Rend. Sem. Mat. Univ. Pol. Torino
Vol. 58, 3 (2000)
Partial Diff. Operators

**J. Mawhin**

# BOUNDED SOLUTIONS OF SECOND ORDER SEMICOERCIVE EVOLUTION EQUATIONS IN A HILBERT SPACE AND OF NONLINEAR TELEGRAPH EQUATIONS

**Abstract.** Motivated by the problem of the existence of a solution of the nonlinear telegraph equation

$$u_{tt} + cu_t - u_{xx} + h(u) = f(t, x),$$

such that $u(t, \cdot)$ satisfies suitable boundary conditions over $(0, \pi)$ and $\|u(t, \cdot)\|$ is bounded over $\mathbb{R}$ for some function space norm $\|\cdot\|$, we prove the dissipativeness and the existence of bounded solutions over $\mathbb{R}$ of semilinear evolution equations in a Hilbert space of the form

$$\ddot{u} + c\dot{u} + Au + g(t, u) = 0,$$

where $c > 0$, $A : D(A) \subset H \to H$ is self-adjoint, semi-positive definite, has compact resolvant and $g : \mathbb{R} \times H \to H$, bounded and sufficiently regular, satisfies some semicoercivity condition.

## 1. Introduction

The problem of the existence of a solution $u(t, \cdot)$ of the nonlinear telegraph equation

$$u_{tt} + cu_t - u_{xx} + h(u) = f(t, x), \tag{1}$$

such that $u(t, \cdot)$ satisfies suitable boundary conditions over a compact interval of $\mathbb{R}$ and $\|u(t, \cdot)\|$ is bounded over $\mathbb{R}$ for a suitable function space norm $\|\cdot\|$, leads to the study of the bounded solutions of evolution equations of the form

$$\ddot{u} + c\dot{u} + Au + g(t, u) = 0, \tag{2}$$

where $u$ takes values in a Hilbert space $H$. Here, $c > 0$, $A : D(A) \subset H \to H$ is self-adjoint, semi-positive definite, has compact resolvant and $g : \mathbb{R} \times H \to H$ is bounded and satisfies suitable regularity conditions. The linear case

$$\ddot{u} + c\dot{u} + Au = f(t), \tag{3}$$

when $c > 0$ and $A$ is a positive definite isomorphism, has been considered by Ghidaglia and Temam [6] (see also [14]). They proved the existence of a solution of (3) bounded over $\mathbb{R}$ in a suitable norm. The positive definiteness of $A$ is satisfied for the special case (1) when $u(t, \cdot)$ satisfies the Dirichlet boundary conditions. The case of Neumann or periodic boundary conditions leads to a semi-positive definite $A$ and is more delicate. This is the one considered in this paper, which essentially summarizes [1], and introduces [11], where complete proofs can be found.

In Theorem 1, we prove that, if $P$ denotes the projector onto $\ker A$, then equation (2) is dissipative, when the condition of *semi-coercivity*

$$(g(t,u),u) \geq \alpha|Pu| - \beta|(I-P)u| - \gamma\,,$$

holds for all $(t,u) \in \mathbb{R} \times H$ and some positive $\alpha, \beta, \gamma$. Notice that this assumption is a consequence of the previous ones if $P = 0$. In Theorem 2, we show that the dissipativeness of equation (2) implies the existence of a solution $u$ such that $u$ and $\dot{u}$ are bounded over $\mathbb{R}$ in a suitable norm. Theorems 1 and 2 are used to prove, in Theorem 3, a necessary and sufficient condition for the existence of a bounded solution of (3) when $A$ is semi-positive definite.

The proofs of Theorems 1 and 2 require a preliminary study of the Cauchy problem for (2) and (3), which is done in Section 3.

For the nonlinear telegraph equation (1) with Neumann boundary conditions in $x$, with

$$\sup_{t\in\mathbb{R}} \int_0^\pi f^2(t,x)\,dx < +\infty\,,$$

and $h$ such that

$$h(-\infty) := \lim_{z\to-\infty} h(z)\,, \qquad h(+\infty) := \lim_{z\to+\infty} h(z)\,,$$

exist, the existence of a solution $u(t,x)$ such that

$$\sup_{t\in\mathbb{R}} \int_0^\pi \left[u(t,x)^2 + u_x(t,x)^2 + u_t(t,x)^2\right] dx < +\infty$$

is proved in Theorem 4, when $f$ satisfies a *Landesman-Lazer type* condition of the form

$$h(-\infty) < A_L\left(\frac{1}{\pi}\int_0^\pi f(t,x)\,dx\right) \leq A_U\left(\frac{1}{\pi}\int_0^\pi f(t,x)\,dx\right) < h(+\infty)\,,$$

where $A_L$ and $A_U$ respectively denote some lower and upper mean values of a bounded continuous function introduced by Tineo [15]. Such a condition was introduced for a second order ordinary differential equations in [12, 13]. We end the paper with some applications to other partial differential equations or boundary conditions, by some remarks about situations where

$$h(-\infty) = h(+\infty)\,.$$

## 2. Fundamental assumptions and concept of solution

Let $A$ be a linear self-adjoint unbounded operator in a Hilbert space $H$, such that, for each $\lambda < 0$, $(A-\lambda I)^{-1} : H \to H$ exists and is compact. We consider the class of evolution equations in the space $H$ of the type (2), where $c > 0$ and $g : \mathbb{R} \times H \to H$ is continuous, Lipschitz continuous with respect to the variable $u$, i.e.,

$$|g(t,x) - g(t,y)| \leq L|x-y|\,, \tag{4}$$

for some $L > 0$ and all $x, y \in H$, $t \in \mathbb{R}$, and bounded, i.e.,

$$\sup_{(t,u)\in\mathbb{R}\times H} |g(t,u)| < +\infty\,.$$

Here $|\cdot|$ denotes the norm associated to the scalar product $(\cdot,\cdot)$ on $H$.

If $\{\lambda_n\}$ denotes the sequence of eigenvalues of $A$ with corresponding eigenvectors $\{\varphi_n\}$, so that

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots, \qquad \lim_{n\to\infty} \lambda_n = +\infty,$$

we consider the subspace of $H$

$$V_1 := \left\{ u \in H : \sum_{n=1}^{\infty} \lambda_n (u,\varphi_n)^2 < +\infty \right\},$$

endowed with the product

$$(u,v)_1 := \sum_{n=1}^{\infty} \lambda_n (u,\varphi_n)(v,\varphi_n), \qquad (u,v \in V_1),$$

and the associated pseudonorm

$$|u|_1 := (u,u)_1^{1/2}, \qquad (u \in V_1).$$

If $P$ denotes the spectral projection from $H$ onto $\ker A$, $V_1$ is a Hilbert space for the scalar product

$$(u,v)_1 + (Pu,Pv). \tag{5}$$

We will use the fact that there exists a constant $R > 0$ such that

$$|u|^2 \le R^2 \left[ |u|_1^2 + |Pu|^2 \right], \tag{6}$$

for all $u \in V_1$.

We denote by $BC(\mathbb{R},H)$ the set of all continuous functions $f : \mathbb{R} \to H$ such that

$$\sup_{t\in\mathbb{R}} |f(t)| < +\infty,$$

and by $BC(\mathbb{R}, V_1 \times H)$ the set of all continuous functions $(u,v) : \mathbb{R} \to V_1 \times H$ such that

$$\sup_{t\in\mathbb{R}} \left[ |u(t)|_1^2 + |Pu(t)|^2 + |v(t)|^2 \right] < +\infty.$$

We say that a function $h \in BC(\mathbb{R},H)$ has a *bounded primitive* if

$$\sup_{t\in\mathbb{R}} \left| \int_0^t h(s)\,ds \right| < +\infty$$

and denote by $BP(\mathbb{R},H)$ the set of those functions. The special cases $BC(\mathbb{R},\mathbb{R})$ and $BP(\mathbb{R},\mathbb{R})$ will be used as well.

This functional setting allows us to make precise the concept of solution of Eq. (2) we are using in this paper.

DEFINITION 1. *We say that $u(t)$ is a solution of Eq. (2) if*

$$u \in C(\mathbb{R}, V_1) \cap C^1(\mathbb{R}, H)$$

*and for each $w \in V_1$ one has*

$$\frac{d^2}{dt^2}(u(t), w) + c\frac{d}{dt}(u(t), w) + (u(t), w)_1 + (g(t, u(t)), w) = 0 \tag{7}$$

*in the sense of distributions, or, equivalently*

$$\frac{d^2}{dt^2}(u(t), w) + c\frac{d}{dt}(u(t), w) + \left(A^{1/2}u(t), A^{1/2}w\right) + (g(t, u(t)), w) = 0\,.$$

DEFINITION 2. *We say that a solution $u(t)$ of Eq. (2) is* bounded *(or bounded on the whole line) if $(u, \dot{u}) \in BC(\mathbb{R}, V_1 \times H)$. We say that a solution $u(t)$ of Eq. (2) is* bounded in the future *if, for each $t_0 \in \mathbb{R}$, one has*

$$\sup_{t \geq t_0} \left[|u(t)|_1^2 + |Pu(t)|^2 + |\dot{u}(t)|^2\right] < +\infty\,.$$

A case in which all the solution of Eq. (2) are bounded in the future is when the equation is dissipative. Among the various notions of dissipativeness which exist for evolution equations (see [3, 7, 8, 9, 16]), we use the following one.

DEFINITION 3. *The equation (2) is called* dissipative *if there exists a constant $\rho > 0$ and a map $T : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for each $M > 0$, each $t_0 \in \mathbb{R}$, and each solution $u(t)$ of (2) with*

$$|u(t_0)|_1^2 + |Pu(t_0)|^2 + |\dot{u}(t_0)|^2 \leq M\,,$$

*one has*

$$|u(t)|_1^2 + |Pu(t)|^2 + |\dot{u}(t)|^2 \leq \rho\,,$$

*for all $t \geq T(M) + t_0$.*

## 3. The Cauchy problem

Under the assumptions of Section 2, let us consider the initial value problem

$$\ddot{u} + c\dot{u} + Au = f(t)\,, \qquad (t \in J)\,, \qquad u(t_0) = u_0\,, \qquad \dot{u}(t_0) = v_0\,, \tag{8}$$

where $J$ is a bounded interval in $\mathbb{R}$, $f \in L^2(J, H)$, $t_0 \in J$, $u_0 \in V_1$ and $v_0 \in H$. It is well known that the problem (8) has a unique solution (see [14]). A proof can be based on Galerkin's method, from which, using the classical theory of ordinary differential equations and Gronwall's Lemma, one can deduce not only the existence of a unique solution $(u, \dot{u}) \in C(J, V_1 \times H)$ of Eq. (8) and its continuous dependence on $u_0$, $v_0$ and $f$ in the strong topologies of $V$, $H$ and $L^2(J, H)$, but also its continuous dependence in the weak topologies.

LEMMA 1. *Let $u(t)$ be the solution of Eq. (8) and let $u_n(t)$ be the solution of*

$$\ddot{u} + c\dot{u} + Au = f_n(t)\,, \qquad (t \in J)\,, \qquad u(t_0) = u_{0n}\,, \qquad \dot{u}(t_0) = v_{0n}\,,$$

*where* $f_n \in L^2(J, H)$. *Assume that*

$$u_{0n} \rightharpoonup u_0 \quad \textit{weak in } V_1\,, \quad v_{0n} \rightharpoonup v_0 \quad \textit{weak in } H\,, \quad f_n \rightharpoonup f \quad \textit{weak in } L^2(J, H)\,.$$

*Then, for each* $t \in J$,

$$u_n(t) \rightharpoonup u(t) \quad \textit{weak in } V_1\,, \quad \dot{u}_n(t) \rightharpoonup \dot{u}(t) \quad \textit{weak in } H\,.$$

The following lemma is useful to construct Lyapunov functions. It follows from the similar result for the Galerkin approximations, and a limit process.

LEMMA 2. *Let* $u(t)$ *be a solution of Eq. (3) and define*

$$\eta(t) = c^2|u(t)|^2 + 2c\,(u(t), \dot{u}(t)) + 2\,|\dot{u}(t)|^2 + 2|u(t)|_1^2\,.$$

*Then* $\eta \in W^{1,1}(J; \mathbb{R})$ *and*

$$\dot{\eta}(t) = -2c\left[|\dot{u}(t)|^2 + |u(t)|_1^2 - \left(f(t), \frac{2}{c}\dot{u}(t) + u(t)\right)\right]$$

*in the sense of distributions on* $J$.

Remark that the derivative $\dot{\eta}(t)$ can be understood in the classical sense (and $\eta \in C^1(J)$) as soon as $f(t)$ is continuous.

Let us consider the initial value problem

$$(9) \qquad \ddot{u} + c\dot{u} + Au + g(t, u) = 0\,, \qquad (t \in J)\,, \qquad u(t_0) = u_0\,, \qquad \dot{u}(t_0) = v_0\,,$$

where $J$ is a bounded interval in $\mathbb{R}$, $t_0 \in J$, $u_0 \in V_1$ and $v_0 \in H$. Let us assume that $A$ and $g(t, u)$ satisfy the hypotheses in Section 2. Under these conditions, the problem (9) possesses a unique solution which is defined in $J$ (see e.g. [14]). The following proposition shows its continuous dependence in the weak topology.

LEMMA 3. *Let* $u(t)$ *be the solution of Eq. (9) and* $u_n$ *be the solution of the same equation with initial conditions* $u_n(t_0) = u_{0n}$, $\dot{u}_n(t_0) = v_{0n}$. *Assume that*

$$u_{0n} \rightharpoonup u_0 \quad \textit{weak in } V_1\,, \qquad v_{0n} \rightharpoonup v_0 \quad \textit{weak in } H\,,$$

*Then, for each* $t \in J$,

$$u_n(t) \rightharpoonup u(t) \quad \textit{weak in } V_1\,, \qquad \dot{u}_n(t) \rightharpoonup \dot{u}(t) \quad \textit{weak in } H\,.$$

## 4. Dissipativeness

Let us consider the evolution equation (2) and assume that all the hypotheses stated in Section 2 on the operator $A$ and the function $g$ hold. The dissipativeness of (2) will follow from a semi-coercivity condition upon $g$.

THEOREM 1. *Assume that there exists $\alpha, \beta, \gamma > 0$ such that*

$$(g(t,u),u) \geq \alpha|Pu| - \beta|(I-P)u| - \gamma\,, \tag{10}$$

*for all $(t,u) \in \mathbb{R} \times H$. Then Eq. (2) is dissipative. Moreover, there exists $\rho > 0$ such that if $u(t)$ is a solution of Eq. (2) and*

$$|u(t_0)|_1^2 + |Pu(t_0)|^2 + |\dot{u}(t_0)|^2 < \rho^2$$

*for some $t_0 \in \mathbb{R}$, then*

$$|u(t)|_1^2 + |Pu(t)|^2 + |\dot{u}(t)|^2 < \rho^2$$

*for all $t \geq t_0$.*

*Proof.* The expression

$$\|(u,v)\| := \left(c^2|u|^2 + 2c(u,v) + 2|v|^2 + 2|u|_1^2\right)^{1/2}\,,$$

defines a norm in $V_1 \times H$ equivalent to the usual one, and can be used in Definition 3. The function

$$\eta(t) := \|(u(t),\dot{u}(t))\|^2$$

is differentiable (see Lemma 2) and

$$\dot{\eta}(t) = -2c\left[|\dot{u}(t)|^2 + |u(t)|_1^2 + \frac{2}{c}\,(g(t,u(t)),\dot{u}(t)) + (g(t,u(t)),u(t))\right].$$

From the boundedness of $g$ and from inequality (6), we obtain that

$$\dot{\eta}(t) \leq -2c\left[|\dot{u}(t)|^2 + |u(t)|_1^2 - \frac{2\widetilde{M}}{c}\,|\dot{u}(t)| + \alpha|Pu(t)| - \beta R|u(t)|_1 - \gamma\right]. \tag{11}$$

Using the fact that

$$\lim_{|x|+|y|+|z|\to\infty}\left[x^2 + y^2 - \frac{2\widetilde{M}}{c}|x| + \alpha|z| - \beta R|y| - \gamma\right] = +\infty\,,$$

it follows that there exist $\rho, \delta > 0$ such that

$$\eta(t) \geq \rho^2 \quad \Longrightarrow \quad \dot{\eta}(t) < -\delta\,. \tag{12}$$

We deduce from (12) that there exists $\tau \geq t_0$ such that

$$\tau \leq t_0 + \max\left\{0, \delta^{-1}\left(\eta(t_0) - \rho^2\right)\right\},$$

and

$$\|(u(\tau),\dot{u}(\tau))\| < \rho\,.$$

Now, we assert that

$$\|(u(t),\dot{u}(t))\| < \rho\,,$$

for all $t > \tau$. Otherwise there must exist $t^* > \tau$ such that

$$\left\|\left(u(t^*),\dot{u}(t^*)\right)\right\|^2 = \rho^2$$

and

$$\|(u(t),\dot{u}(t))\|^2 < \rho^2\,,$$

for all $t \in [\tau, t^*)$. In consequence, $\dot{\eta}(t^*) \geq 0$, a contradiction with (12). $\square$

## 5. Bounded solutions

We use the results obtained in Section 4 to prove the existence of a solution of Eq. (2) that is bounded on the whole line.

THEOREM 2. *If Eq. (2) is dissipative, it has a solution $u(t)$ such that*

$$(u, \dot{u}) \in BC(\mathbb{R}, V_1 \times H)\,. \tag{13}$$

*Proof.* Let $u_n(t)$ be the solution of Eq. (2) with initial conditions

$$u_n(-n) = 0\,, \quad \dot{u}_n(-n) = 0\,.$$

By definition, there exists $T, \rho > 0$ such that

$$|u_n(t)|_1^2 + |Pu_n(t)|^2 + |\dot{u}_n(t)|^2 < \rho^2\,, \tag{14}$$

for all $t \geq T - n$. We can assume, without loss of generality, that there exists $u_0 \in V_1$ and $v_0 \in H$ such that

$$u_n(0) \rightharpoonup u_0 \quad \text{weak in } V_1, \qquad \dot{u}_n(0) \rightharpoonup v_0 \quad \text{weak in } H.$$

Let $u(t)$ be the solution of (2) with initial conditions

$$u(0) = u_0\,, \quad \dot{u}(0) = v_0\,.$$

Lemma 3 applies and we obtain for each $t \in \mathbb{R}$

$$u_n(t) \rightharpoonup u(t) \quad \text{weak in } V_1, \qquad \dot{u}_n(t) \rightharpoonup \dot{u}(t) \quad \text{weak in } H.$$

Moreover it follows from (14) that

$$|u(t)|_1^2 + |Pu(t)|^2 + |\dot{u}(t)|^2 \leq \rho^2\,,$$

for all $t \in \mathbb{R}$, and therefore (13) holds. □

We make a first use of Theorem 2 to prove the existence of a bounded solution of the linear equation (3) where $f \in BC(\mathbb{R}, H)$, a problem studied in [6, 14] when $\lambda_1 > 0$. In the resonant case $\lambda_1 = 0$, an additional hypothesis is required. We treat both cases in the following Theorem 3, whose proof requires a result of Ortega [12] on second order linear ordinary differential equations, that we include here for completeness.

LEMMA 4. *Let $p : \mathbb{R} \to \mathbb{R}$ be continuous and $c \neq 0$. Then the equation*

$$y''(t) + cy'(t) = p(t) \tag{15}$$

*has a bounded solution if and only if $p \in BP(\mathbb{R}, \mathbb{R})$.*

*Proof.* Necessity. Let $y$ be a bounded solution of Eq. (15) (i.e. $y$ and $y'$ are bounded on $\mathbb{R}$), and set

$$P(t) = \int_0^t p(s)\, ds\,. \tag{16}$$

Then

$$y'(t) - y'(0) + c[y(t) - y(0)] = P(t)\,,$$

and so $P$ is bounded.

Sufficiency. Let $p \in BP(\mathbb{R}, \mathbb{R})$ and consider the equation

$$u'(t) + cu(t) = P(t)\,, \tag{17}$$

where $P$ is defined in (16). By a classical result (see e.g. [4]), equation (17) has a unique bounded solution $U$. From the equation, we see immediately that $U'$ is also bounded. As $P \in C^1$, $U \in C^1$, and satisfies the differential equation (15). □

THEOREM 3. *If $\lambda_1 > 0$, all solutions of Eq. (3) are bounded in the future and Eq. (3) has a solution $u(t)$ which satisfies $(u, \dot{u}) \in BC(\mathbb{R}, V_1 \times H)$. If $\lambda_1 = 0$, the same statement is valid if and only if*

$$Pf \in BP(\mathbb{R}, \ker A)\,. \tag{18}$$

*Proof.* If $\lambda_1 > 0$, condition (10) with $P = 0$ holds for $g(t, u) = -f(t)$ taking $\alpha = 1$, $\beta = \sup_{t\in\mathbb{R}} |f(t)|$, $\gamma = 1$. Consequently, Theorems 1 and 2 apply.

If $\lambda_1 = 0$, and $m = \dim \ker A$, let $\widetilde{H} := \operatorname{span}\{\varphi_{m+1}, \varphi_{m+2}, \dots\}$ be the orthogonal complement of $\ker A$. The restriction $\widetilde{A}$ of the operator $A$ to $\widetilde{H} \cap D(A)$ is positive definite and we can apply the first assertion to deduce that the equation

$$\ddot{u} + c\dot{u} + \widetilde{A}u = (I - P)f(t)$$

has a bounded solution $\widetilde{u}(t)$ which satisfies

$$\left(\widetilde{u}, \dot{\widetilde{u}}\right) \in BC\left(\mathbb{R}, \widetilde{V}_1 \times \widetilde{H}\right)$$

where $\widetilde{V}_1 = V_1 \cap \widetilde{H}$ is endowed with the norm $|\cdot|_1$. On the other hand, by Lemma 4, the equation

$$\ddot{u} + c\dot{u} = Pf(t) \tag{19}$$

in the finite dimensional space $\ker A$ has a bounded solution, denoted by $u_0(t)$, if and only if $Pf \in BP(\mathbb{R}, \ker A)$. Now, the function $u(t) = u_0(t) + \tilde{u}(t)$ is a solution of Eq. (3) which satisfies (13). In addition, all the solutions of the autonomous equation

$$\ddot{u} + c\dot{u} + Au = 0$$

are bounded in the future and this implies that all the solutions of Eq. (3) are also bounded in the future. Conversely, if Eq. (3) has a bounded solution $u(t)$, then $Pu(t)$ is a bounded solution of Eq. (19). Because condition (18) is both necessary and sufficient for the existence of a bounded solutions of Eq. (19), we deduce that (18) holds. □

## 6. Nonlinear telegraph equation

We use the previous results to study the boundedness of the solutions of the nonlinear telegraph equation with Neumann boundary conditions

$$u_{tt} + cu_t - u_{xx} + h(u) = f(t, x)\,, \qquad (t \in \mathbb{R},\ x \in (0, \pi))\,, \tag{20}$$

$$u_x(t, 0) = u_x(t, \pi) = 0\,, \qquad (t \in \mathbb{R})\,, \tag{21}$$

where $c$ is a real positive constant, $h : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and $f : \mathbb{R} \times (0, \pi) \to \mathbb{R}$ is a function in the space $BC(\mathbb{R}, L^2(0, \pi))$. We also assume that $h$ is bounded and that the limits

$$h(-\infty) := \lim_{z \to -\infty} h(z), \qquad h(+\infty) := \lim_{z \to +\infty} h(z) \tag{22}$$

exist. The abstract framework in Section 2 applies in this case taking $H = L^2(0, \pi)$ and the operator $Au = -u_{xx}$ defined in

$$D(A) = \left\{ u \in H^2(0, \pi) : u_x(0) = u_x(\pi) = 0 \right\}.$$

The operator $A^{1/2}$ is given by $A^{1/2}u = u_x$ and its domain is the space $V_1 = H^1(0, \pi)$. Thus, a *solution* of Eq. (20)-(21) is a function $u(t, x)$ which satisfies

$$u \in C\left(\mathbb{R}, H^1(0, \pi)\right) \cap C^1\left(\mathbb{R}, L^2(0, \pi)\right),$$

such that, for each $w \in H^1(0, \pi)$, one has

$$\frac{d^2}{dt^2} \int_0^\pi u(t, x)w(x)\, dx + c \frac{d}{dt} \int_0^\pi u(t, x)w(x)\, dx + \int_0^\pi u_x(t, x)w_x(x)\, dx$$
$$+ \int_0^\pi h(u(t, x))w(x)\, dx = \int_0^\pi f(t, x)w(x)\, dx.$$

The space $\ker A$ is the space of constant functions on $(0, \pi)$, and the spectral projection from $L^2(0, \pi)$ onto $\ker A$ is given by the mean value

$$Pu = \frac{1}{\pi} \int_0^\pi u(x)\, dx, \qquad \left(u \in L^2(0, \pi)\right).$$

When the function $f(t, x)$ is $2\pi$-periodic in $t$ and $x$, it is proved in [10, 5] that Eq. (20) has at least one solution $u(t, x)$ $2\pi$-periodic in $t$ and $x$ provided the following condition of the *Landesman-Lazer type* is fulfilled

$$h(-\infty) < (2\pi)^{-2} \int_0^{2\pi} \int_0^{2\pi} f(t, x)\, dx dt < h(+\infty). \tag{23}$$

To find a condition similar to (23), that guarantees the existence of a bounded solution of problem (20)-(21) when $f(t, x)$ is bounded but not necessarily periodic, we introduce the *lower* and *upper mean values* of a given function $e \in BP(\mathbb{R}, \mathbb{R}) + BC(\mathbb{R}, \mathbb{R})$ as in [15],

$$\begin{aligned} A_L(e) &:= \lim_{r \to +\infty} \inf_{t-s \geq r} \frac{1}{t-s} \int_s^t e(\tau)\, d\tau, \\ A_U(e) &:= \lim_{r \to +\infty} \sup_{t-s \geq r} \frac{1}{t-s} \int_s^t e(\tau)\, d\tau, \end{aligned}$$

which both coincide with the mean value if the function $e(t)$ is periodic. Elementary considerations show that if $e = e^* + e^{**}$ is any decomposition of $e \in BP(\mathbb{R}, \mathbb{R}) + BC(\mathbb{R}, \mathbb{R})$ with $e^* \in BP(\mathbb{R}, \mathbb{R})$ and $e^{**} \in BC(\mathbb{R}, \mathbb{R})$, then one has

$$\inf e^{**} \leq A_L\left(e^{**}\right) = A_L(e) \leq A_U(e) = A_U\left(e^{**}\right) \leq \sup e^{**}. \tag{24}$$

The following result is due to Ortega and Tineo [13]. We include its proof for the reader's convenience.

LEMMA 5. *Let $e \in BP(\mathbb{R},\mathbb{R}) + BC(\mathbb{R},\mathbb{R})$ be a given function and $\alpha < \beta$ real numbers. Then the following statements are equivalent:*

*(i)* $\alpha < A_L(e) \le A_U(e) < \beta$.

*(ii) There exists a decomposition $e = e^* + e^{**}$ with $e^* \in BP(\mathbb{R},\mathbb{R})$, $e^{**} \in BC(\mathbb{R},\mathbb{R})$ and*

$$\alpha < \inf e^{**} \le \sup e^{**} < \beta \,. \tag{25}$$

*Proof.* If (ii) holds, then, using (24), we immediately obtain (i). Conversely, assume that (i) holds, write $e = e_1 + e_2$, with $e_1 \in BP(\mathbb{R},\mathbb{R})$ and $e_2 \in BC(\mathbb{R},\mathbb{R})$, and let

$$E_i(t) = \int_0^t e_i(u)\,du\,, \quad (i = 1, 2)\,, \quad E(t) = E_1(t) + E_2(t)\,.$$

If $t_1, t_2 \in \mathbb{R}$, the Lagrange mean value theorem applied to $E - E_1$ implies that

$$|E(t_1) - E(t_2)| \le b + a|t_1 - t_2|\,, \tag{26}$$

where $b = 2|E_1|_{L^\infty}$ and $a = |E_2|_{L^\infty}$. Let $\epsilon > 0$ be such that

$$\alpha < A_L(e) - 2\epsilon < A_U(e) + 2\epsilon < \beta\,,$$

and $T > 0$ such that

$$\inf_{t-s\ge r} \frac{1}{t-s}\int_s^t e(u)\,du > A_L(e) - \epsilon > \alpha + \epsilon\,,$$
$$\sup_{t-s\ge r} \frac{1}{t-s}\int_s^t e(u)\,du < A_U(e) + \epsilon < \beta - \epsilon\,,$$

whenever $r \ge T$. Hence we have, for all $t \in \mathbb{R}$,

$$\alpha + \epsilon < \frac{1}{T}\int_t^{t+T} e(u)\,du < \beta - \epsilon\,. \tag{27}$$

Defining

$$e^{**}(t) = \frac{1}{T}\int_t^{t+T} e(u)\,du\,, \qquad e^*(t) = e(t) - e^{**}(t)\,,$$

we see that $e^{**} \in BC(\mathbb{R},\mathbb{R})$ and (25) holds. To prove that $e^* \in BP(\mathbb{R},\mathbb{R})$, let

$$E^*(t) = E(t) - \frac{1}{T}\int_t^{t+T} E(u)\,du\,.$$

Then,

$$\begin{aligned}\left(E^*\right)'(t) &= e(t) - \frac{1}{T}[E(t+T) - E(t)]\\ &= e(t) - \frac{1}{T}\int_t^{t+T} e(u)\,du = e(t) - e^{**}(t) = e^*(t)\,,\end{aligned}$$

so that $E^*$ is a primitive of $e^*$. Now

$$E^*(t) = E(t) - E(\tau)$$

for some $\tau \in ]t, t+T[$, and hence, using (26), we get

$$|E^*(t)| \le b + a|t - \tau| \le b + aT\,,$$

for all $t \in \mathbb{R}$, which shows that $e^* \in BP(\mathbb{R},\mathbb{R})$. ◻

THEOREM 4. *If*

$$h(-\infty) < A_L\left(\frac{1}{\pi}\int_0^\pi f(t,x)\,dx\right) \le A_U\left(\frac{1}{\pi}\int_0^\pi f(t,x)\,dx\right) < h(+\infty)\,, \tag{28}$$

*then Eq. (20)-(21) is dissipative and has a solution $u(t,x)$ such that*

$$\sup_{t\in\mathbb{R}}\int_0^\pi \left[u(t,x)^2 + u_x(t,x)^2 + u_t(t,x)^2\right] dx < +\infty\,. \tag{29}$$

*Proof.* Since $\ker A$ is a one dimensional space, we can use Lemma 5 to deduce that $Pf$ admits a decomposition of the form $Pf = f^* + f^{**}$ with $f^*, f^{**} \in BC(\mathbb{R}, \ker A)$ such that

$$f^* \in BP(\mathbb{R}, \ker A)$$

and

$$h(-\infty) < \inf_{t\in\mathbb{R}} f^{**}(t) \le \sup_{t\in\mathbb{R}} f^{**}(t) < h(+\infty) \tag{30}$$

for all $t \in \mathbb{R}$. Thus we can write

$$f = f^* + f^{**} + (I - P)f\,,$$

and by the second assertion in Theorem 3, there exists a bounded solution $\varphi(t)$ of

$$\ddot{u} + c\dot{u} + Au = f^*(t) + (I - P)f(t)\,.$$

Now, the change of variables $u = z + \varphi(t)$ reduces the abstract form of problem (20)-(21) to the equation

$$\ddot{z} + c\dot{z} + Az + h(z + \varphi(t)) = f^{**}(t)\,. \tag{31}$$

On the other hand, it follows from (22) and (30) that there exists two positive real constants $a$ and $b$ such that

$$z\left(h(z) - f^{**}(t)\right) \ge a|z| - b$$

for all $z \in \mathbb{R}$. From this, it is not too difficult to show that we have, for all $u \in L^2(0,\pi)$,

$$\left(u, h(u) - f^{**}(t)\right)_{L^2(0,\pi)} \ge a\sqrt{\pi}|Pu|_{L^2(0,\pi)} - a\sqrt{\pi}|(I - P)u|_{L^2(0,\pi)} - \pi b\,.$$

Since we are assuming that $\varphi$, $h$ and $f^{**}$ are bounded, it follows that condition (10) holds. We deduce, by Theorem 1, that Eq. (31) is dissipative and, by Theorem 2, that Eq. (31) has a solution $z(t)$ which is bounded in the whole line. Now it is clear that $u(t) = \varphi(t) + z(t)$ is a solution of Eq. (20) that is bounded in the whole line. □

REMARK 1. The condition (28) is also *necessary* for the existence of a bounded solution when $h$ is such that

$$h(-\infty) < h(z) < h(+\infty)$$

for all $z \in \mathbb{R}$, and hence is a characterization of the dissipativeness of Eq. (20)-(21) for this class of nonlinearities.

EXAMPLE 1. The equation

$$u_{tt} + cu_t - u_{xx} + \arctan u = \left(\mu \arctan t + \sin t^2\right)(1 + 7\cos 7x)$$

with Neumann boundary condition (21) is dissipative and possesses a bounded solution if and only if

$$|\mu| < 1\,. \tag{32}$$

To prove this fact, note that a primitive of $\sin t^2$ is a Fresnel type function and it remains bounded on the whole line; in consequence, the upper and lower mean values of $\sin t^2$ are both 0. On the other hand, the lower and upper mean values of $\arctan t$ are $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ respectively. Thus, the condition (28) becomes condition (32) and Theorem 4 and Remark 1 apply.

The monotone character of the function $h$ is by no means necessary for the sufficiency condition, as shown by the following example, whose nonlinear term has, besides 0, $m$ negative and $m$ positive zeros, but is chosen to have the same limits at $\pm\infty$ as the nonlinear term in Example 1.

EXAMPLE 2. The equation

$$u_{tt} + cu_t - u_{xx} + \frac{\pi}{2}\sin[(4m+1)\arctan u] = \left(\mu \arctan t + \sin t^2\right)(1 + 7\cos 7x)$$

with Neumann boundary conditions (21) and $m \geq 0$ an integer, is dissipative and possesses a bounded solution if condition (32) holds.

REMARK 2. Similar results can be obtained for the telegraph equation (20) with the *periodic boundary conditions* in $x$ on $[0, 2\pi]$

$$u(t,0) = u(t,2\pi)\,, \qquad u_x(t,0) = u_x(t,2\pi)\,, \qquad (t \in \mathbb{R})\,, \tag{33}$$

or for the *damped beam equation*

$$u_{tt} + cu_t + u_{xxxx} + h(x) = f(t,x)\,, \tag{34}$$

with the *periodic boundary conditions* in $x$ on $[0, 2\pi]$

$$\begin{aligned} u(t,0) &= u(t,2\pi)\,, & u_x(t,0) &= u_x(t,2\pi) \\ u_{xx}(t,0) &= u_{xx}(t,2\pi)\,, & u_{xxx}(t,0) &= u_{xxx}(t,2\pi)\,, \quad (t \in \mathbb{R})\,. \end{aligned}$$

REMARK 3. The assumptions of Theorem 4 require that $h(-\infty) < h(+\infty)$ and one can raise the question of obtaining existence theorems in situations where $h(-\infty) = h(+\infty)$ (for example $h(u) = \frac{u}{1+u^2}$). The recent paper [11] proposes in particular a *method of weak upper and lower solutions* for the bounded solutions $u \in L^\infty(\mathbb{R} \times \mathbb{T})$, of equation

$$u_{tt} + cu_t - u_{xx} = F(t,x,u)\,,$$

satisfying periodic boundary conditions in $x$ on $[0, 2\pi]$, when

$$\frac{F(t,x,u) - F(t,x,v)}{u - v} \geq -\frac{c^2}{4} \quad \text{whenever} \quad u \geq v\,.$$

This approach allows to prove the following existence result:
*If h is Lipschitzian with constant*

$$L \leq \frac{c^2}{4} \tag{35}$$

*and if there exists $R > 0$ such that*

$$h(u)u \geq 0 \quad \textit{whenever} \quad |u| \geq R\,, \tag{36}$$

*then the problem (20)-(33) has at least one solution $u \in L^\infty(\mathbb{R}\times\mathbb{T})$ for each $f \in BC(\mathbb{R}\times\mathbb{T},\mathbb{R})$ such that*

$$\frac{1}{2\pi}\int_0^{2\pi} f(t,x)\,dx \in BP(\mathbb{R},\mathbb{R})\,.$$

The proof consists in making the change of variable $u = \varphi + v$, where $\varphi$ is the unique bounded solution of the problem

$$u_{tt} + cu_t - u_{xx} = f(t,x)$$

which is $2\pi$-periodic in $x$, (it exists by the second assertion in Theorem 3 and belongs to $L^\infty(\mathbb{R},\mathbb{T})$ by Sobolev's imbedding theorem), and showing that, because of condition (36), the equivalent equation

$$v_{tt} + cv_t - v_{xx} + h(\varphi(t,x) + v) = 0$$

admits, if $R^* > 0$ is sufficiently large, $-R^*$ as a lower solution and $R^*$ as an upper solution. It is an open problem to know if condition (35) can be dropped.

## References

[1] ALONSO J.M., MAWHIN J. AND ORTEGA R., *Bounded solutions of second order semilinear evolution equations and applications to the telegraph equation*, J. Math. Pures Appl. **78** (1999), 49–63.

[2] ALONSO J.M. AND ORTEGA R., *Global asymptotic stability of a forced Newtonian system with dissipation*, J. Math. Anal. Appl. **196** (1995), 965–986.

[3] BABIN A.V. AND VISHIK M.I., *Attractors of evolution equations*, North-Holland, Amsterdam 1992.

[4] COPPEL W.A., *Stability and asymptotic behavior of differential equations*, Heath, Boston 1967.

[5] FUČIK S. AND MAWHIN J., *Generalized periodic solutions of nonlinear telegraph equations*, Nonlinear Anal. **2** (1978), 609–617.

[6] GHIDAGLIA J.M. AND TEMAM R., *Attractors for damped nonlinear hyperbolic equations*, J. Math. Pures Appl. **66** (1987), 273–319.

[7] HALE J.K., *Asymptotic behavior of dissipative system*, Mathematical Surveys and Monographs, Providence 1988.

[8] HARAUX A., *Systèmes dynamiques dissipatifs et applications*, Masson, Paris 1991.

[9] LADYZHENSKAYA O., *Attractors for semigroups and evolution equations*, Lezioni Lincee, Cambridge University Press, Cambridge 1991.

[10] MAWHIN J., *Periodic solutions of nonlinear telegraph equations*, in: "Dynamical systems" (Eds. Bednarek and Cesari), Academic Press, New York 1977, 193–210.

[11] MAWHIN J., ORTEGA R. AND ROBLES-PÉREZ A., *A maximum principle for bounded solutions of the telegraph equations and applications to nonlinear forcings*, J. Math. Anal. Appl. **251** (2000), 695–709.

[12] ORTEGA R., *A boundedness result of Landesman-Lazer type*, Differential and Integral Equations **8** (1995), 729–734.

[13] ORTEGA R. AND TINEO A., *Resonance and non-resonance in a problem of boundedness*, Proc. Amer. Math. Soc. **124** (1996), 2089–2096.

[14] TEMAM R., *Infinite dimensional dynamical systems in mechanics and physics*, Springer-Verlag, New York 1988.

[15] TINEO A., *An iterative scheme for periodic solutions of ordinary differential equations*, J. Differential Equations **116** (1995), 1–15.

[16] VISHIK M.I., *Asymptotic behavior of solutions of evolutionary equations*, Lezioni Lincee, Cambridge University Press, Cambridge 1992.

**AMS Subject Classification: 35B35, 34G20.**

Jean MAWHIN
Université Catholique de Louvain
U.C.L., Département de mathématique
Chemin du cyclotron, 2
B-1348 Louvain-la-Neuve, Belgique
e-mail: mawhin@amm.ucl.ac.be