic structures used in Lagrangian dynamics are defined on a tangent manifold TN, and consist of symplectic forms of the type

$$\omega_{\mathcal{L}} = \frac{1}{2}\left(\frac{\partial^2\mathcal{L}}{\partial q^i \partial u^j} - \frac{\partial^2\mathcal{L}}{\partial q^j \partial u^i}\right) dq^i \wedge dq^j + \frac{\partial^2\mathcal{L}}{\partial u^i \partial u^j} du^i \wedge dq^j. \tag{1}$$

In (1), $(q^i)_{i=1}^n$ ($n = dim\, N$) are local coordinates in the configuration space N, (u^i) are the corresponding natural coordinates in the fibers of TN, and $\mathcal{L}$ is the non degenerate Lagrangian function on TN (i.e., $\mathcal{L} \in C^\infty(TN)$, $rank(\partial^2\mathcal{L}/\partial u^i \partial u^j) = n$). (In this paper, everything is C^∞.)

The most known geometric description of $\omega_{\mathcal{L}}$ is that it is the pullback of the canonical symplectic form of T^*N by the Legendre transformation defined by $\mathcal{L}$. But, geometrically, it is more significant that $\omega_{\mathcal{L}}$ is related with the *tangent structure* of the manifold TN. The latter is the bundle morphism $S : TTN \to TTN$ defined by

$$SX \in T\mathcal{V},\ (SX)(<\alpha_{\pi(u)}, u>) = <\alpha_{\pi(u)}, \pi_* X_u>, \tag{2}$$

where $\mathcal{V}$ is the foliation of TN by fibers (the *vertical foliation*), $u \in TN$, $X \in \Gamma TTN$, $\alpha \in \Gamma T^*N$, and $\pi : TN \to N$ is the natural projection. (Γ denotes spaces of global cross sections.) Formulas (2) define the action of SX on q^i, u^i, and one has

$$S\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial u^i},\ S\left(\frac{\partial}{\partial u^i}\right) = 0. \tag{3}$$

The relation between $\omega_{\mathcal{L}}$ and S is e.g., [4]

$$\omega_{\mathcal{L}} = d\theta_{\mathcal{L}},\ \theta_{\mathcal{L}} = d\mathcal{L} \circ S. \tag{4}$$

A symplectic form given by (4) is called a *global Lagrangian symplectic (g.L.s.) structure*. The Lagrangian $\mathcal{L}$ on TN uniquely defines $\omega_{\mathcal{L}}$ but, if (and only if) we change $\mathcal{L}$ by

$$\mathcal{L}' = \mathcal{L} + f(q) + \alpha_i(q)u^i, \tag{5}$$

where $f \in C^\infty(N)$ and $\alpha_i(q)dq^i$ is a closed 1-form on N, we get the same symplectic form $\omega_{\mathcal{L}'} = \omega_{\mathcal{L}}$.

Accordingly, it is natural to study locally Lagrangian symplectic manifolds M i.e., manifolds which have an open covering $M = \cup_{\alpha \in A} U_\alpha$, with local Lagrangian functions $\mathcal{L}_\alpha \in C^\infty(U_\alpha)$ that give rise to a global symplectic form which locally is of the type (4). In particular, this is interesting since we may expect to also have compact manifolds of this type while, a g.L.s. manifold cannot be compact since its symplectic form is exact.

First, we must ask our manifold M to carry a *tangent structure* [1] i.e., an *almost tangent structure* $S \in \Gamma End(TM)$ where

$$S^2 = 0, \quad im\, S = ker\, S,$$

which is *integrable* i.e., locally defined by (3). The existence of S implies the even-dimensionality of M, say $dim\, M = 2n$, whence $rank\, S = n$, and integrability is equivalent with the vanishing of the Nijenhuis tensor

$$\mathcal{N}_S(X,Y) = [SX, SY] - S[SX,Y] - S[X,SY] + S^2[X,Y] = 0. \tag{6}$$

It is important to notice that S defines the *vertical distribution* $V := im\, S = ker\, S$, which is tangent to a foliation $\mathcal{V}$ if S is integrable. Moreover, in the latter case $\mathcal{V}$ is *locally leafwise affine* since a change of local coordinates where (3) holds is of the form

$$\tilde{q}^i = \tilde{q}^i(q^j), \quad \tilde{u}^i = \frac{\partial \tilde{q}^i}{\partial q^j} u^j + \theta^i(q^j). \tag{7}$$

(We use the Einstein summation convention.) The parallel vector fields of the locally affine structure of the leaves of $\mathcal{V}$ are the vector fields SX where X is a $\mathcal{V}$-projectable vector field on M.

Now, we can give the formal definition: a *locally Lagrangian symplectic (l.L.s.) manifold* is a manifold M endowed with a tangent structure S and a symplectic structure ω of the form (4), where $\mathcal{L}$ are local functions on M.

A simple example can be obtained as follows. Take $N = I\!R^n$, $TN = I\!R^{2n}$, and the Lagrangian of the *modified harmonic oscillator* [6]

$$\mathcal{L} = \frac{1}{2}(\delta_{ij}u^i u^j + \alpha_{ij} q^i q^j),$$

where (δ_{ij}) is the unit matrix and (α_{ij}) is a constant symmetric matrix. Then, quotientize by the group

$$q^i \mapsto q^i + m^i,\ u^i \mapsto u^i + s^i \quad (m^i, s^i \in \mathbf{Z})$$

to get the torus T^{2n}. The tangent structure of $I\!R^{2n}$ projects to T^{2n}, and the function $\mathcal{L}$ yields local functions on T^{2n} which have transition relations (5) hence, define the same Lagrangian symplectic form namely, the canonical symplectic form of the torus T^{2n} . Notice that this example is on a compact manifold.

Another interesting example is that of the tangent bundle TN of a symplectic manifold N with symplectic form σ. Namely, the form $\omega = d(\flat_\sigma^* \lambda)$, where λ is the Liouville form (e.g., [9]) of T^*N and $\flat_\sigma : TN \to T^*N$ is the usual index-lowering isomorphism, is an l.L.s. form on TN with its canonical tangent structure. This follows by an easy computation in local Darboux coordinates on (N, σ).

Further examples will be given later on.

Following is a coordinate-free characterization of the l.L.s. manifolds [13]:

PROPOSITION 1. *Let M be a manifold with a tangent structure S and a symplectic form ω. Then, ω is locally Lagrangian with respect to S iff*

$$\omega(X, SY) = \omega(Y, SX) \qquad (\forall X, Y \in \Gamma TM). \tag{8}$$

Proof. Using local coordinates where (3) and (1) hold, it is easy to check (8). Conversely, from (8), it follows that the vertical foliation $\mathcal{V}$ of S ($T\mathcal{V} = im\, S$) is ω-Lagrangian. Hence, M may be covered by local charts (U, x^i, y^i) such that

$$V := T\mathcal{V} = span\left\{\frac{\partial}{\partial y^i}\right\}, \quad \omega = \sum_i dx^i \wedge dy^i. \tag{9}$$

Furthermore, we must also have

$$S\left(\frac{\partial}{\partial x^i}\right) = \sum_k \lambda_{ik} \frac{\partial}{\partial y^k}, \tag{10}$$

where, because of (8), $\lambda_{ik} = \lambda_{ki}$.

Now, (6) implies

$$[S\frac{\partial}{\partial x^i}, S\frac{\partial}{\partial x^j}] = 0$$

hence, there are new coordinates

$$q^i = x^i, \; u^i = u^i(x^j, y^j)$$

such that

$$S\left(\frac{\partial}{\partial q^i}\right) = S\left(\frac{\partial}{\partial x^i} + \frac{\partial y^k}{\partial q^i}\frac{\partial}{\partial y^k}\right) = S\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial u^i}. \tag{11}$$

From (10) and (11) we get $\partial y^i/\partial u^k = \lambda_{ki}$, and, since $\lambda_{ki} = \lambda_{ik}$, there exist local functions $\mathcal{L}$ such that $y^i = -\partial\mathcal{L}/\partial u^i$. Using (9), we get (1).

□

(8) is the *compatibility condition* between S and ω. It allows us to notice one more interesting object namely,

$$\Theta([X]_V, [Y]_V) := \omega(SX, Y), \tag{12}$$

where the arguments are cross sections of the transversal bundle $\nu\mathcal{V} := TM/V$ of the foliation $\mathcal{V}$. Θ is a well defined pseudo-Euclidean metric with the local components $(\partial^2\mathcal{L}/\partial u^i\partial u^j)$. If this metric is positive definite, we say that the manifold (M, S, ω) is of the *elliptic type*.

PROPOSITION 2. *Let (M, ω) be a symplectic manifold endowed with a Lagrangian foliation $\mathcal{V}$ $(T\mathcal{V} = V)$, and a $\mathcal{V}$-projectable pseudo-Euclidean metric Θ on $\nu\mathcal{V} = TM/V$. Then, there exists a unique ω-compatible tangent structure S on M for which Θ is the metric* (12).

Proof. Split $TM = V' \oplus V$, where V' also is a ω-Lagrangian distribution, and identify $\nu\mathcal{V}$ with V'. Then define

$$S/_V = 0, \quad S/_{V'} = \sharp_\omega \circ \flat_\Theta,$$

where the musical isomorphisms are defined as in Riemannian geometry. It is easy to check that S is the required tangent structure. In particular, we must check that $\mathcal{N}_S(X, Y) = 0$, and it suffices to look at the various cases where the arguments are in V, V' while, if in V', they are $\mathcal{V}$-projectable vector fields. The only non trivial case $X, Y \in \Gamma V'$ is settled by noticing that, whenever $X, Y, Z \in \Gamma_{proj} V'$, one has

$$d\omega(Z, SX, SY) = \omega(Z, [SX, SY]) = 0,$$

whence $[SX, SY] = 0$.

$\square$

Proposition 2 allows us to find more examples of l.L.s. manifolds. Let

$$H(1,p) := \{ \begin{pmatrix} Id_p & X & Z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \; / \; X, Z \in I\!R^p, y \in I\!R\} \tag{13}$$

be the *generalized Heisenberg group*, and take the quotients

$$M(p,q) := \Gamma(p,q)\backslash(H(1,p) \times H(1,q)),$$

where $\Gamma(p,q)$ consists of pairs of matrices of type (13) with integer entries. Then, with the notation of (13) on the two factors, the form

$$\omega = {}^t dX_1 \wedge (dZ_1 - X_1 dy_1) + {}^t dX_2 \wedge (dZ_2 - X_2 dy_2) + dy_1 \wedge dy_2$$

(t denotes matrix transposition) defines a symplectic structure on $M(p,q)$, and the equations

$$X_1 = const., \; X_2 = const., \; y_1 - \alpha y_2 = const. \quad (\alpha \in I\!R),$$

define an ω-Lagrangian foliation $\mathcal{V}$ [2, 10]. Furthermore

$$g := {}^t dX_1 \otimes dX_1 + {}^t dX_2 \otimes dX_2 + [d(y_1 - \alpha y_2] \otimes [d(y_1 - \alpha y_2]$$

is a projectable metric on the transversal bundle of $\mathcal{V}$.

Correspondingly, the construction of Proposition 2 yields a l.L.s. structure of elliptic type on $M(p,q)$.

A similar construction holds on the so-called Iwasawa manifolds

$$I(p) = \Gamma_c(1,p)\backslash H_c(1,p),$$

where $H_c(1,p)$ is given by (13) with $I\!R$ replaced by $\mathbf{C}$, and $\Gamma_c(1,p)$ consists of matrices with Gauss integers entries [2, 10].

We end this section by

PROPOSITION 3. *The l.L.s. manifold (M,S,ω) is globally Lagrange symplectic iff $\omega = d\epsilon$ for some global 1-form ϵ on M such that (i) ϵ vanishes on the vertical leaves of S, and (ii) if η is the cross section of V^* ($V = T\mathcal{V}$, $\mathcal{V}$ is the vertical foliation of S) which satisfies $\eta \circ S = \epsilon$, then $\eta = d_{\mathcal{V}}\mathcal{L}$, where $d_{\mathcal{V}}$ is the differential along the leaves of $\mathcal{V}$ and $\mathcal{L} \in C^\infty(M)$.*

Proof. Notice the exact sequence

$$0 \to V \stackrel{\subseteq}{\to} TM \stackrel{\pi}{\to} TM/V \approx V \to 0$$

which shows that V^* can be identified with the subbundle $(TM/V)^*$ of T^*M. Accordingly, we see that a unique leafwise form η as required is associated with each ϵ that satisfies (i) namely, using the local coordinates of (3), we have

$$\epsilon = \epsilon_i dq^i \mapsto \eta = \epsilon_i [du^i]_{(TM/V)^*}.$$

If we are in the g.L.s. case, we have (4), and $\epsilon = \theta_{\mathcal{L}}$ is the required form.

Conversely, if $\omega = d\epsilon$, where (i), (ii) hold, we must locally have

$$\epsilon = (\frac{\partial \mathcal{L}}{\partial u^i} du^i) \circ S = \frac{\partial \mathcal{L}}{\partial u^i} dq^i,$$

and $\mathcal{L}$ is the required global Lagrangian. □

Notice that condition (ii) has a cohomological meaning. If the l.L.s. form $\omega = d\epsilon$ where ϵ satisfies (i) then (4) implies

$$\epsilon/_U = \theta_{\mathcal{L}_U} + \lambda_U, \qquad d\lambda_U = 0,$$

where U is an open neighborhood on which the local Lagrangian $\mathcal{L}_U$ exists, and λ is a 1-form which vanishes on the leaves of $\mathcal{V}$. Thus,

$$\lambda = d(\varphi(q)), \quad \eta = \left(\frac{\partial \mathcal{L}_U}{\partial u^i} + \frac{\partial \varphi(q)}{\partial q^i} \right) [du^i],$$

and we get $d_{\mathcal{V}}\eta = 0$. In foliation theory (e.g., [8]), it is known that if $H^1(M, \Phi_{\mathcal{V}}) = 0$, where $\Phi_{\mathcal{V}}$ is the sheaf of germs of functions $f(q^i)$, then $d_{\mathcal{V}}\eta = 0$ implies $\eta = d_{\mathcal{V}}\mathcal{L}$ for a global function $\mathcal{L}$.

2. Further geometric results

a). Let (M,S) be a manifold endowed with an integrable tangent structure S. Then, the following problem is of an obvious interest: find the symplectic structures ω on M which are compatible with S hence, are l.L.s. forms.

Proposition 4. *The symplectic form ω on M is S-compatible iff*

$$\omega = \frac{1}{2}\varphi_{ij}(q)dq^i \wedge dq^j + d(\zeta_i(q,u)dq^i), \tag{14}$$

where (q^i, u^i) are the local coordinates of (3), the first term of (14) is a local closed 2-form and $\zeta = \zeta_i dq^i$ is a local 1-form such that $d_{\mathcal{V}}(\zeta_i[du^i]) = 0$ and $d\zeta$ is non degenerate.

Proof. If ω is S-compatible, ω is of the form (4), which is (14) with a vanishing first term.

For the converse result, we notice that our hypotheses indeed imply that the form ω is closed and non degenerate, while the compatibility condition (8) is equivalent to $\partial\zeta_i/\partial u^j = \partial\zeta_j/\partial u^i$ i.e., $d_{\mathcal{V}}(\zeta_i[du^i]) = 0$.

□

The local Lagrangians of the form ω of (14) are obtained by putting (locally)

$$\frac{1}{2}\varphi_{ij}(q)dq^i \wedge dq^j = d(\alpha_i(q)dq^i), \;\; \zeta_i[du^i] = d_{\mathcal{V}}f, \;\; f \in C^\infty(M).$$

Then the Lagrangians are

$$\mathcal{L} = f + \alpha_i u^i. \tag{15}$$

A coordinate-free criterion which ensures (14) is given by

Proposition 5. *The symplectic form ω on (M,S) has the local form (14), possibly without $d_{\mathcal{V}}(\zeta_i[du^i]) = 0$, iff the vertical foliation $\mathcal{V}$ of S is a Lagrangian foliation with respect to ω.*

Proof. Consider the vertical foliation $\mathcal{V}$ of S, denote $V = T\mathcal{V}$, and let V' be a transversal distribution. We will use a well known technique of foliation theory namely, the bigrading of differential forms and multivector fields associated with the decomposition $TM = V' \oplus V$. In particular, one has

$$d = d'_{(1,0)} + d''_{(0,1)} + \partial_{(2,-1)},$$

where d'' may be identified with $d_{\mathcal{V}}$, and $d^2 = 0$ becomes

$$d''^2 = 0, \; \partial^2 = 0, \; d'd'' + d''d' = 0, \; d'\partial + \partial d' = 0, \; d'^2 + d''\partial + \partial d'' = 0. \tag{16}$$

Furthermore, d'' is the coboundary of the leafwise de Rham cohomology, and it satisfies a Poincaré lemma e.g., [8].

Clearly, $\mathcal{V}$ is Lagrangian with respect to (14).

Conversely, if the foliation $\mathcal{V}$ is Lagrangian for a symplectic form ω, we must have a decomposition

$$\omega = \omega_{(2,0)} + \omega_{(1,1)},$$

and $d\omega = 0$ means

$$d''\omega_{(1,1)} = 0, \; d''\omega_{(2,0)} + d'\omega_{(1,1)} = 0, \; d'\omega_{(2,0)} + \partial\omega_{(1,1)} = 0. \tag{17}$$

Accordingly, there exists a local $(1,0)$-form ζ such that $\omega_{(1,1)} = d''\zeta$ (the d''-Poincaré lemma), and (with (16)) the last two conditions (17) become

$$\begin{aligned} d''\omega_{(2,0)} + d'd''\zeta = &\quad d''(\omega_{(2,0)} - d'\zeta) = 0, \\ d'\omega_{(2,0)} + \partial d''\zeta = &\quad d'\omega_{(2,0)} - d'^2\zeta - d''\partial\zeta = d'(\omega_{(2,0)} - d'\zeta) = 0, \end{aligned}$$

whence,

$$\Phi := \omega_{(2,0)} - d'\zeta$$

is a closed 2-form of bidegree $(2,0)$. Therefore, ω has the local expression (14).

□

The simplest geometric case is that of a tangent bundle $M = TN$ with the canonical tangent structure (3). In this case, the local expressions (14) can be glued up by means of a partition of unity on N, and we get all the global S-compatible symplectic forms [12]

$$\omega = \pi^*\Phi + d\zeta, \tag{18}$$

where $\pi : TN \to N$, Φ is a closed 2-form on N, and ζ is a 1-form on TN which satisfies the hypotheses of Proposition 2.1. In particular, $d_{\mathcal{V}}(\zeta_i[du^i]) = 0$, and the contractibility of the fibers of TN allows to conclude that $\zeta_i[du^i] = d_{\mathcal{V}}\varphi$, $\varphi \in C^\infty(TN)$. This also yields another expression of the S-compatible forms on TN:

$$\omega = \pi^*\Phi + \omega_\varphi, \tag{19}$$

where ω_φ is given by (1).

Furthermore, if we also use Proposition 3, we see that ω of (18) is globally Lagrangian iff it is exact. Indeed,if ω of (18) is $\omega = d\xi$, where $\xi = \xi_{(1,0)} + \xi_{(0,1)}$, we must have $d''\xi_{(0,1)} = 0$ since ω has no $(0,2)$-component. Then, because the fibers of TN are contractible, $\xi_{(0,1)} = d''\varphi$, $\varphi \in C^\infty(TN)$, and

$$\omega = d\xi_{(1,0)} + dd''\varphi = d\xi_{(1,0)} + d(d-d')\varphi = d(\xi_{(1,0)} - d'\varphi) = d\epsilon,$$

where $\epsilon = \xi_{(1,0)} - d'\varphi$ is a $(1,0)$-form. If (18) is l.L.s., then it is clear that ϵ satisfies the hypotheses of Proposition 3, with a global Lagrangian of the form (15).

In this context it is interesting to mention that a criterion to distinguish the tangent bundles in the class of the manifolds (M, S) with an integrable tangent structure is available. Namely, M is a tangent bundle iff the following three conditions hold: (i) the vertical foliation $\mathcal{V}$ of S has simply connected leaves, (ii) the flat affine connection induced by S on the leaves of $\mathcal{V}$ is complete, (iii) $\mathcal{E}(\mathcal{V}) = 0$, where $\mathcal{E}(\mathcal{V})$ is the 1-dimensional cohomology class with coefficients in the sheaf of germs of $\mathcal{V}$-projectable cross sections of $T\mathcal{V}$ produced by the difference cocycle of the local *Euler vector fields* $E := u^i(\partial/\partial u^i)$ [3, 10].

b). At this point we restrict ourselves to the case of a tangent bundle $M = TN$ with the canonical tangent structure S of (3). Then, we may speak of *second order vector fields* X on TN i.e., vector fields X such that the projection on N of their trajectories satisfy an autonomous system of ordinary differential equations of the second order. With respect to the local coordinates (3) such a vector field has the form

$$X = u^i \frac{\partial}{\partial q^i} + \gamma^i(q,u)\frac{\partial}{\partial u^i}. \tag{20}$$

For TN, the local coordinate transformations (7) of u^i are linear (i.e., $\theta^i = 0$ in (7)) and $E = u^i(\partial/\partial u^i)$ is a well defined global vector field on TN (the infinitesimal generator of the homotheties) called the *Euler vector field.* We see that the vector field X is of the second order iff $SX = E$.

PROPOSITION 6. *The symplectic form ω on TN is S-compatible iff the following conditions are satisfied: (i) the vertical foliation $\mathcal{V}$ by the fibers of TN is ω-Lagrangian, (ii) there exist ω-Hamiltonian vector fields which are second order vector fields.*

Proof. ([12]) If ω is S-compatible, it must be of the form (19), and (i) holds. Concerning (ii), it is known in Lagrangian dynamics that, if we consider the energy associated with the Lagrangian φ of (19) given by

$$\mathcal{E}_\varphi = E\varphi - \varphi,$$

its ω_φ-Hamiltonian vector field X_φ $(i(X_\varphi)\omega_\varphi = -d\mathcal{E}_\varphi)$ is of the second order.

Furthermore, all the second order vector fields are given by $X = X_\varphi + Z$, where Z is an arbitrary vertical vector field, and the 1-form $\Psi = i(X)\pi^*\Phi$ (Φ of (19)) is independent on Z.

If we want a function $h \in C^\infty(TN)$ such that

$$-dh = i(X)\omega = i(X_\varphi + Z)\omega = \Psi - d\mathcal{E}_\varphi + i(Z)\omega_\varphi,$$

it means we want a relation of the form

$$\Psi + i(Z)d''\zeta = df, \tag{21}$$

where ζ is the 1-form of (18), d'' is defined using an arbitrary complementary distribution V' of $V = T\mathcal{V}$, and $f = \mathcal{E}_\varphi - h$ must be the lift of a function on N, since the left hand side of (21) is of bidegree $(1,0)$.

Because the Lagrangian φ is non degenerate, (21) has a solution Z for any $f \in C^\infty(N)$. Therefore, condition (ii) of Proposition 6 is satisfied and we even know how to find all the ω-Hamiltonian, second order vector fields.

Conversely, condition (i) implies (18) (see Proposition 5), and, if a vector field X of the form (20) such that $i(X)\omega = -dh$ $(h \in C^\infty(TN))$ exists, the equality of the $(0,1)$-components yields

$$u^i\frac{\partial\zeta_i}{\partial u^k} = \frac{\partial h}{\partial u^k}$$

whence the derivatives

$$\frac{\partial\zeta_j}{\partial u^k} = \frac{\partial^2 h}{\partial u^j\partial u^k} - u^i\frac{\partial^2\zeta_i}{\partial u^j\partial u^k}$$

are symmetric, and we are done (see Proposition 4). □

Concerning second order vector fields on a tangent bundle TN, the following problem is important:

Problem 1. Let X be a second order vector field on TN. Study the existence and generality of the Poisson structures P on TN such that X is a P-Hamiltonian vector field.

Conversely, we can formulate:

Problem 2. If P is a Poisson structure on TN, study the existence and generality of second order P-Hamiltonian vector fields.

In analogy with the variational calculus problems, Problem 2 can be called the *direct problem*, and Problem 1 the *inverse problem* for Hamiltonian second order vector fields. Proposition 6 gives the solution of the direct problem for symplectic structures ω on TN for which the vertical foliation $\mathcal{V}$ is Lagrangian [12]. But, it is easy to check that if W is a Poisson structure on N its lift P to TN has no second order Hamiltonian vector fields. P is defined by $\{f \circ \pi, g \circ \pi\} = 0$ for all $f, g \in C^\infty(N)$, $\pi : TN \to N$, and

$$\{\alpha, f \circ \pi\} = (\sharp_W \alpha) f, \ \{\alpha, \beta\} = L_{\sharp_W \alpha}\beta - L_{\sharp_W \beta}\alpha - d(W(\alpha, \beta)),$$

for fiberwise linear functions identified with 1-forms α, β. We again refer to [12] for an iterative method of solving the inverse problem for Poisson structures P such that $\{f \circ \pi, g \circ \pi\} = 0$.

It is interesting to notice that the notion of a second order vector field may also be defined for arbitrary manifolds endowed with an integrable almost tangent structure.

Let (M, S) be a manifold with the integrable structure (3). Then, (7) shows the existence of a natural foliated structure on the vector bundle $V = T\mathcal{V}$ ($\mathcal{V}$ is the vertical foliation of S), and it makes sense to speak of $\mathcal{V}$-projectable cross sections of V. A vector field $X \in \Gamma TM$ will be called a *second order vector field* if for each canonical coordinate neighborhood (U, q^i, u^i), $SX - E_U$, where $E_U = u^i(\partial/\partial u^i)$ is the local Euler vector field, is a $\mathcal{V}$-projectable cross section of V. The condition is invariant by the coordinate transformations (7), and the local expression of X is of the form

$$X = (u^i + \alpha^i(q))\frac{\partial}{\partial q^i} + \beta^i(q, u)\frac{\partial}{\partial u^i}, \tag{22}$$

which yields

$$\frac{d^2 q^i}{dt^2} = \beta^i(q^j, \frac{dq^j}{dt} - \alpha^i(q)) + \frac{\partial \alpha^i}{\partial q^j}\frac{dq^j}{dt} \tag{23}$$

along the trajectories of X. Equation (23) explains the name.

The following fact, which is known for tangent bundles (e.g. [6]) is also true in the general case: if X is a second order vector field then $(L_X S)^2 = Id$, where L denotes the Lie derivative. This is easily checked by using (22) and by acting on $\partial/\partial q^i$, $\partial/\partial u^i$. It follows that $F := L_X S$ is an almost product structure on M which has the $(+1)$-eigenspace equal to $V = T\mathcal{V}$. The complementary distribution

$$V' = span\left\{\frac{\partial}{\partial q^i} - \frac{1}{2}\left(\frac{\partial \alpha^j}{\partial q^i} - \frac{\partial \beta^j}{\partial u^i}\right)\frac{\partial}{\partial u^j}\right\}$$

is the (-1)-eigenspace of F.

In particular, if the functions β^i of (22) are quadratic with respect to u^i (a condition which is invariant by (7)), V' is an *affine distribution* transversal to $\mathcal{V}$ i.e., the process of lifting paths of transversal submanifolds of $\mathcal{V}$ to paths tangent to V' yields affine mappings between the leaves of $\mathcal{V}$ [5, 10].

c). In what follows, we give a result concerning symplectic reduction of locally Lagrangian symplectic manifolds.

Proposition 7. *Let (M, S, ω) be a l.L.s. manifold and N a coisotropic submanifold with the kernel foliation $C = (TN)^{\perp_\omega}$. Assume that the following conditions are satisfied:*
(i) $S(TN) \subseteq TN$, $V \cap TN \subseteq S(TN) + C$ $(V = im\, S)$,
(ii) the leaves of C are the fibers of a submersion $r : N \to P$,
(iii) the restriction of S to N sends C-projectable vector fields to C-projectable vector fields.
Then S induces a tangent structure S' on P which is compatible with the reduction ω' of the symplectic structure ω to P.

Proof. (i) and the compatibility condition (8) imply $S(C) \subseteq C$, therefore, we get an induced morphism $\tilde{S} : TN/C \to TN/C$, such that $\tilde{S}^2 = 0$ and $im\, \tilde{S} = (V \cap TN)/(V \cap C)$.

Since the quotient is Lagrangian for the reduction of ω, we get

$$rank\, \tilde{S} = \frac{1}{2} dim\, (TN/C).$$

Condition (ii) allows us to reduce ω to a symplectic structure ω' of P, and (iii) ensures that $\tilde{S}$ projects to an ω'-compatible tangent structure S'. $\square$

3. Maslov Classes

Since a l.L.s. manifold (M, S, ω) has a canonical Lagrangian foliation $\mathcal{V} = im\, S$, any Lagrangian submanifold L of M has Maslov classes, which are cohomological obstructions to the transversality of L and $\mathcal{V}$. In this section, we will indicate a differential geometric way of computation of these Maslov classes.

We begin by a brief recall of the general definition of the Maslov classes [9]. Let $\pi : E \to M$ be a vector bundle of rank $2n$, and $\sigma \in \Gamma \wedge^2 E^*$ a nowhere degenerate cross section. Then (E, σ) is a *symplectic vector bundle*. Furthermore, (E, σ) has a reduction of its structure group $Sp(2n, \mathbb{R})$ to the unitary group $U(n)$ which is defined up to homotopy, and can be fixed by the choice of a complex structure J *calibrated* by σ (i.e., σ-compatible: $\sigma(Js_1, Js_2) = \sigma(s_1, s_2)$, and such that $g(s_1, s_2) := \sigma(s_1, Js_2)$ is positive definite, $s_1, s_2 \in \Gamma E$).

If $L_0 \to M$ is a Lagrangian subbundle of (E, σ), the complex version of (E, J, g) has local unitary bases of the form

$$\epsilon_i = \frac{1}{\sqrt{2}}(e_i - \sqrt{-1} J e_i), \tag{24}$$

where $(e_i)_{i=1}^n$ is a local, real, g-orthonormal basis of L_0 (L_0-*orthonormal*, J-*unitary bases*).

Correspondingly, E admits L_0-orthogonal, J-unitary connections ∇^0 of local expressions

$$\nabla^0 \epsilon_i = \theta^j_{0i} \epsilon_j \qquad (\theta^j_{0i} + \theta^i_{0j} = 0),$$

and a local curvature matrix

$$\Theta^0 = d\theta^0 + \theta^0 \wedge \theta^0.$$

Furthermore, if $L_1 \to M$ is a second Lagrangian subbundle of (E, σ), and ∇^1 is a L_1-orthogonal, J-unitary connection, the following objects exist:
i) the *difference tensor* $\alpha := \nabla^1 - \nabla^0$,
ii) the *curvature variation*

$$\Theta_t := (1-t)\Theta_0 + t\Theta_1 + t(1-t)\alpha \wedge \alpha,$$

$0 \leq t \leq 1$ (of course, α and Θ are matrices),
iii) the *Chern-Weil-Bott forms*

$$\begin{aligned}(25)\qquad & \Delta(\nabla^0, \nabla^1)c_{2h-1} := \\ & = (-1)^{h+1}\frac{\sqrt{-1}}{(2\pi)^{2h-1}(2h-2)!}\int_0^1 (\delta^{j_1 \ldots j_{2h-1}}_{i_1 \ldots i_{2h-1}} \tilde{\alpha}^{i_1}_{j_1} \wedge \tilde{\Theta}^{i_2}_{tj_2} \wedge \ldots \wedge \tilde{\Theta}^{i_{2h-1}}_{tj_{2h-1}})dt,\end{aligned}$$

where the components $\tilde{\alpha}^i_j$, $\tilde{\Theta}^i_{tj}$ of α and Θ_t are taken with respect to any *common*, local, J-unitary bases of (E, J, g) (which may not be L_a-orthogonal $(a = 0, 1)$).

It turns out that the forms (25) are closed, and define cohomology classes

$$(26)\qquad \mu_h(E, L_0, L_1) := [\Delta(\nabla^0, \nabla^1)c_{2h-1}] \in H^{4h-3}(M, I\!R),$$

$(h = 1, 2, \ldots)$, which do not depend on the choice of J, ∇^0, ∇^1, and are invariant by homotopy deformations of (L_0, L_1) via Lagrangian subbundles.

We refer to [9] for details. The classes (26) are called the *Maslov classes* of the pair (L_0, L_1). If $L_0 \oplus L_1 = E$, one may use $\nabla^1 = \nabla^0$ and $\mu_h = 0$. The following two properties are also important:

$$\mu_h(E, L_0, L_1) = -\mu_h(E, L_1, L_0),$$

$$\mu_h(E, L_0, L_1) + \mu_h(E, L_1, L_2) + \mu_h(E, L_2, L_0) = 0.$$

If (M, ω) is a symplectic manifold endowed with a Lagrangian foliation $\mathcal{V}$, and if L is a Lagrangian submanifold of M, we have *Maslov classes* of L defined by

$$(27)\qquad \mu_h(L) = \mu_h(TM/_L, T\mathcal{V}/_L, TL)) \in H^{4h-3}(L, I\!R).$$

For $h = 1$, and if L is a Lagrangian submanifold of $\mathbf{R}^{2n} = T^*I\!R^n$, $\mu_1(L)$ is the double of the original class defined by Maslov.

In particular, if (M, S, ω) is a l.L.s. manifold, and $\mathcal{V}$ is its vertical Lagrangian foliation, formula (27) will be the definition of the Maslov classes of the Lagrangian submanifold L of M. We will discuss a way of computing these classes.

Let V' be a transversal Lagrangian distribution of $V = im\,S$. Then $F := S/_{V'}$ is an isomorphism $V' \approx V$ with the inverse $F^{-1} : V \approx V'$, and it is easy to check that, if F^{-1} is extended to a morphism $S' : TM \to TM$ by asking $S'/_{V'} = 0$, then S' is an almost tangent structure on M which satisfies the compatibility condition

$$\omega(X, S'Y) = \omega(Y, S'X).$$

Furthermore, it is also easy to check that $J := S' - S$ is an ω-compatible almost complex structure, which is positive iff (M, S, ω) is of the elliptic type (i.e., $\omega(SX, Y)$) is a positive definite bilinear form on V'; see Section 1).

Therefore, on l.L.s. manifolds of the elliptic type there is an easy construction of an almost complex structure J as needed in the computation of the Maslov classes, and we also get the corresponding Riemannian metric

$$g(X, Y) := \omega(X, JY).$$

We will say that g is the *Riemannian metric associated with* V', and the restriction of g to V' is the metric Θ defined in Section 1.

Now, we need connections as required in (25). We can obtain such connections in the following way. Start with a metric connection ∇^0 of the vector bundle V, and extend ∇^0 to TM by asking

$$\nabla^0_X Z = S'\nabla^0_X Y,$$

if $Z = S'Y \in \Gamma V'$ $(Y \in \Gamma V)$. It follows easily that the extended connection, also denoted by ∇^0, is a V-orthogonal, J-unitary connection, and we will use it on $TM/_L$.

Furthermore, since L is a Lagrangian submanifold and J is compatible with ω, the g-normal bundle of L is JTL, and we may write down *Gauss-Weingarten equations* of the form [9]

$$\begin{aligned} \nabla^0 e_i &= \lambda_i^j e_j + b_i^j (Je_j), \\ \nabla^0 (Je_i) &= -b_i^j e_j + \lambda_i^j (Je_j). \end{aligned} \tag{28}$$

In (28), (e_i) is a g-orthonormal basis tangent to L, the coefficients are 1-forms and the second equation is obtained by acting by J on the first equation. Moreover, the metric character of ∇^0 implies

$$\lambda_i^j + \lambda_j^i = 0,\ b_i^j = b_j^i.$$

The coefficients λ_i^j are the local connection forms of the metric connection ∇^1 induced by ∇^0 in L, and b_i^j are the local components of the *second fundamental form* of L.

The connection ∇^1 extends to TM by putting

$$\nabla^1_X JZ = J\nabla^1_X Z \qquad (Z \in \Gamma TL),$$

and it becomes the connection ∇^1 which is required in (25).

In order to use (25) we need local unitary bases, and we may use the bases (24) where e_i are those of (28). Accordingly, we get

$$\nabla^0 \epsilon_i = (\lambda_i^j + \sqrt{-1} b_i^j)\epsilon_j,\ \nabla^1 \epsilon_i = \lambda_i^j \epsilon_j.$$

The difference tensor has the components

$$\alpha_i^j = -\sqrt{-1} b_i^j.$$

The curvature variation can also be obtained by a technical computation [9]. In particular, we get

Proposition 8. *The first Maslov class $\mu_1(L)$ is represented by the differential 1-form $(1/2\pi) b_i^i$.*

This is a generalization of a result due to J. M. Morvan in $\mathbb{R}^{2n}$ [7], where Proposition 8 yields a nice relationship between the first Maslov class and the mean curvature vector of a Lagrangian submanifold $L \subseteq \mathbb{R}^{2n}$.

An interesting situation where the calculations above can be used is that of a symplectic manifold (M, ω) endowed with a transversally Riemannian Lagrangian foliation $\mathcal{V}$. The tangent structure is provided by Proposition 2, and the natural connection

∇^0 to be used is that for which $\nabla^0/_{V'}$ is the Levi-Civita connection of the transversal metric of the foliation. In particular, we can apply these choices to the case of a Lagrangian submanifold of a tangent bundle TN with a global Lagrangian symplectic structure. What we will get is a translation of known calculations on cotangent bundles via a Legendre transformation [9, 11].

4. Locally Lagrangian Poisson Manifolds

The aim of this section is to suggest an open problem namely, the study of Poisson manifolds such that their symplectic leaves are l.L.s manifolds. Besides, we would also like to have the l.L.s. structure of the leaves vary smoothly, in a reasonable sense. This leads to the following

DEFINITION 1. *A* locally Lagrangian Poisson (l.L.P.) manifold *is a triple* (M, P, S) *where* P *is a Poisson bivector field on* M, *and* $S \in \Gamma End TM$ *and satisfies the properties:*

$$P(\alpha, \beta \circ S) = P(\beta, \alpha \circ S), \tag{29}$$

$$P(\alpha \circ S, \beta \circ S) = 0, \tag{30}$$

$$rank_x S/_{im\, \sharp_P} = \frac{1}{2} rank_x P, \tag{31}$$

$$\mathcal{N}_S(X, Y) = 0, \qquad \forall X, Y \in \Gamma(im\, \sharp_P), \tag{32}$$

where $\alpha, \beta \in \Gamma T^*M$, $x \in M$, $\sharp_P : T^*M \to TM$ *is defined by* $< \sharp_P \alpha, \beta >= P(\alpha, \beta)$, *and* $\mathcal{N}_S$ *is the Nijenhuis tensor (6). If all these conditions, with the exception of (32) hold,* (M, P, S) *is an* almost l.L.P. manifold.

The name is justified by

PROPOSITION 9. *The symplectic leaves of a l.L.P. manifold are locally Lagrangian symplectic manifolds.*

Proof. Formula (29) also reads

$$S \sharp_P \alpha = -\sharp_P(\alpha \circ S),$$

and this shows that the tangent spaces of the symplectic leaves are S-invariant. From (29), (30), it follows that for $F := S/_{im\, \sharp_P}$, $F^2 = 0$ which, together with (31) and (32), shows that F defines a tangent structure on every symplectic leaf $\mathcal{S}$ of P. Furthermore, if we look at the symplectic structure

$$\omega_{\mathcal{S}}(\sharp_P \alpha, \sharp_P \beta) := -P(\alpha, \beta)$$

of the leaf $\mathcal{S}$, we see that (29) implies the compatibility condition (8).

□

An easy example is provided by a manifold

$$M = TN^n \times_f TN^n, \tag{33}$$

which is the fibered product of two copies of a tangent bundle. In this case, M has an atlas of local coordinates $(x^i, y^i, z^i)_{i=1}^n$ with the coordinate transformations

$$\tilde{x}^i = \tilde{x}^i(x^j),\ \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j,\ \tilde{z}^i = \frac{\partial \tilde{x}^i}{\partial x^j} z^j, \tag{34}$$

and we get a tensor field $S \in \Gamma End\, TM$ if we ask

$$S(X_i) = 0, S\left(\frac{\partial}{\partial y^i}\right) = \frac{\partial}{\partial z^i},\ S\left(\frac{\partial}{\partial z^i}\right) = 0,$$

where

$$X_i = \frac{\partial}{\partial x^i} + t_i^j \frac{\partial}{\partial z^j}$$

are chosen such as to span a fixed transversal distribution of the fibers of the second copy of TN in (33).

The definition of S is inspired by (3), and it is easy to check that S is invariant by (34) and satisfies

$$S^2 = 0,\ \mathcal{N}_S = 0.$$

Now, let P be a 2-contravariant non degenerate symmetric tensor field on N. Then,

$$\Pi = P^{ij} \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial z^j}$$

is a Poisson bivector field on M with the symplectic leaves given by the fibers of the natural fibration $M \to N$, and the conditions (29)-(32) are satisfied.

Therefore, (M, Π, S) is a l.L.P. manifold. Moreover, if we define the covariant tensor Λ such that $\Lambda_{ij} P^{jk} = \delta_i^k$, the symplectic structure of the leaf over any fixed point $x \in M$ is $\Lambda_{ij}(x) dz^i \wedge dy^j$, and the leaf has the global Lagrangian function $\mathcal{L} = \Lambda_{ij}(x) z^i y^j$, which also is a global function on M.

In agreement with this example we give

DEFINITION 2. *A l.L.P.manifold* (M, P, S) *is a* globally Lagrangian Poisson (g.L.P.) manifold *if there exists a global function* $\mathcal{L} \in C^\infty(M)$ *such that its restriction to the symplectic leaves of* P *is a global Lagrangian function of the induced symplectic structure.*

Remark. If N is a locally affine manifold with the affine local coordinates (x^i), the manifold (33) can also be seen as $M = T^{(2)}N$, the second order osculating bundle of N (i.e., the bundle of the second order jets at 0 of the mappings in $C^\infty(\mathbb{R}, M)$).

Now, the following question is natural: on a tangent bundle TN, find all the Poisson structures P such that (M, P, S) where S is the canonical tangent structure of TN is a l.L.P. or a g.L.P. manifold.

If we use the coordinates of (3), it follows

$$dq^i \circ S = 0,\ du^i \circ S = dq^i, \tag{35}$$

and (30) with $\alpha = du^i, \beta = du^j$ yields $P(df, dg) = 0$ for all $f, g \in C^\infty(M)$. This means that P must be a zero-related structure i.e., such that $\pi : (TN, P) \to (N, 0)$ is a Poisson mapping [12]. Furthermore, again using (35), we see that (29) reduces to

$$P(dq^i, du^j) = P(dq^j, du^i).$$

Hence, P must be of the form

$$P = P^{ij}\frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial u^j} + \frac{1}{2}A^{ij}\frac{\partial}{\partial u^i} \wedge \frac{\partial}{\partial u^j}, \tag{36}$$

where $P^{ij} = P^{ji}$, $A^{ij} = -A^{ji}$.

It is an easy consequence of (35), (36) that

$$S(im\,\sharp_P) = span\left\{P^{ij}\frac{\partial}{\partial u^j}\right\}$$

hence, $rank\,S = rank(P^{ij})$. Therefore, the problem becomes that of finding the Poisson bivector fields (36) which satisfy the condition

$$rank\,P = 2rank(P^{ij})$$

at each point of TN.

For instance, if we start with a Poisson structure w on the manifold N, the complete lift $P = \tilde{w}$ of w to TN, e.g. [12], is an l.L.P. structure on (TN, S). If w is non degenerate this construction again yields the example of the l.L.s. structure of the tangent bundle of a symplectic manifold indicated in Section 1.

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