Rend. Sem. Mat. Univ. Pol. Torino
Vol. 58, 4 (2000)

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GEVREY HYPOELLIPTICITY FOR PARTIAL DIFFERENTIAL EQUATIONS WITH CHARACTERISTICS OF HIGHER MULTIPLICITY

Abstract. We consider a class of partial differential equations with characteristics of constant multiplicity $m \geq 4$. We prove for these equations a result of hypoellipticity and Gevrey hypoellipticity, by using classical Fourier integral operators and $S^m_{\rho,\delta}$ arguments.

1. Introduction and statement of the result

This paper concerns the Gevrey hypoellipticity of linear partial differential operators:

$$P = \sum_{|\alpha|\leq M} c_\alpha(x)\, D^\alpha \,. \tag{1}$$

We use in (1) standard notations, and we assume that the coefficients $c_\alpha(x)$ are analytic, defined in a neighborhood Ω of a point $x_0 \in \mathbb{R}^\mathbf{n}$. More generally, P could be assumed in the following to be a classical analytic pseudo-differential operator, defined as, for example, in Rodino [15], Trèves [16].
We recall that P is said to be hypoelliptic at (a neighborhood Ω of) the point x_0 when

$$sing\, supp\; Pu = sing\, supp\, u \quad for\ all\ u \in D'(\Omega) \tag{2}$$

and Gevrey d-hypoelliptic, $1 < d < +\infty$, when

$$d - sing\, supp\; Pu = d - sing\, supp\, u \quad for\ all\ u \in D'(\Omega). \tag{3}$$

In (3) the d-singular support of a distribution u is defined as the smallest closed set in the complement of which u is a G^d function, $1 < d < +\infty$, i.e.: it satisfies locally estimates of the type

$$|D^\alpha f(x)| \leq C^{|\alpha|+1} (\alpha!)^d.$$

We want to study the multiple characteristics case. Namely, consider the principal symbol:

$$p_M(x,\xi) = \sum_{|\alpha|=M} c_\alpha(x)\, \xi^\alpha.$$

Arguing microlocally, we fix $\xi_0 \neq 0$ and set:

DEFINITION 2. *We say that P is an operator with characteristics of constant multiplicity $m \geq 2$ at (x_0, ξ_0) if in a conic neighborhood $\Gamma \subset \Omega \times (\mathbb{R}^{\mathbf{n}} \backslash 0)$ of (x_0, ξ_0) we may write*

$$p_M(x, \xi) = e_{M-m}(x, \xi)\, a_1(x, \xi)^m ,$$

where $e_{M-m}(x, \xi)$ is an analytic elliptic symbol, homogeneous of order $M - m$, and the first order analytic symbol $a_1(x, \xi)$ is real- valued and of microlocal principal type, i.e. $d_{x,\xi}\, a_1(x, \xi)$ never vanishes and it is not parallel to $\sum_{j=1}^{n} \xi_j\, dx_j$ on

$$\Sigma = \{(x, \xi) \in \Gamma,\ a_1(x, \xi) = 0\}.$$

Observe that Σ is also characteristic manifold of $p_M(x, \xi)$; we understand $(x_0, \xi_0) \in \Sigma$. For P satisfying such definition, we want to study hypoellipticity or, more precisely, micro-hypoellipticity at (x_0, ξ_0), defined by

$$\Gamma \bigcap WF\, Pu = \Gamma \bigcap WF\, u \quad for\ all\ u \in D'(\Omega) \tag{4}$$

and d-micro-hypoellipticity, defined by

$$\Gamma \bigcap WF_d\, Pu = \Gamma \bigcap WF_d\, u \quad for\ all\ u \in D'(\Omega), \tag{5}$$

for a sufficiently small neighborhood Γ of (x_0, ξ_0). See for example Hörmander [4], Rodino [15] for the definition of the wave front set $WF\, u$ and Gevrey wave front set $WF_d\, u$ of a distribution u. We observe that (4), (5) imply respectively (2), (3), when satisfied in a conic neighborhood Γ of (x_0, ξ_0) for all $\xi_0 \neq 0$.

To express our result we need the so-called subprincipal symbol of P:

$$p'_{M-1}(x, \xi) = \sum_{|\alpha|=M-1} c_\alpha(x)\, \xi^\alpha - \frac{1}{2i} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j\, \partial \xi_j}\, p_M(x, \xi).$$

We recall that p'_{M-1} has a geometric invariant meaning at Σ, see for example Hörmander [4]; we shall write in the following

$$J^0(x, \xi) = p'_{M-1}(x, \xi)|_\Sigma .$$

Let us assume for simplicity in Γ :

$$p_M(x, \xi)\ is\ real-valued\ and,\ when\ m\ is\ even,\ non-negative \tag{6}$$

(this is not restrictive, if we are allowed to multiply by an elliptic factor passing to the pseudo-differential frame).

It is then known from Liess-Rodino [6] that in the case $\Im J^0(x_0, \xi_0) \neq 0$ we have micro-hypoellipticity and d-micro-hypoellipticity at (x_0, ξ_0) for $d \geq \frac{m}{m-1}$. In this paper we shall allow $\Im J^0(x_0, \xi_0) = 0$, but assume $\Re J^0(x_0, \xi_0) \neq 0$. To be definite, let us set

$$\Re J^0(x, \xi) < 0 \quad for\ (x, \xi) \in \Sigma. \tag{7}$$

Fixing attention here on the higher multiplicity case $m \geq 3$, we need to consider some other invariants associated to p'_{M-1}, cf. Liess-Rodino [7], Mascarello-Rodino [8]:

$$J^r(x, \xi, X) = \frac{1}{r!}\, \chi^r\, p'_{M-1}(x, \xi),$$

for $(x, \xi, X) \in N(\Sigma)$, $1 \leq r \leq m-2$, where $N(\Sigma)$ is the normal bundle to the characteristic manifold Σ and χ is a vector field in Γ such that $\chi(x, \xi)$ at $(x, \xi) \in \Sigma$ is in the equivalence class of $X \in N_{(x,\xi)}(\Sigma)$.
Obviously we have:

$$J^r(x, \xi, -X) = (-1)^r J^r(x, \xi, X). \tag{8}$$

For uniformity of notation we shall also regard J^0 as a function on $N(\Sigma)$, independent of X at (x, ξ).

THEOREM 1. *Let P be an operator with characteristics of constant multiplicity m, satisfying* (6), (7). *Assume moreover there exists r^*, $0 < r^* < \frac{(m-1)}{2}$, such that*

i) $\Im J^{r^*}(x, \xi, X) \neq 0$, *for all* $(x, \xi, X) \in N(\Sigma)$, $X \neq 0$,

ii) $\Im J^{r^*}(x, \xi, X) \Im J^r(x, \xi, X) \geq 0$, *for all* $(x, \xi, X) \in N(\Sigma)$, $0 \leq r < r^*$.

Then P is micro-hypoelliptic and d-micro-hypoelliptic for $d \geq \frac{m}{m-1-r^*}$.

Let us compare our result with the existing literature. For the sake of brevity, we limit attention to some models in $\mathbb{R}^2$, satisfying (6), (7) at $x_1^0 = 0$, $x_2^0 = 0$, $\xi_1^0 = 0$, $\xi_2^0 > 0$. We list first the following examples, representative of general classes already considered by other authors:

$$D_{x_1}^m - D_{x_2}^{m-1} \quad (m \geq 2)\,, \tag{9}$$

$$D_{x_1}^m - D_{x_2}^{m-1} + i\, x_1\, D_{x_1}\, D_{x_2}^{m-2} \quad (m \geq 3)\,, \tag{10}$$

$$D_{x_1}^m - D_{x_2}^{m-1} + i\, x_1^{2h}\, D_{x_2}^{m-1} \quad (m \geq 2)\,, \tag{11}$$

$$D_{x_1}^m - D_{x_2}^{m-1} + i\, x_1^{2h}\, D_{x_2}^{m-1} + i\, x_1^l\, D_{x_1}\, D_{x_2}^{m-2}\ (m \geq 3)\,. \tag{12}$$

The operators (9), (10) are not hypoelliptic; observe also that (10) is not locally solvable, cf. Corli [1]. The operator (11) is hypoelliptic for any $h \geq 1$, despite the fact that $\Im J^0(x_0, \xi_0) = 0$, cf. Menikoff [9], Popivanov [10], Roberts [14]; the operator (12), having the same J^0 as (11), is not hypoelliptic if h is sufficiently large with respect to $l \geq 1$, cf. Popivanov-Popov [12], Popivanov [11].
Theorem 1 gives new conditions on J^r, i.e. on the coefficient of the terms
$D_{x_1}^r\, D_{x_2}^{m-r-1}$ for models of the preceding type, to guarantee hypoellipticity and Gevrey hypoellipticity. We have to assumenow $m \geq 4$.
Let us observe that, if r^* is odd, then *i), ii)* in Theorem 1 and (8) actually imply $\Im J^r \equiv 0$ for even $r < r^*$; as examples of hypoelliptic operators characterized by Theorem 1 consider in this case

$$D_{x_1}^m - D_{x_2}^{m-1} + i\, D_{x_1}\, D_{x_2}^{m-2} \qquad (r^* = 1), \tag{13}$$

$$D_{x_1}^m - D_{x_2}^{m-1} + i\, x_1^{2h}\, D_{x_1}\, D_{x_2}^{m-2} + i\, D_{x_1}^3\, D_{x_2}^{m-4} \qquad (r^* = 3), \tag{14}$$

having the same J^0 as the non-hypoelliptic operators (9), (10). If r^* is even, then *i), ii)* and (8) imply $\Im J^r \equiv 0$ for odd $r < r^*$; as corresponding example of hypoelliptic operator consider

$$D_{x_1}^m - D_{x_2}^{m-1} + i\, x_1^{2h} D_{x_2}^{m-1} + i\, D_{x_1}^2 D_{x_2}^{m-3} \qquad (r^* = 2), \tag{15}$$

having the same J^0 as (11), (12). In (13), (14), (15), the order m has to be chosen sufficiently large, to satisfy the assumption $\frac{m-1}{2} > r^*$. Returning to general operators, we may regard Theorem 1 as extension of a result of Liess-Rodino([7],Theorem 6.3), which prove the same order of Gevrey hypoellipticity, requiring *i)* and $\Im J^r = 0$ for all $r < r^*$, which is stronger than *ii)*; see also Tulovsky [17] for hypoellipticity in the C^∞-sense. Observe however that Liess-Rodino [7] allow $0 < r^* \leq m-2$, whereas we do not know whether our result is valid for $\frac{m-1}{2} \leq r^* \leq m-2$.

The proof of Theorem 1 will be reduced, after conjugation by classical Fourier integral operators, to a simple $S^m_{\rho,\delta}$ argument (let us refer in particular to the result of Kajitani-Wakabayashi [5] in the Gevrey frame).

2. Gevrey hypoellipticity for a class of differential polynomials.

In this section we begin to study a pseudo-differential model in suitable simplectic co-ordinates. The conclusion of the proof of Theorem 1 will be given in the subsequent Section 3. As before we denote by $x = (x_1, ..., x_n)$ the real variables in Ω, open subset of $\mathbb{R}^{\mathbf{n}}$; $\xi = (\xi_1, \xi_2, ..., \xi_n)$, $\xi_2 > 0$, the dual variables of x. We consider the conic neighborhood $\Lambda = \{0 < \xi_1^2 + |\zeta|^2 < C\xi_2^2\}$ of the axis $\xi_2 > 0$, where $\zeta = (\xi_3, ..., \xi_n) \in \mathbb{R}^{\mathbf{n-2}}$, for a suitable constant C. Moreover we take $q, m, r, s \in \mathbb{N}$ such that $m \geq 2$, $1 \leq q < m$, and (r, s) belong to the set $I = \{(r, s) \in \mathbb{N}^{\mathbf{2}} : 0 < qr + ms < qm\}$.

Let the function in $\Omega \times \Lambda = \Gamma$

$$p(x,\xi) = \xi_1^m - h_{0,q}(x,\xi)\,\xi_2^q + \sum_{(r,s)\in I} h_{r,s}(x,\xi)\,\xi_1^r \xi_2^s\,, \tag{16}$$

be a differential polynomial, symbol of a (micro) pseudo-differential operator $P(x, D)$, where $h_{(\cdot,\cdot)} : \Gamma \to \mathbb{C}$, $h_{(\cdot,\cdot)} = \Re h_{(\cdot,\cdot)} + i\Im h_{(\cdot,\cdot)}$, $\Re h_{(\cdot,\cdot)}$, $\Im h_{(\cdot,\cdot)} : \Gamma \to \mathbb{R}$, $\Re h_{(\cdot,\cdot)}$, $\Im h_{(\cdot,\cdot)} \in G^1(\Gamma)$, see below.
We define the sets, for $k \in \mathbb{N}$, $0 < k < qm$:

$$I_k = \{(r, s) \in \mathbb{N}^{\mathbf{2}} : qr + ms = k\}$$

and fix $k = k^*$ such that $q(m - \frac{1}{2}) < k^* < qm$. We use the notation k^- for all $k < k^*$ and k^+ for all $k > k^*$. We may split $I = I_- \bigcup I_{k^*} \bigcup I_+$, with $I_- = \bigcup I_{k^-}$, $I_+ = \bigcup I_{k^+}$.

LEMMA 1. *Let $p(x,\xi)$ be the function* (16),*where $h_{(\cdot,\cdot)}$ is assumed to be homogeneous of order zero with respect to ξ and analytic, which implies for some constant $L > 0$*

$$|D_x^\alpha D_\xi^\beta h_{(\cdot,\cdot)}| \leq L^{|\alpha|+|\beta|+1} \alpha!\beta!\,(1 + |\xi|)^{-|\beta|}. \tag{17}$$

Assume moreover I_{k^*} consists of one couple (r^*, s^*), $k^* = qr^* + ms^*$, such that:

(i) $\Im h_{r^*,s^*}(x,\xi) \neq 0$, *for all* $(x,\xi) \in \Gamma$,

(ii) $\Im h_{r^*,s^*}(x,\xi)\,\Im h_{r,s}(x,\xi)\,\xi_1^{r^*+r}\xi_2^{s+s^*} \geq 0$, *for all* $(x,\xi) \in \Gamma$, $k^* < k^+ = qr+ms < qm$,

(iii) $\Im h_{0,q}(x,\xi)\,\Im h_{r^*,s^*}(x,\xi)\,\xi_1^{r^*}\xi_2^{q+s^*} \leq 0$, *for all* $(x,\xi) \in \Gamma$,

(iv) $\Re h_{0,q}(x,\xi) \neq 0$, *for all* $(x,\xi) \in \Gamma$.

Then for all $\alpha, \beta \in \mathbb{Z}_+^{\mathbf{n}}$, *for all* $K \subset\subset \Omega$, *we have for new positive constants L and B independent of* α, β*:*

$$\frac{|D_x^\alpha D_\xi^\beta\, p(x,\xi)|\,|\xi|^{\rho|\beta|-\delta|\alpha|}}{|p(x,\xi)|} \leq L^{|\alpha|+|\beta|+1}\alpha!\beta!\,,\quad |\xi| > B\,, \tag{18}$$

where $\rho = \frac{k^*-q\,(m-1)}{m}$, $\delta = \frac{qm-k^*}{m}$. *Observe that we have* $\delta < \rho$, *since we have assumed* $k^* > q(m-\frac{1}{2})$

REMARK 1. Hypothesis *(ii)* implies that $\Im h_{r^*,s^*}(x,\xi)$ and $\Im h_{r,s}(x,\xi)$ are both positive or both negative ($\Im h_{r,s}(x,\xi)$ may vanish, too), and that r is according (both even or both odd) to r^* for all r such that $k^* < k^+$. Otherwise (r is not according to r^*), $\Im h_{r,s}(x,\xi)$ has to vanish in Γ.
Hypothesis *(iii)* induces $\Im h_{0,q}(x,\xi) \equiv 0$ if r^* is odd.

REMARK 2. By formula (18) and by Kajitani-Wakabayashi([5], Theorem 1.9), we have that the operator $P(x,D)$, associated to the symbol $p(x,\xi)$ in (16), is G^d-microlocally hypoelliptic in Γ for $d \geq max\left\{\frac{1}{\rho}\,,\,\frac{1}{1-\delta}\right\} = \frac{1}{\rho}$.

REMARK 3. When $\rho < 1$, and $\delta > 0$, one can prove by means of interpolation theory as in Wakabayashi([18], Theorem 2.6) that (18) is valid for any $\alpha,\beta \in \mathbb{Z}_+^{\mathbf{n}}$, if (18) holds for $|\alpha+\beta| = 1$. Hence it is sufficient to verify (18) for $|\alpha+\beta| = 1$ because $\rho = \frac{k^*-q(m-1)}{m} < \frac{q}{m} < 1$, and $\delta = \frac{qm-k^*}{m} > 0$.

REMARK 4. For the proof of Theorem 1 it will be sufficient to apply Lemma 1 for $q = m-1$. The general case $1 \leq q < m$ leads to a more involved geometric invariant statement, which we shall detail in a future paper.

Proof of Lemma 1. We first estimate the numerator of (18), then we give some lemmas to estimate the denominator of (18).
If $|\alpha| = 1$, $|\beta| = 0$, we get

$$\begin{aligned}|D_{x_j}\,p(x,\xi)|\,|\xi|^{-\delta} &= \left|\textstyle\sum_{(r,s)\in I} D_{x_j} h_{r,s}(x,\xi)\,\xi_1^r\xi_2^s - D_{x_j} h_{0,q}(x,\xi)\,\xi_2^q\right|\,|\xi|^{-\delta}\\ &\leq L_1\left(\textstyle\sum_{(r,s)\in I}|\xi_1|^r\xi_2^s + \xi_2^q\right)|\xi|^{-\delta},\quad j = 1,\dots,n;\end{aligned}$$

for a suitable constant L_1 in view of the assuption (17).
If $|\alpha| = 0$, $|\beta| = 1$, then

$$|D_{\xi_j}\,p(x,\xi)|\,|\xi|^\rho \leq L_2\left(\sum_{(r,s)\in I}|\xi_1|^r\xi_2^s + \xi_2^q\right)|\xi|^\rho\,(1+|\xi|)^{-1}, \tag{19}$$

$$j = 3,\dots,n;$$

for a suitable constant L_2, in view of (17).
Moreover:

$$(20)\qquad \begin{aligned} |D_{\xi_1}\, p(x,\xi)|\,|\xi|^{\rho} \;\le\; & \left(m|\xi_1|^{m-1} + L_3 \textstyle\sum_{(r,s)\in I} |\xi_1|^{r-1}\xi_2^{s}\right) |\xi|^{\rho} \\ & + L_4 \left(\textstyle\sum_{(r,s)\in I} |\xi_1|^{r}\xi_2^{s} + \xi_2^{q}\right) |\xi|^{\rho}\,(1+|\xi|)^{-1} \end{aligned}$$

and

$$(21)\qquad \begin{aligned} |D_{\xi_2}\, p(x,\xi)|\,|\xi|^{\rho} \;\le\; & \left(qL_{0,q}\xi_2{}^{q-1} + L_5 \textstyle\sum_{(r,s)\in I} |\xi_1|^{r}\xi_2^{s-1}\right) |\xi|^{\rho} \\ & + L_6 \left(\textstyle\sum_{(r,s)\in I} |\xi_1|^{r}\xi_2^{s} + \xi_2^{q}\right) |\xi|^{\rho}\,(1+|\xi|)^{-1}, \end{aligned}$$

for suitable constants $L_3, L_4, L_5, L_6, L_{0,q}$ in view of (17).
On the other hand, we have:

$$(22)\qquad |\xi|^{\rho}\,(1+|\xi|)^{-1} \;\le\; |\xi|^{-\delta}, \quad for\ all\ \xi \in \Lambda,$$

in fact, by multiplying by $|\xi|^{\delta}(1+|\xi|)$ on both sides of (22), we obtain

$$|\xi| \,-\, |\xi|^{p} \,+\, 1 \,\ge\, 0\,, \quad for\ all\ \xi \in \Lambda,$$

where $p = \rho + \delta < 1$.
Then in the right-hand side of (19), (20), (21) we may further estimate $|\xi|^{\rho}\,(1+|\xi|)^{-1}$ by $|\xi|^{-\delta}$. Therefore, to prove (18), it will be sufficient to show the boundedness in Γ, for $|\xi| > B$, of the functions

$$Q_1(\xi) \;=\; \frac{\left(\sum_{(r,s)\in I} |\xi_1|^{r}\xi_2^{s} + \xi_2^{q}\right)|\xi|^{-\delta}}{|p(x,\xi)|}\,,$$

$$Q_2(\xi) \;=\; \frac{\left(m|\xi_1|^{m-1} + L_3 \sum_{(r,s)\in I} |\xi_1|^{r-1}\xi_2^{s}\right)|\xi|^{\rho}}{|p(x,\xi)|}\,,$$

$$Q_3(\xi) \;=\; \frac{\left(qL_{0,q}|\xi_2|^{q-1} + L_5 \sum_{(r,s)\in I} |\xi_1|^{r}\xi_2^{s-1}\right)|\xi|^{\rho}}{|p(x,\xi)|}$$

(we observe that terms of the type $Q_2,\ Q_3$ were already considered in De Donno [2]).
First introduce in the cone Λ, three regions:

$$(23)\qquad \begin{aligned} R_1: &\quad c\,\xi_2^{q} \le |\xi_1|^{m} \le C\,\xi_2^{q}\,, \\ R_2: &\quad |\xi_1|^{m} \ge C\,\xi_2^{q}\,, \\ R_3: &\quad |\xi_1|^{m} \le c\,\xi_2^{q}\,; \end{aligned}$$

where the constants c, C satisfy $c \ll \min\left\{\frac{1}{2}\, min_{(x,\xi)\in\Gamma} |\Re\, h_{0,q}(x,\xi)|, 1\right\}$, and $C \gg \max\left\{2\, max_{(x,\xi)\in\Gamma} |\Re\, h_{0,q}(x,\xi)|, 1\right\}$.
The following inequalities then hold:

$$(24)\qquad |\xi|^{-\delta} \;\le\; \begin{cases} C^{\frac{\delta}{q}}\, |\xi_1|^{-\delta\frac{m}{q}} & ,\ \xi \in \Lambda \bigcap R_1 \qquad (I) \\ |\xi_1|^{-\delta} & ,\ \xi \in \Lambda \bigcap R_2 \qquad (II) \\ \xi_2^{-\delta} & ,\ \xi \in \Lambda \bigcap R_3\,; \qquad (III) \end{cases}$$

note that *(II)* and *(III)* hold for all $\xi \in \Lambda$, but for our aim we may limit ourselves to consider them respectively in $\Lambda \bigcap R_2$ and in $\Lambda \bigcap R_3$. By abuse of notation, in the following we shall also denote by R_1, R_2, R_3 the sets $\Omega \times R_1, \Omega \times R_2, \Omega \times R_3$; recall that $\Gamma = \Omega \times \Lambda$.

We will show in Lemma 2, Lemma 3 and Lemma 4, that there are positive constants $K_1 < 1$, $K_2 < 1$, $K_3 < 1$, B, such that:

$$|p(x,\xi)| \geq K_1 \left|\Im h_{r^*,s^*}(x,\xi)\right| |\xi_1|^{r^*} \xi_2^{s^*}, \quad in\ \Gamma \bigcap R_1,\ |\xi| > B, \tag{25}$$

$$|p(x,\xi)| \geq K_2 |\xi_1|^m, \quad in\ \Gamma \bigcap R_2,\ |\xi| > B, \tag{26}$$

$$|p(x,\xi)| \geq K_3\, \xi_2{}^q, \quad in\ \Gamma \bigcap R_3,\ |\xi| > B. \tag{27}$$

In (25) we may further estimate $\left|\Im h_{r^*,s^*}(x,\xi)\right| > \lambda$ for $\lambda > 0$, in view of *(i)* in Lemma 1.
We first consider $Q_1(\xi)$ separately in the regions $R_1,\ R_2,\ R_3$, to prove boundedness.
In R_1 by (24),(25), we get easily, writing as before $k = qr + ms$:

$$Q_1(\xi) \leq const \left(\sum_k \frac{1}{|\xi_1|^{m-\frac{k}{q}}} + 1 \right),\ |\xi| > B,$$

where $m - \frac{k}{q} > 0$ by definition of I and $I_k - sets$.
In the regions R_2, R_3 by using respectively (24),(26) and (24),(27), we have for a constant $\epsilon > 0$ which we may take as small as we want by fixing B sufficiently large:

$$Q_1(\xi) \leq const \left(\sum_k \frac{1}{|\xi_1|^{m+q-\frac{k}{q}-\frac{k^*}{m}}} + \frac{1}{|\xi_1|^\delta} \right) < \epsilon,\ |\xi| > B,$$

and

$$Q_1(\xi) \leq const \left(\sum_k \frac{1}{|\xi_2|^{2q-\frac{k}{m}-\frac{k^*}{m}}} + \frac{1}{|\xi_2|^\delta} \right) < \epsilon,\ |\xi| > B.$$

We have therefore proved that $Q_1(\xi)$ is bounded. Let us estimate $Q_2(\xi)$, $Q_3(\xi)$. As above in the regions $R_1,\ R_2,\ R_3$, we obtain

$$Q_2(\xi) \leq const \left(1 + \sum_k \frac{1}{|\xi_1|^{m-\frac{k}{q}}} \right),$$

$$Q_3(\xi) \leq const \left(\frac{1}{|\xi_1|^{\frac{m}{q}-1}} + \sum_k \frac{1}{|\xi_1|^{(m+\frac{m}{q}-1)-\frac{k}{q}}} \right) < \epsilon,$$

in R_1 for $|\xi| > B$,

$$Q_2(\xi) \leq const \left(\frac{1}{|\xi_1|^{m-\frac{k^*}{q}}} + \sum_k \frac{1}{|\xi_1|^{2m-\frac{k}{q}-\frac{k^*}{q}}} \right) < \epsilon,$$

$$Q_3(\xi) \leq const \left(\frac{1}{|\xi_1|^{(m+\frac{m}{q}-1)-\frac{k^*}{q}}} + \sum_k \frac{1}{|\xi_1|^{(2m+\frac{m}{q}-1)-(\frac{k}{q}+\frac{k^*}{q})}} \right) < \epsilon,$$

in R_2 for $|\xi| > B$,

$$Q_2(\xi) \le const \left(\frac{1}{|\xi_2|^{q-\frac{k^*}{m}}} + \sum_k \frac{1}{|\xi_2|^{2q-\frac{k}{m}-\frac{k^*}{m}}} \right) < \epsilon\,,$$

$$Q_3(\xi) \le const \left(\frac{1}{|\xi_2|^{(1+q-\frac{q}{m})-\frac{k^*}{m}}} + \sum_k \frac{1}{|\xi_2|^{(2q+1-\frac{q}{m})-(\frac{k}{m}-\frac{k^*}{m})}} \right) < \epsilon\,,$$

in R_3 for $|\xi| > B$.
Now Lemma 2, Lemma 3 and Lemma 4 complete the proof.

□

LEMMA 2. *Let $p(x,\xi)$ be the function* (16), *such that* (17) *and (i), (ii), (iii) in Lemma* 1 *hold. Then there are positive constants $K_1 < 1$, B, such that:*

$$|p(x,\xi)| \ge K_1 \left|\Im h_{r^*,s^*}(x,\xi)\right| |\xi_1|^{r^*} \xi_2^{s^*}, \quad (x,\xi) \in \Gamma \bigcap R_1,\ |\xi| > B.$$

Proof. We have that

$$\begin{aligned} |p(x,\xi)|^2 &= \Big(\xi_1^m - \Re h_{0,q}(x,\xi)\,\xi_2^q + \textstyle\sum_{(r,s)\in I} \Re h_{r,s}(x,\xi)\,\xi_1^r\xi_2^s\Big)^2 + \\ &+ \Big(\Im h_{r^*,s^*}(x,\xi)\xi_1^{r^*}\xi_2^{s^*} + \textstyle\sum_{(r,s)\in I_-} \Im h_{r,s}(x,\xi)\,\xi_1^r\xi_2^s + \\ &+ \textstyle\sum_{(r,s)\in I_+} \Im h_{r,s}(x,\xi)\,\xi_1^r\xi_2^s - \Im h_{0,q}(x,\xi)\,\xi_2^q\Big)^2\,; \end{aligned} \tag{28}$$

by removing the terms rising from the real part of $p(x,\xi)$, we can write

$$|p(x,\xi)|^2 \ge \Im h_{r^*,s^*}(x,\xi)^2\,\xi_1^{2r^*}\xi_2^{2s^*} + \sum_{j=1}^4 J_j(x,\xi)$$

where

$$\begin{aligned} J_1(x,\xi) &= \Bigg(\sum_{(r,s)\in I_-} \Im h_{r,s}(x,\xi)\xi_1^r\xi_2^s + \\ &\qquad \sum_{(r,s)\in I_+} \Im h_{r,s}(x,\xi)\xi_1^r\xi_2^s - \Im h_{0,q}(x,\xi)\xi_2^q\Bigg)^2, \end{aligned} \tag{29}$$

$$J_2(x,\xi) = 2\Im h_{r^*,s^*}(x,\xi) \sum_{(r,s)\in I_-} \Im h_{r,s}(x,\xi)\,\xi_1^{r^*+r}\xi_2^{s^*+s}\,, \tag{30}$$

$$J_3(x,\xi) = 2\Im h_{r^*,s^*}(x,\xi) \sum_{(r,s)\in I_+} \Im h_{r,s}(x,\xi)\,\xi_1^{r^*+r}\xi_2^{s^*+s}\,, \tag{31}$$

$$J_4(x,\xi) = -2\Im h_{r^*,s^*}(x,\xi)\,\Im h_{0,q}(x,\xi)\,\xi_1^{r^*}\xi_2^{s^*+q}\,. \tag{32}$$

(29) is non-negative for all $(x,\xi) \in \Gamma$, (31) and (32) are also non negative by hypotheses (ii), (iii) for all $(x,\xi) \in \Gamma$.
Let us fix attention on $J_2(x,\xi)$ defined by (30). We have for all $\epsilon > 0$

$$\big(\Im h_{r^*,s^*}(x,\xi)\big)^2 \, {\xi_1}^{2r^*} \xi_2^{2s^*} + J_2(x,\xi) \geq (1-\epsilon)\big(\Im h_{r^*,s^*}(x,\xi)\big)^2 \, {\xi_1}^{2r^*} \xi_2^{2s^*} ,$$

in $\Gamma \bigcap R_1$, $|\xi| > B$. In fact, assuming for simplicity $\xi_1 \geq 0$,by (17), (23) in $\Gamma \bigcap R_1$ and hypothesis (i), for all $\epsilon > 0$ we get for B sufficiently large

$$\frac{|J_2(x,\xi)|}{\big(\Im h_{r^*,s^*}(x,\xi)\big)^2 \, {\xi_1}^{2r^*} \xi_2^{2s^*}} \leq const \sum_{(r,s)\in I_-} \frac{\xi_1^{r^*+r} \xi_2^{s^*+s}}{{\xi_1}^{2r^*} \xi_2^{2s^*}} \leq$$

$$\leq const \sum_{(r,s)\in I_-} \frac{\xi_1^{r^*+r+(s^*+s)\frac{m}{q}}}{\xi_1^{2r^*+2s^*\frac{m}{q}}} < \epsilon , \quad |\xi| > B;$$

we remark that $k^* = qr^* + ms^* > k^- = qr + ms$.
Then,

$$|p(x,\xi)| \geq K_1 \left|\Im h_{r^*,s^*}(x,\xi)\right| |\xi_1|^{r^*} \xi_2^{s^*}, \quad (x,\xi) \in \Gamma \bigcap R_1, \ |\xi| > B,$$

for a suitable constant K_1.

□

LEMMA 3. *Let $p(x,\xi)$ be the function* (16),*such that* (17) *holds. Then there are positive constants $K_2 < 1$, B, such that:*

$$|p(x,\xi)| \geq K_2 \, |\xi_1|^m , \quad (x,\xi) \in \Gamma \bigcap R_2, \ |\xi| > B.$$

Proof. We write $|p(x,\xi)|^2$ as in (28); by removing the terms arising from the imaginary part of $p(x,\xi)$, we get

$$|p(x,\xi)|^2 \geq \Big(\xi_1^m - \Re h_{0,q}(x,\xi)\xi_2^q\Big)^2 + W_1(x,\xi) + W_2(x,\xi) \tag{33}$$

where

$$W_1(x,\xi) = \left(\sum_{(r,s)\in I} \Re h_{r,s}(x,\xi) \, \xi_1^r \xi_2^s\right)^2 , \tag{34}$$

$$W_2(x,\xi) = 2\sum_{(r,s)\in I} \Re h_{r,s}(x,\xi) \, \xi_1^{r+m} \xi_2^s - 2\Re h_{0,q}(x,\xi) \sum_{(r,s)\in I} \Re h_{r,s}(x,\xi) \, \xi_1^r \xi_2^{s+q} . \tag{35}$$

Observe first that for $\lambda > 0$ sufficiently small

$$\Big(\xi_1^m - \Re h_{0,q}(x,\xi)\xi_2^q\Big)^2 > \lambda \, \xi_1^{2m} ;$$

in fact

$$\Big(\xi_1^m - \Re h_{0,q}(x,\xi)\xi_2^q\Big)^2 \geq \xi_1^{2m} - 2\Re h_{0,q}(x,\xi) \, \xi_1^m \xi_2^q ,$$

and using (23) in $\Gamma \bigcap R_2$, we have for $\Re\, h_{0,q}\, \xi_1 \geq 0$

$$\xi_1^{2m} - 2\Re\, h_{0,q}(x,\xi)\, \xi_1^m \xi_2^q \geq \left(1 - \frac{2}{C}\, \Re\, h_{0,q}(x,\xi)\right) \xi_1^{2m} > \lambda\, \xi_1^{2m},$$

since $C > 2 \max_{(x,\xi)\in\Gamma} |\Re\, h_{0,q}(x,\xi)|$.
(34) is non negative for all $(x,\xi) \in \Gamma$. We denote (35) by $\Upsilon_1(x,\xi) - \Upsilon_2(x,\xi)$, then

$$|p(x,\xi)|^2 \geq \lambda \xi_1^{2m} + \Upsilon_1(x,\xi) - \Upsilon_2(x,\xi)\,.$$

Arguing on Υ_1, Υ_2 in the same way as we have done in Lemma 2, it is possible to show that for all $\epsilon > 0$

$$\lambda \xi_1^{2m} + \Upsilon_1(x,\xi) - \Upsilon_2(x,\xi) \geq (\lambda - \epsilon)\xi_1^{2m}, \quad (x,\xi) \in \Gamma \bigcap R_2,\ |\xi| > B\,,$$

then

$$|p(x,\xi)| \geq K_2\, |\xi_1|^m\,, \quad (x,\xi) \in \Gamma \bigcap R_2,\ |\xi| > B\,,$$

where $K_2 = (\lambda - \epsilon)^{\frac{1}{2}}$.

$\square$

LEMMA 4. *Let* $p(x,\xi)$ *be the function* (16), *such that* (17) *and (iv) in Lemma* 1 *hold. Then there are positive constants* $K_3 < 1$, B, *such that:*

$$|p(x,\xi)| \geq K_3\, \xi_2{}^q\,, \quad (x,\xi) \in \Gamma \bigcap R_3,\ |\xi| > B.$$

Proof. We apply again (33), (34), (35) to $|p(x,\xi)|^2$. Observe that in $\Gamma \bigcap R_3$, arguing as above, since $c < \frac{1}{2} \min_{(x,\xi)\in\Gamma} |\Re\, h_{0,q}(x,\xi)|$, we obtain for a suitable constant $\mu > 0$

$$\left(\xi_1^m - \Re\, h_{0,q}(x,\xi)\xi_2^q\right)^2 > \mu\, \xi_2^{2q}\,.$$

About the terms in (34) and (35), the remarks we have done in Lemma 3 hold by replacing $\lambda\, \xi_1^{2m}$ with $\mu\, \xi_2^{2q}$, then we have

$$|p(x,\xi)| \geq K_3\, \xi_2{}^q\,, \quad (x,\xi) \in \Gamma \bigcap R_3,\ |\xi| > B\,,$$

where $K_3 = (\mu - \epsilon)^{\frac{1}{2}}$.

$\square$

3. Fourier integral operators and proof of Theorem 1

We consider in this section an operator mapping a fuction (or distribution, or ultradistribution) u into

$$(2\pi)^{-n} \int a(x,\xi)\widehat{u}(\xi)\, e^{i\varphi(x,\xi)}\, d\xi\ . \tag{36}$$

The phase function $\varphi(x,\xi)$ is assumed to be analytic real-valued, homogenuous of degree 1 with respect to ξ; (36) is called a Fourier integral operator (F.I.O.). Concerning the symbol $a(x,\xi)$, we suppose it belongs to $S^k(\Omega)$, the space of the classical analytic symbols of order k. The

function $\widehat{u}(\xi)$ is the Fourier transform of the function u. The particular case $\varphi(x, \xi) = x \cdot \xi$ corresponds to the usual pseudo-differential operators.
The machinery of the F.I.O.'s (see Hörmander [4], Trèves [16], Rodino [15]) may lead to relevant simplifications in the study of the micro-operator $P = P(x, D)$ in (1). Precisely, let χ be a homogeneous analytic canonical transformation acting from the conic neighborhood Γ of the point $\rho_0 = (x_0, \xi_0)$ to a conic neighborhood Γ' of the point $\chi(\rho_0) = (y_0, \eta_0)$; that χ is canonical means that it preserves the symplectic two-form $\sigma = \sum_{j=1}^{n} dx_j \wedge d\xi_j$.
Then we may consider the Fourier integral operator F with phase function φ corresponding to χ; this is a map $F : M^d(\Gamma) \to M^d(\Gamma')$, $1 < d \leq \infty$ with inverse $F^{-1} : M^d(\Gamma') \to M^d(\Gamma)$ where $M^d(\Gamma)$ denotes the factor space $D'(\Omega)/\sim$, where $u \sim v$ means that $\Gamma \bigcap WF_d(u-v) = \emptyset$, for $u, v \in D'(\Omega)$, with $WF_\infty u = WF\, u$. More details are, for example, in Rodino [14].
We then have:

$$WF_d\,(Fu) = \chi(WF_d\, u), \quad WF_d\,(F^{-1}v) = \chi^{-1}\,(WF_d\, v), \tag{37}$$

moreover

$$\tilde{P} = F\,P\,F^{-1} : M^d(\Gamma') \to M^d(\Gamma')$$

is a micro-pseudo-differential operator, with homogeneous analytic principal symbol $\tilde{p_m}(y, \eta) = p_m\left(\chi^{-1}(y, \eta)\right)$.
On the other hand, as it follows from (37)

$$\begin{array}{c} \tilde{P}\ is\ micro-hypoelliptic\ or\ d-micro-hypoelliptic \\ if\ and\ only\ if\ P\ is\ such. \end{array} \tag{38}$$

Moreover, if we assume $\rho_0 \in \Sigma$ and denote by $\tilde{\Sigma}$ the characteristic manifold of $\tilde{P}$, then $\chi(\rho_0) \in \tilde{\Sigma}$ and $\tilde{\Sigma} = \chi(\Sigma)$ in Γ'.
In this way, by fixing a suitable canonical transformation χ, we may reduce ourselves to the study of operators $\tilde{P}$ of a truly elementary form. Particular simplification in the expression of $\tilde{P}$ can be obtained by means of the following theorem.

THEOREM 2. *Let A be a classical pseudo-differential operator of microlocal principal type of first order, the function a_1 (principal symbol of A) be real and $a_1(x_0, \xi_0) = 0$, $x_0 \in \Omega$, $\xi_0 \neq 0$. Then there exists a F.I.O. F, such that $\tilde{A} = FAF^{-1}$, and $\tilde{A}$ is a pseudo-differential operator of first order, whose symbol is equal to η_k in a conic neighborhood of the point (y_0, η_0) corresponding to (x_0, ξ_0) for some k, $1 \leq k \leq n$.*

For the proof see, for example in the C^∞ frame, Egorov-Schulze([3], cap. 6, Theorem 9).

We apply Theorem 2 to the operator $P(x, D)$ with characteristics of constant multiplicity at (x_0, ξ_0), such that in a conic neighborhood Γ its principal symbol admits a decomposition as in Definition 2:

$$p_M(x, \xi) = e_{M-m}\,(x, \xi)\,a_1\,(x, \xi)^m\,.$$

The symbol of $P(x, D)$ is given by

$$p(x, \xi) = e_{M-m}\,(x, \xi)\,a_1\,(x, \xi)^m + P_{M-1}(x, \xi)$$

where $P_{M-1}(x, \xi)$ is of order $M - 1$ and, by passing to the operators:

$$P(x, D) = e_{M-m}\,(x, D)\,a_1\,(x, D)^m + R(x, D),$$

or

$$e_{M-m}(x,D)^{-1}\,P(x,D) \;=\; a_1\,(x,D)^m + e_{M-m}(x,D)^{-1}\,R(x,D),$$

where $R(x,D)$ is of order $M-1$.
$P(x,D)$ is micro-hypoelliptic if and only if $a_1\,(x,D)^m + e_{M-m}(x,D)^{-1}\,R(x,D)$ is micro-hypoelliptic, then by (38) if and only if

$$Q(y,D) \;=\; F^{-1}a_1\,(x,D)^m\,F + F^{-1}e_{M-m}(x,D)^{-1}\,R(x,D)\,F$$

is micro-hypoelliptic, and by Theorem 2 we get that:

$$F\,a_1\,(x,D)^m\,F^{-1} \;=\; \underbrace{F\,a_1\,(x,D)\,F^{-1}\cdots F\,a_1\,(x,D)\,F^{-1}}_{m\ times} \;=\; b(y,D),$$

such that $b(y,\eta) \;=\; \eta_k^m$ for some $k,\ 1 \,\le\, k \,\le\, n$. Then

$$q(y,\eta) \;\sim\; \eta_k^m \;+\; \sum_{j=1}^{\infty} q_{m-j}(y,\eta).$$

Let us assume $k \;=\; 1$ and use again the notation $p(x,\xi)$ in the role of $q(y,\eta)$; we may also suppose $\xi_2 \ge 0$ in the corresponding Γ. We can rewrite further $p(x,\xi)$ as:

$$p(x,\xi) \;=\; \xi_1^m \;+\; \sum_{j=1}^{m-1} p_{m-j}(x,\xi) \;+\; \underbrace{p_0(x,\xi)}_{order\ 0};$$

that becomes for Taylor formula stopped at order m-j

$$\xi_1^m \;+\; \sum_{j=1}^{m-1}\;\sum_{r=0}^{m-j-1}\left[\frac{1}{r!}\,\frac{\partial_{\xi_1}^r\,p_{m-j}(x,\xi)|_{\xi_1=0}}{\xi_2^s}\,\xi_1^r\xi_2^s + \xi_1^{m-j}\,\underbrace{r_{(m-j)}(x,\xi)}_{order\ 0}\right] \;+\; p_0(x,\xi),$$

with $r+s=m-j$.
Let us set:

$$h_{r,s}(x,\xi) \;=\; \frac{1}{r!}\,\frac{\partial_{\xi_1}^r\,p_{m-j}(x,\xi)|_{\xi_1=0}}{\xi_2^s},$$

so, we have:

$$\xi_1^m \;+\; h_{0,m-1}(x,\xi)\xi_2^{m-1} \;+\; \sum_{r+s\le m-1} h_{r,s}(x,\xi)\,\xi_1^r\xi_2^s\,, \tag{39}$$

where $(r,s) \ne (0,m-1)$ in the sum and $h_{m-j,0}(x,\xi) \;=\; r_{(m-j)}(x,\xi)$, $h_{0,0}(x,\xi) \;=\; p_0(x,\xi)$.
All the terms $h_{r,s}(x,\xi)$ are homogeneous of order zero, but $h_{0,0}$, which will not play any role when checking the $S^m_{\rho,\delta}$ estimates; observe also that for $(r,s) \ne (m-j,0)$ the symbol $h_{r,s}(x,\xi)$ is actually ξ_1-independent.
Formula (39) gives the model that we have studied in Section 2 with $q=m-1$.

The characteristic manifold of $p(x,\xi)$, in the new symplectic co-ordinates, is the subset $\Sigma' \;=\; \{\xi_1 \;=\; 0\}$ of $\mathbb{R}^{2n}$, so in this case we obtain $p'_{m-1} \;=\; p_{m-1}$ and $J^0(x,\xi) \;=\; p_{m-1}(x,\xi)|_{\xi_1=0} \;=\; h_{0,m-1}(x,\xi)\xi_2^{m-1}$.

Hypotheses (6), (7) and *i), ii)* in Theorem 1 are clearly transported by symplectic transformations and multiplication by elliptic factors. Moreover it is simple to verify that, taking χ proportional to $\frac{\partial}{\partial \xi_1}$ by a factor which we again denote ξ_1 after differentation:

$$\frac{1}{r!}\chi^r\, p'_{m-1}(x,\xi) = \frac{1}{r!}\partial^r_{\xi_1}\, p_{m-1}(x,\xi)|_{\xi_1=0}\, \xi_1^r = h_{r,s}(x,\xi)\, \xi_1^r\, \xi_2^s,$$

with $r+s = m-1$.
Immediately we can see that the hypotheses of the Theorem 1 are equivalent to the hypotheses of the Lemma 1, that gives our result.

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AMS Subject Classification: 35S05.

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Lavoro pervenuto in redazione il 17.05.1999 e in forma definitiva il 20.01.2000.