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D. Penazzi*

AN EXTENSION TO THE NON-METRIC CASE OF A THEOREM OF GLASNER

Abstract. In [1] the Furstenberg Structure Theorem for flows was extended from the metric to the non-metric case by means of a special construction of minimal flows. We are able to use this construction to generalize another theorem from the metric to the non-metric case: Glasner proved in [4] that if the space of regular Borel probability measures of a flow is distal, then it is equicontinuous, provided the flow is a metric minimal flow. We are able here to remove the metric condition.

Let X be a compact Hausdorff space, and let $\mathcal{M}(X)$ be the set of regular Borel probability measures on X.
$\mathcal{M}(X)$ will always be assumed to have the weak-* topology induced as a subset of the dual of $\mathcal{C}(X)$, that is, $\mu_i \to \mu \iff \int f\ d\mu_i \to \int f\ d\mu\ \forall\ f \in \mathcal{C}(X)$. With this topology, $\mathcal{M}(X)$ is compact Hausdorff. Moreover, if X is metric, then so is $\mathcal{M}(X)$.
A *Dirac measure* is a measure of the form δ_x, where δ_x is defined to be $\int f\ d\delta_x = f(x)$. The function $\delta : X \mapsto \mathcal{M}(X)$ that sends x to δ_x is a homeomorphism onto its image. We'll sometimes identify X with $\delta(X)$.
If $\pi : X \mapsto Y$ is continuous, we define $\hat{\pi} : \mathcal{M}(X) \mapsto \mathcal{M}(Y)$ by $\int f\ d\hat{\pi}(\mu) = \int (f \circ \pi) d\mu$.
Assume now that (X, T) is a flow. The action of T on X induces an action of T on $\mathcal{M}(X)$ in the following way: first, if f is a measurable function, define tf to be $tf(x) = f(xt)$. Then, define μt as the measure given by $\int f\ d(\mu t) = \int (tf)\ d\mu$. This is an action and $(\mathcal{M}(X), T)$ is a flow.

DEFINITION 1. *A (not-necessarily minimal) flow (X, T) is called* strongly distal *(or sd for short) if $(\mathcal{M}(X), T)$ is distal.*

REMARK 1. Strongly distal implies distal since X is a closed T-invariant subset of $\mathcal{M}(X)$ (by means of the identification $X = \delta_X = \{\delta_x : x \in X\}$).

LEMMA 1. *If $\pi : X \mapsto Y$ is an epimorphism of T-flows and (X, T) is strongly distal, then so is (Y, T).*

Proof. If $\pi : X \to Y$ is a homomorphism onto, then so is $\hat{\pi} : \mathcal{M}(X) \mapsto \mathcal{M}(Y)$. Thus, $\mathcal{M}(X)$ distal implies that $\mathcal{M}(Y)$ is distal.

□

DEFINITION 2. *Let $S := \{H : H$ is a countable subgroup of $T\}$. Let ρ be a*

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continuous pseudometric on X. Define, for $H \in \mathcal{S}$

$$\begin{aligned} R(H) = R(\rho, H) &= \{(x,y) \in X \times X \,:\, \rho(xh, yh) = 0 \forall h \in H\} \\ X_H &= X/R(H) \end{aligned}$$

and $\pi_H : X \mapsto X_H$ the canonical projection. This construction is called the Ellis' Construction

Lemma 2. *Let $H \in \mathcal{S}$. Then:*
i) X_H is compact Hausdorff and X_H is metrizable.
In fact, ρ induces a metric ρ_H on X_H by $\rho_H(\pi_H(x), \pi_H(y)) = \sum_{i=0}^{\infty} 2^{-i} \rho(xh_i, yh_i)$, where $H = \{h_i\}_{i=0}^{\infty}$.
ii) H acts on X_H and π_H is an H-homomorphism.
iii) If (X,T) is minimal, then $\exists\, K \in \mathcal{S} \,:\, H \subset K$ and (X_K, K) is minimal.

Proof. Lemma 1.2 and Proposition 1.6 of [1].

□

Remark 2. For technical reasons , we'll assume that the elements of H are numbered such that $h_0 = e$. This gives the property $\rho(x,y) \leq \rho_H(\pi_H(x), \pi_H(y))$, since : $\rho(x,y) = \rho(xe, ye) = 2^{-0}\rho(xh_0, yh_0) \leq \sum_{i=0}^{\infty} 2^{-i}\rho(xh_i, yh_i) = \rho_H(\pi_H(x), \pi_H(y))$.

Remark 3. The following theorem was proved in the metric case by Glasner. (see either Theorem 1.1 or 5.2 of [4]). Here we prove it in the non-metric case.

Theorem 1. *Every strongly distal minimal flow is equicontinuous.*

Proof. Suppose that X is not metric, not equicontinuous and strongly distal. Then T is not totally bounded in $C(X,X)$ and so there exists a pseudometric ρ on X and $r > 0$ such that

$$\forall \{t_1, t_2, \ldots, t_n\} \subset T,\ T \neq \cup_{i=1}^{n} \hat{\rho}(t_i, r) \tag{1}$$

where $\hat{\rho}(f,r) = \{g \in C(X,X) \,:\, \hat{\rho}(f,g) < r\}$ and $\hat{\rho}(f,g) = \sup_{x \in X} \rho(f(x), g(x))$.
Now, choose any element $t_1 \in T$. By (1), $T \neq \hat{\rho}(t_1, r)$, so $\exists\, t_2 \in T$ with $t_2 \notin \hat{\rho}(t_1, r)$. Again by (1), $\hat{\rho}(t_1, r) \cup \hat{\rho}(t_2, r) \neq T$ and thus $\exists\, t_3 \in T \,:\, t_3 \notin \hat{\rho}(t_1, r) \cup \hat{\rho}(t_2, r)$. Continuing in this way we find that

$$\exists \{t_i\}\_i \subset T \text{ with } t_n \notin \cup_{i=1}^{n-1} \hat{\rho}(t_i, r) \tag{2}$$

Let H_0 be the subgroup of T generated by $\{t_n\}_{n \geq 1}$.
H_0 is countable and thus by Lemma 2 there exists $H \in \mathcal{S}$ with (X_H, H) minimal and $H_0 \subseteq H$, where (X_H, H) and $\pi_H : X \to X_H$ are as in Definition 2 . Now $(\mathcal{M}(X), T)$ is distal since (X,T) is strongly distal, hence $(\mathcal{M}(X), H)$ is also distal, that is (X, H) is strongly distal. Thus by Lemmas 2 *ii)* and 1 , we have that (X_H, H) is also strongly distal. Since X_H is metric (by Lemma 2 *i)*), then (X_H, H) is equicontinuous, since the theorem is true in the metric case. (Theorem 1.1 or 5.2 of [4]). Let ρ_H be the metric on X_H induced by ρ, and $\hat{\rho}_H$ the metric on $C(X_H, X_H)$ induced by ρ_H. H is totally bounded in $C(X_H, X_H)$ since (X_H, H) is equicontinuous. So, $\exists\, h_1, h_2, \ldots, h_k \in H$ with $H = \cup_{i=1}^{k} \hat{\rho}_H(h_i, r/2)$. Since $H_0 \subseteq H$, we have by the previous that

$$(3) \qquad \forall\ n\ \exists\ i_n \in \{1,\ldots,k\}\ :\ t_n \in \hat{\rho}_H(h_{i_n}, r/2)$$

Since the number of t_n's is infinite $\exists\ n > m\ :\ i_n = i_m$ Denote $i = i_n$. By (3) we have that $t_n, t_m \in \hat{\rho}_H(h_i, r/2)$ and so $t_n \in \hat{\rho}_H(t_m, r)$. However $t_n \notin \hat{\rho}(t_m, r)$ by (2). So $\exists\ x \in X\ :\ \rho(xt_n, xt_m) \geq r$. But $\rho(xt_n, xt_m) \leq \rho_H(\pi_H(x)t_n, \pi_H(x)t_m)$ (by Remark 2) and $\rho_H(\pi_H(x)t_n, \pi_H(x)t_m) \leq \hat{\rho}_H(t_n, t_m) < r$. Thus $r < r$.

□

An important consequence of the generalization to the non-metric case is that we can then apply the theorem to the enveloping semigroup of X, (which need not be metric even if X is), obtaining the following:

COROLLARY 1. *(X,T) is equicontinuous iff $(E(X),T)$ is strongly distal. (no minimality or point-transitivity assumption)*

Proof. Let $E = E(X)$. If (E,T) is strongly distal, then it is distal (by Remark 1), hence it is minimal (if $p, q \in E$, then take $\{t_i\}_i \subset T\ :\ t_i \to p^{-1}q$. Then, $pt_i \to pp^{-1}q = q$ and $q \in \overline{pT}$). But, then, (E,T) is strongly distal and minimal, so by Theorem 1 we have that (E,T) is equicontinuous, hence so is (X,T). (e.g., because $E(E,T) \simeq E$). On the other hand, if X is equicontinuous, then so is E, and thus, since (E,T) is point-transitive, then it is minimal. Thus, (E,T) is minimal and equicontinuous, hence, $(\mathcal{M}(E),T)$ is equicontinuous, in particular, it is distal, and so (E,T) is strongly distal.

□

Although the previous work applies to non-metric flows, the following proves that we can't have X metric and $E(X)$ non-metric in some cases:

COROLLARY 2. *Let X be metric with $E(X)$ strongly distal. Then $E(X)$ is metric.*

Proof. By the previous corollary, if $E(X)$ is strongly distal, then X is equicontinuous. In particular, every element of $E(X)$ is continuous. Thus, since X is metric, $E(X)$ is also metric.

□

Note that the proof of the main theorem goes along these lines: first, from the fact that (X,T) is not equicontinuous, we construct a certain subgroup H of T, which gives us in turn X_H. Then we prove that (X,H) is strongly distal, and hence that (X_H,H) is strongly distal, applying then the theorem for the metric case, obtaining that (X_H,H) is equicontinuous, a contradiction because of how H was constructed. This means that the only properties of "strongly distal" that we used in the previous theorem were that :
I) If a flow is strongly distal, any factor of it is strongly distal;
and:
II) If (X,T) is strongly distal and H is a subgroup of T, then (X,H) is strongly distal. (and of course, the fact that every strongly distal *metric* minimal flow is equicontinuous).

Hence, we can state:

Proposition 1. *If a property P of a flow satisfies* I) *and* II) *above, that is, if a flow (X,T) has property P then any factor of it has property P, and if H is a subgroup of T then (X,H) has property P; then every metric flow with property P is equicontinuous if and only if every flow with property P is equicontinuous.*

Properties satisfying similar conditions to I) and II) have appeared in the literature. In particular, in [5] a property P of flows is called *transferable* if:
(1) P is preserved by tranformation group homomorphisms onto minimal sets, and
(2) if (X,T) has property P and S is a subgroup of T then there is a point $x^* \in X$ such that $(\overline{x^*S}, S)$ has property P.

Theorem 2. *If all metric flows with property P are equicontinuous and P is transferable, then all flows with property P are equicontinuous.*

Proof. We proceed as in the proof of the main theorem, constructing H from X, and hence X_H. Now, we cannot conclude that (X,H) has property H, only that there is a point x^* such that $(\overline{x^*H}, H)$ has property P. But since $\overline{x^*H}$ is H-invariant and closed, its image on X_H under the map $\pi|_{\overline{x^*H}}$ is also H-invariant and closed (the spaces are compact Hausdorff). Since X_H is H-minimal, we conclude $\pi(\overline{x^*H}) = X_H$, i.e., $\pi|_{\overline{x^*H}}$ is an epimorphism. The rest of the proof is the same.

□

Glasner proved another theorem in [4], namely, that if $\mathcal{M}(X)$ is semisimple (pointwise almost periodic) and X is metric minimal, then X is equicontinuous. (see Theorem 5.1, page 120 of [4]). We cannot generalize this theorem, but we can obtain:

Theorem 3. *If X is a minimal flow such that $\mathcal{M}(X)$ is H-semisimple for all subgroups H of T, then X is equicontinuous.*

Proof. The property thus defined clearly satisfies II), and if $X \mapsto Y$ is an extension, and $\mathcal{M}(X)$ is H-semisimple, then so is $\mathcal{M}(Y)$. Clearly, if a metric flow satifies this property, it satisfies the condition of Theorem 5.1 of [4], thus it is equicontinuous.

□

An important property is whether the minimal subflows of a flow are distal. (We'll call this property "f-distal"). The property P_0: "every minimal flow of $\mathcal{M}(X)$ is distal" (that is, $\mathcal{M}(X)$ is f-distal, or, we may say, X is strongly f-distal) is conjectured by Glasner to be equivalent to distal. It is clear that it is a property "between" distal and equicontinuity. We wil strenghten this property as we did with semisimple, requiring: P_1: "for every subgroup H of T, every H-minimal flow of $\mathcal{M}(X)$ is H-distal". In this case, we have:

Proposition 2. *If every metric minimal flow having property P_1 above is equicontinuous, then P_1 is "equicontinuity" for minimal flows.*

Proof. As said above, every equicontinuous flow satisfies P_0, hence P_1 since "equicontinuous" is group-hereditary. Hence we need to show only that every flow with P_1 is equicontinuous, assuming that this is true for metric flows. We will show that P_1 satisfies properties I) and II), so by Proposition 1 we will be done. Clearly, by our

definition, P_1 satisfies II). Let's see that it satisfies I.
Let $\pi : X \to Y$ be an epimorphism, X satisfying P_1. Hence $\hat{\pi}$ is onto. Let H subgroup of T and $N \subseteq \mathcal{M}(Y)$ be a H-minimal. Let $P = \overline{co}(N)$. Then $\overline{ex}(P)$ is a closed H-invariant subset of N (Milman), and since N is an H-minimal flow, we have that $\overline{ex}(P) = N$, hence P is a minimally generated affine flow. Hence, there exists a P-irreducible subflow $\mathcal{M}_N(X)$ of $\mathcal{M}(X)$. (2.1 of [2]). Let $X_N = \overline{ex}(\mathcal{M}_N(X))$. Then, also by 2.1 of [2], X_N is minimal, and $\hat{\pi}(X_N) = N$. Since X_N is H-minimal and X satisfies P_1, X_N is H-distal. Thus, so is N and we are done.

□

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Daniel PENAZZI
FaMAF, Universidad Nacional de Cordoba
Cordoba, ARGENTINA
e-mail: penazzi@mate.uncor.edu.ar

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