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GEOMETRY OF THE BOUNDARY AND DOUBLING PROPERTY OF THE HARMONIC MEASURE FOR GRUSHIN TYPE OPERATORS

Abstract. In this note we prove a doubling formula near the boundary for the harmonic measure associated with a class of degenerate elliptic equations known in the literature as Grushin type operators.

1. Introduction

The aim of this note is to apply new boundary regularity results for the harmonic measure associated with subelliptic operators recently obtained in [7] to the study of a class of Hörmander's sum-of-squares operators known sometimes in the literature as Grushin type operators, since their hypoellipticity was first proved by Grushin in [11]. Let us start by summarizing the main results of [7]; to this end, we introduce now the main notations. A more complete introduction to the subject, together with a richer list of references can be found in [7].

Let $X = \{X_1, \dots, X_p\}$ be a family of smooth vector fields in $\mathbb{R}^n$, $n \geq 3$, satisfying Hörmander's rank condition

$$\text{rank Lie}\,(X_1, \dots, X_p) = n\,, \tag{1}$$

where Lie $(X_1, \dots, X_p)$ is the Lie algebra generated by $X_1, \dots, X_p$. In other words, we assume that there exist n linearly independent commutators of $X_1, \dots, X_p$ of order less or equal to r for some $r \in \mathbb{N}$, and, to fix our notations, from now on we shall indicate by m the minimum natural number r that enjoys uniformly this property in a fixed compact set. We shall denote by $\mathcal{L}$ the differential operator

$$\mathcal{L} = \sum_{j=1}^{p} X_j^* X j\,,$$

and we shall say that a function u is $\mathcal{L}$-harmonic if $\mathcal{L}u = 0$.

It is well known that we can associate with X the so-called Carnot–Carathéodory distance (CC–distance), see e.g. [15], where in particular the authors prove that metric balls enjoy the doubling property with respect to the Lebesgue measure, i.e., denoting by $B_{x,s}$ the ρ-metric ball of center x and radius s, if $K \subset \mathbb{R}^n$ is a compact set, then there exist positive constants $A = A(K)$ and $s_0 = s_0(K)$ such that

$$|B_{x,2s}| \leq A|B_{x,s}|\,, \tag{2}$$

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for every $x \in K$ and for every $0 < s < s_0$, where $|E|$ denotes the Lebesgue measure of the set E. It follows that

$$|B_{x,\delta s}| \geq \delta^Q |B_{x,s}| \tag{3}$$

for $x \in K$ and $\delta < 1$, where $Q = \log_2 A$ is called the *local homogeneous dimension* of $\mathbb{R}^n$ endowed with the CC–distance.

Throughout this paper, Ω will be an open bounded subset of $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, and we shall write $y = (y', y_n) \in \mathbb{R}^n$, where $y' \in \mathbb{R}^{n-1}$, $y_n \in \mathbb{R}$. We assume that the boundary of Ω is locally the graph of a function $f \in C^{1,\alpha}(\mathbb{R}^{n-1})$, $\alpha \in]0, 1]$, such that $f(0) = 0$ and $\nabla f(0) = 0$. In particular, since the following assumption is not restrictive for our goal, we suppose that

$$\Omega = \left\{(y', y_n) \in \mathbb{R}^n : f(y') < y_n < f(y') + M_1,\ |y'| < M_2\right\}$$

for some positive numbers M_1 and M_2.

If m is defined above, let b and r be positive numbers; we set

$$F_b^m = \left\{(y', y_n) \in \mathbb{R}^n : f(y') < y_n < f(y') + b^m\right\},$$

while for every $x = (x', x_n) \in \mathbb{R}^n$ we put

$$\begin{aligned} \triangle^m(x, b, r) &= \left\{(y', y_n) \in \mathbb{R}^n : f(y') < y_n < f(y') + b^m,\ |y' - x'| < r\right\}, \\ \partial^s \triangle(x, b, r) &= \left\{(y', y_n) \in \mathbb{R}^n : f(y') \leq y_n \leq f(y') + b^m,\ |y' - x'| = r\right\}, \\ \partial^u \triangle(x, b, r) &= \left\{(y', y_n + b^m) \in \mathbb{R}^n : |y' - x'| \leq r\right\} \end{aligned} \tag{4}$$

and $\delta(x) = x_n - f(x')$.

Moreover, for every $\sigma \in F_b^m$ we set $A_{\sigma,b}^m = F_b^m \cap B_{\sigma,c_1 b}$, where $c > 1$ is a constant depending only on the operator $\mathcal{L}$.

Finally, throughout this paper, the symbols C, c, C', c' will indicate a constant that can change from formula to formula and even in the same string, whereas c_0, C_0, c_0', C_0', $a, \cdots$ will denote fixed constants.

It follows from Bony's classical results ([2], [3]) that we can associate with $\mathcal{L}$, with a bounded connected open set $U \subset \mathbb{R}^n$, and with a point $x \in U$ a probability measure on ∂U (the $\mathcal{L}$-harmonic measure, or, shortly, the harmonic measure if there is no way to misunderstandings) denoted by ω_U^x such that for any $\phi \in C(\partial U)$ we have

$$H_\phi^U(x) = \int_{\partial U} \phi(\sigma)\, d\omega_U^x(\sigma),$$

where H_U^ϕ is the Perron–Wiener–Brelot solution (PWB-solution) of the Dirichlet problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } U, \\ u \equiv \phi & \text{on } \partial U \end{cases}$$

(see the Appendix of [7] for more details). The following property of the $\mathcal{L}$-harmonic measure will be used through the paper.

THEOREM 1. *If $V \subseteq \partial U$ is a Borel set, then $\omega_U^x(V)$ is the PWB-solution evaluated at the point x of the Dirichlet Problem*

$$\begin{cases} \mathcal{L}u = 0 & \text{in } U, \\ u \equiv 1 & \text{on } \partial V, \\ u \equiv 0 & \text{on } \partial U \setminus V. \end{cases}$$

When dealing with the Laplace operator, the regularity of the harmonic measure has been intensively studied in the last years (see for instance [13], [5] and [14] for different crucial steps in the theory). Recently, in [6] the first author proved regularity results for Hölder domains adapting a probabilistic approach ([1]). Basically, our approach relies on the analogy between cusps for elliptic operators and characteristic points for subelliptic operators.

We can state now our first result, i.e. the upper estimate of

$$\omega^y_{\triangle^m(x,b,r)}\left(\partial^s \triangle^m (x,b,r)\right).$$

PROPOSITION 1. *There exists a positive real constant $c\in]0,1[$ such that for every $x\in\Omega$, for every positive real numbers b and r, $r\geq c_2 b$, and for every $y\in\triangle^m(x,b,r)$, if $y'=x'$, then*

$$\omega^y_{\triangle^m(x,b,r)}\left(\partial^s \triangle^m (x,b,r)\right)\leq c^{[r/b]}.$$

Let us state now a lower bound for the harmonic measure proved in [7] that, together with Proposition 1, will yield the doubling estimate for the harmonic measure itself. To this end, let us introduce the definition of ϕ-Harnack's domains. Let us start with some preliminary definitions.

DEFINITION 1. *We shall say that a quasi-metric $\tilde\rho$ is compatible with $\mathcal{L}$ if $\tilde\rho$ is equivalent to the Carnot–Carathéodory metric ρ, i.e. there exist constants c, C such that*

$$c\rho(x,y)\leq\tilde\rho(x,y)\leq C\rho(x,y) \tag{5}$$

for all $x,y\in\mathbb{R}^n$.

DEFINITION 2. *Let $\tilde\rho$ be a compatible quasi-metric; a $\tilde\rho$-Harnack's chain of balls of length ν connecting $x_1,x_2\in\Omega$ is a finite sequence of open $\tilde\rho$ balls contained in Ω, with centers $y_1=x_1$, $y_2,\dots,y_\nu=x_2$ and radii $r_1,\dots,r_\nu$, $r_j\leq dist_{\tilde\rho}(y_j,\partial\Omega)$, $j=1,\dots,\nu$, such that*

$$\tilde\rho\left(y_j,y_{j+1}\right)\leq\theta\min\left\{r_j,r_{j+1}\right\}, \tag{6}$$

where θ is a given geometric constant, that will be chosen small, see the Remark below.

REMARK 1. The meaning of θ in Definition 2 must be better explained. To this end, we recall that it is well known that positive $\mathcal{L}$-harmonic functions on ρ-balls satisfy an invariant Harnack's inequality, and the same result can be extended to $\tilde\rho$-balls, where $\tilde\rho$ is compatible with $\mathcal{L}$. The above constant θ depends on such inequality, since it satisfies the following property: if u is $\mathcal{L}$-harmonic in $B_{x,r}$, then

$$\sup_{B_{x,\theta r}} u\leq C\inf_{B_{x,\theta r}} u,$$

where C is a positive constant independent of u, x and $r\leq r_0$.

If x and y can be connected by a chain of $\tilde\rho$-balls of length ν then, by the Harnack's principle

$$\frac{h(x)}{h(y)}>c^\nu$$

for every positive $\mathcal{L}$-harmonic function h in Ω, where $c>0$ is independent on x, y, and h.

REMARK 2. In fact, in the sequel we shall not always use Harnack's chains of balls; instead we shall use families of sets depending on a point (the center) and a positive real number (the radius) that are in a natural sense equivalent to metric balls (see [10], [8]). This will not change the result.

We want now to classify open sets Ω according the possibility of connecting a point $x \in \Omega$ close to the boundary with another point that is 'away enough from the boundary' by means of a $\tilde{\rho}$-Harnack's chain whose length is controlled in terms of a given function of $\delta(x_1)$, where $\delta(x) = x_n - f(x')$. We shall see that, even if the boundary of our domain Ω is a smooth manifold, nevertheless the estimate of the length of a Harnack's chain may be that of Hölder domains for uniformly elliptic operators.

Thus, we introduce now the definition of ϕ-Harnack domain.

DEFINITION 3. *Let $\tilde{\rho}$ be a compatible quasi-metric, and let ϕ :$]0,\infty[\to]0,\infty[$ be a non-increasing function. We shall say that Ω is a ϕ-Harnack's domain in a neighborhood U of the origin if there exist $c(U) > 0$, $c_2(U), c_3(U), c_4(U) > 1$, such that, for every $r, b > 0$ sufficiently small, $b \le c_4(U)r$, and for every $w \in U \cap \Omega$ with $\triangle^m(w, c_2(U)b, c_3(U)r) \subset U \cap \Omega$, if $x \in \triangle^m(w, b, r)$, then there exist $y \in \Omega$, $\delta(y) > (c_2(U)b)^m$, $|y' - w'| < c_3(U)r$, and a $\tilde{\rho}$-Harnack's chain $H = \left\{\widetilde{B}_j, 1 \le j \le \nu\right\}$ connecting x with y such that:*

i) $\widetilde{B}_j \subset \triangle^m(w, 1, c_3(U)r)$;

ii) $\nu \le \phi(\delta(x)) - \phi\left((c_2(U)b)^m\right)$;

iii) *if we set* $S = \bigcup_{j=1}^{\nu} \widetilde{B}_j$, *then*

$$\omega_S^y\left(\partial S \setminus F^m_{c_2(U)b}\right) \ge c(U)\,.$$

In fact, it would be more precise to speak of ϕ-Harnack's domain with respect to $\tilde{\rho}$ and to a given choice of cylinders $\triangle^m$, see [7], Remark 3.2.

We can state now the second crucial estimate, i.e. the estimate from below of

$$\omega^x_{\triangle^m(0,c_2(U)b,c_3(U)r)}\left(\partial^u \triangle^m(0, c_2(U)b, c_3(U)r)\right)$$

in ϕ-Harnack's domains.

PROPOSITION 2. *Let Ω be a ϕ-Harnack's domain. With the notations of Definition 3, if $b, r > 0$ are such that $\triangle^m(0, c_2(U)b, c_3(U)r) \subset U$, then for $x \in \triangle^m(0, b, r)$ we have*

$$\omega^x_{\triangle^m(0,c_2(U)b,c_3(U)r)}\left(\partial^u \triangle^m(0, c_2(U)b, c_3(U)r)\right) \ge C(U)c^{-(\phi(\delta(x))-\phi((2b)^m))}\,,$$

where $C(U) > 0$ is independent of u, b and r.

For every positive real number γ let us denote by $\partial^g \triangle^m(x, \gamma b, \gamma r)$ the following subset of the boundary of the cylinder $\triangle^m(x, \gamma b, \gamma r)$,

$$\partial^g \triangle^m(x, \gamma b, \gamma r) = \partial \triangle^m(x, \gamma b, \gamma r) \setminus \left(\partial\Omega \cup \partial^s \triangle^m(x, b, \gamma r)\right).$$

To avoid cumbersome notations, from now on we shall take $c_2(U) = c_3(U) = 2$; our doubling estimate for $\mathcal{L}$-harmonic functions reads as follows.

THEOREM 2. *Suppose Ω is a ϕ-Harnack domain and let $\{b_k\}_{k\in\mathbb{N}}$, $\{r_k\}_{k\in\mathbb{N}}$ be decreasing sequences of real positive numbers such that:*

i) $r_0 = 2r$, $r_k \to (1+\theta)r$, $\theta > 0$ *and* $b_0 = b$, $b_k \to 0$*;*

ii) $r_k - r_{k+1} \geq c_2 b_{k+1}$, *where* c_2 *is defined in Proposition 1;*

iii) $\sum_{j=0}^{\infty} \exp\left(\phi\left(b_{j+2}^m\right) - \phi\left((2b)^m\right) - \frac{\tilde{c}(r_j - r_{j+1})}{b_{j+1}}\right) < \infty$, *where* $\tilde{c} = |\log c|$, c *being defined in Proposition 1.*

If $b, r > 0$, *then there exists a positive constant* $c = c(r,b)$ *(independent of* u*) such that for every* $y \in \triangle^m(x,b,r)$, x *in a neighborhood* U *of the origin, we have*

$$\omega^y_{\triangle^m(x,2b,2r)}\left(\partial^s \triangle^m(x,b,2r)\right) \leq c(b,r)\omega^y_{\triangle^m(x,2b,2r)}\left(\partial^g \triangle^m(x,2b,2r)\right),$$

where

$$\begin{aligned} c(b,r) \quad \leq \quad & C(U)\Bigg\{\exp\left(\phi\left(b_1^m\right) - \phi\left((2b)^m\right)\right) \\ & + \sum_{k=0}^{\infty} \exp\left(\phi\left(b_{k+2}^m\right) - \phi\left((2b)^m\right) - \tilde{c}\left(r_k - r_{k+1}\right)/b_k\right)\Bigg\}. \end{aligned}$$

Hence the following doubling formula holds

$$(7)\quad \omega^y_{\triangle^m(x,2b,2r)}\left(\partial \triangle^m(x,2b,2r) \setminus \partial\Omega\right) \leq (1 + c(b,r))\,\omega^y_{\triangle^m(x,2b,2r)}\left(\partial^g \triangle^m(x,2b,2r)\right).$$

The role of the function ϕ and in particular the importance of the logarithmic case is illustrated by the following corollaries.

COROLLARY 1. *With the notations of Theorem 2, if* $\phi(st) \leq s^{-\alpha}\phi(t)$ *for* $t > 0$, $0 < s < 1$, $0 < \alpha < 1/m$, *then the sum in* iii) *converges by choosing* $b_k = 2^{-k/m}b$ *and* $r_k = 2r - \frac{r}{8}\sum_{j=0}^{k}(j+1)^{-2}$ *for* $k \geq 0$.

Proof. Indeed

$$\begin{aligned} &\phi\left(b_{k+2}^m\right) - \phi\left((2b)^m\right) - \tilde{c}\,\frac{r_{k+1} - r_k}{b_{k+1}} \\ &\leq \left(2^{(k+m+2)\alpha} - 1\right)\phi\left((2b)^m\right) - \tilde{c}\,\frac{r}{b}2^{(k+1)/m}\frac{1}{(k+1)^2} \\ &= -\frac{2^{(k+1)/m}}{k^2}\left(\tilde{c}\,\frac{r}{b} + \phi\left((2b)^m\right)o(1)\right) \end{aligned}$$

as $k \to \infty$, so that the series in iii) converges. □

COROLLARY 2. *With the notations of Theorem 2, if* $\phi(s) = c|\log s|$, *then the sum in iii) converges and in addition when* $r \approx b$ *the constant* $c(r,b)$ *can be bounded by an universal constant, yielding a scale–invariant doubling inequality.*

Proof. We have

$$\begin{aligned} \phi\left(b_{k+2}^m\right) - \phi\left((2b)^m\right) - \tilde{c}\,\frac{r_{k+1} - r_k}{b_{k+1}} \quad &\leq \quad (k+m+2)\log 2 - \tilde{c}'\,\frac{r}{b}2^{(k+1)/m}\frac{1}{(k+1)^2} \\ &\leq \quad -\frac{2^{(k+1)/m}}{k^2}\left(\tilde{c}'\,\frac{r}{b} + o(1)\right), \end{aligned}$$

as $k \to \infty$ and we are done. □

2. Geometry for Grushin operators

In this section, we shall examine in particular the notion of ϕ-Harnack's domain associated with the family $X = \{X_1, \cdots, X_n\}$ of vector fields in $\mathbb{R}^n_{(x,y)} = \mathbb{R}^{n-1}_x \times \mathbb{R}_y$

$$X_j = \frac{\partial}{\partial x_j}, \; j = 1, \dots, n-1, \; X_n = |x|^\ell \frac{\partial}{\partial y},$$

for $\ell \in \mathbb{N}$. Clearly, these vector fields are smooth vector fields satisfying Hörmander's condition only if ℓ is even; nevertheless, the geometry of the associated Carnot-Carathéodory distance is fully understood for $\ell \geq 1$ ([10], [8]). Before giving a precise description of the metric balls associated with $X_1, \dots, X_n$, we recall that the sub-Laplacian $\mathcal{L} = \sum_j X_j^2$ is invariant under vertical translation, so that, if u is $\mathcal{L}$-harmonic, then $u(x, y+k)$ is $\mathcal{L}$-harmonic too.

THEOREM 3 ([8], [10]). *If $x_0 \in \mathbb{R}^{n-1}$, $y_0 \in \mathbb{R}$, $z_0 = (x_0, y_0)$, $r > 0$, put*

i) $F(x_0, r) = r(|x_0| + r)^\ell$;

ii) $Q(z_0, r) = B_{\text{euc}}(x_0, r) \times (y_0 - F(x_0, r), y_0 + F(x_0, r))$,

where B_{euc} is the $(n-1)$-dimensional Euclidean ball of radius r centred at x_0. Then there exists $b > 1$ such that

$$Q(z_0, r/b) \subset B_{z_0,r} \subset Q(z_0, br)$$

when $0 < r < 1$. In addition

iii) $\rho((x_1, y_1), (x_2, y_2))$

$$\begin{aligned} &\approx \rho_1((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, \varphi(x_1, |y_1 - y_2|), \varphi(x_2, |y_1 - y_2|)\} \\ &\approx |x_1 - x_2| + \varphi(x_1, |y_1 - y_2|) \approx |x_1 - y_1| + \varphi(x_2, |y_1 - y_2|), \end{aligned}$$

where

$$\varphi(x, t) = F(x, \cdot)^{-1}(t)$$

for $t > 0$.

iv) Moreover, there exists $\beta \geq 1$ such that

$$F(x, \sigma r) \geq \sigma^\beta F(x, r)$$

for $\sigma \in]0, 1[$.

In the sequel, we shall call $Q(z, r)$ 'metric cubes'.

We state now an estimate from below of the $\mathcal{L}$-harmonic measure that we shall use in the sequel when proving $iii)$ of Definition 3 for special classes of open domains.

LEMMA 1. *For every $Q = Q(\bar{z}, r)$, $\bar{z} = (\bar{x}, \bar{y})$,*

$$\omega_Q^{\bar{z}}(\partial Q \cap \{y > \bar{y} - F(\bar{x}, r)\}) \geq \frac{1}{2}.$$

Proof. By definition, $z \to u(z) = \omega_Q^z(\partial Q \cap \{y > \bar{y} - F(\bar{x}, r)\})$ is the PWB-solution of the Dirichlet problem:

$$\begin{cases} \mathcal{L}u = 0 & \text{on } Q \\ u = \chi_{\{y > \bar{y} - F(\bar{x}, r)\}} & \text{in } \partial Q\,. \end{cases}$$

On the other hand, the function

$$l(y) = \frac{y + F(\bar{x}, r)}{2F(\bar{x}, r)}$$

is $\mathcal{L}$-harmonic and $0 \le l(y) \le \chi_{\{y > \bar{y} - F(\bar{x}, r)\}}$ on ∂Q. By the maximum principle, $u(z) \ge l(z)$ for $z \in Q$, and the assertion follows. □

DEFINITION 4. *If $\alpha > 0$ and $z_0 = (x_0, y_0) \in \mathbb{R}^n$, we put*

$$\Gamma_{z_0,\alpha} = \left\{(x, y) \in \mathbb{R}^n : \alpha F(x_0, |x - x_0|) < y - y_0\right\}.$$

We shall say that open set Ω is $X - Lipschitz$ if there exists $\alpha > 0$ such that (locally)

$$\Gamma_{z_0,\alpha} \subset \Omega$$

for all $z_0 \in \Omega$.

We notice that, if $\Omega = \{y > f(x)\}$, then the above condition is equivalent to

(8) $$f(x) - f(\xi) \le \alpha F(x, |x - \xi|) \qquad \textit{for } x, \xi \in \mathbb{R}^{n-1}\,.$$

LEMMA 2. *Let $\Omega = \{y > f(x)\} \subset \mathbb{R}^n$ be such that $f \in C^{\ell,1}\big(\mathbb{R}^{n-1}\big)$. If $\frac{\partial^\gamma f}{\partial x^\gamma} = O(|x|)$, as $x \to 0$ when $|\gamma| \le \ell$, then Ω is a X-Lipschitz domain in a neighborhood of the origin.*

Proof. From Taylor formula it follows that there exists $\xi = \xi(x)$, $|\xi - x_0| \le |x - x_0|$ such that

$$\begin{aligned} f(x) - f(x_0) &= \sum_{k=1}^{\ell} \frac{1}{k!} \sum_{|\gamma|=k} \frac{\partial^\gamma f}{\partial x^\gamma}(x_0)(x - x_0)^\gamma \\ &\quad + \frac{1}{\ell!} \sum_{|\gamma|=\ell} \left(\frac{\partial^\gamma f}{\partial x^\gamma}(\xi) - \frac{\partial^\gamma f}{\partial x^\gamma}(x_0)\right)(x - x_0)^\gamma \\ &\le |x - x_0| \left(c \sum_{k=1}^{\ell} \left|\frac{\partial^\gamma f}{\partial x^\gamma}(x_0)\right| |x - x_0|^{k-1} + c'|x - x_0|^\ell\right). \end{aligned}$$

Notice now $\left|\frac{\partial^\gamma f}{\partial x^\gamma}(x)\right| = \left(|x|^{l-|\gamma|+1}\right)$, by iteration. Hence

$$\begin{aligned} f(x) - f(x_0) &\le c''|x - x_0| \left(|x_0|^\ell + |x - x_0|^\ell\right) \\ &\le c'''|x - x_0|\,(|x_0| + |x - x_0|)^\ell = c''' F(x_0, |x - x_0|)\,, \end{aligned}$$

and we are done. □

LEMMA 3. *Let $\Omega = \{y > f(x)\} \subset \mathbb{R}^n$ be a X-Lipschitz domain. Then there exists a positive constant C such that, for every $z = (x, y)$, $w = (x, y + h) \in \Omega$, $h > 0$, then there exists a Harnack's chain in Ω of length ν connecting z and w such that*

(9) $$\nu \le C \left|\log \frac{h}{y - f(x)}\right|$$

Proof. By assumption, there exists $\alpha > 0$ such that $\Gamma_{\alpha,(x,f(x))} \subset \Omega$ and $z, w \in \Gamma_{\alpha,(x,f(x))}$. Thus, (9) will be proved by constructing a Harnack's chain contained in $\Gamma_{\alpha,(x,f(x))}$. To this end, we define a sequence of metric cubes $\{Q(z_j, r_j),\ j = 1, \dots, \nu\}$ such that $z_0 = z$, $z_\nu = w$, $z_j = (x, y_j)$, with $y_{j+1} = y_j + F(x, \sigma r_j)$, where σ is a positive (small) constant to be chosen in such a way (6) holds. The sequence $(r_j)_{j\in\mathbb{N}}$ is defined as follows:

$$r_j = \varphi\left(x, \frac{y_j - f(x)}{\alpha+1}\right), \quad j = 1, \dots, \nu\,. \tag{10}$$

Let us prove that $Q(z_j, r_j) \subset \Gamma_{\alpha,(x,f(x))}$; indeed, if $\zeta = (\xi, \eta) \in Q(z_j, r_j)$, then, by (10),

$$\begin{aligned}\eta - f(x) &= \eta - y_j + y_j - f(x) \geq -F(x, r_j) + y_j - f(x)\\ &= \alpha F(x, r_j) = \alpha r_j(|x| + r_j)^\ell\\ &> \alpha|\xi - x|(|x| + |\xi - x|)^\ell\,,\end{aligned}$$

as we held. Now the proof will be accomplished by proving a bound for ν; to this end, if we put $\delta_j = y_j - f(x)$, again by (10), we have

$$\begin{aligned}\delta_{j+1} &= \delta_j + F(x, \sigma r_j) \geq \delta_j + \sigma^\beta F(x, r_j)\\ &= \delta_j\left(1 + \frac{\sigma^\beta}{1+\alpha}\right).\end{aligned}$$

By iteration (9) follows. □

THEOREM 4. *X-Lipschitz domains are ϕ-Harnack's domains, with $\phi(t) = C|\log t|$.*

Proof. Let r, b be given $0 < b \leq r$. Take $w = (w', w'')$ in a neighborhood of the origin, and let $z = (x, y)$ be such that $|x - w'| < r$, $f(x) < y < f(x) + b^{\ell+1}$, and consider the sequence of metric cubes described in Lemma 3 with $h = (2b)^{\ell+1} - y + f(x)$. Let us prove that this chain satisfies condition $i)$ of Definition 3. If (ξ, η) belongs to some $Q_j = Q(z_j, r_j)$, $j \leq n$, see Lemma 3, then $|\xi - x| < r_\nu \varphi\left(x, \frac{y_\nu - f(x)}{\alpha+1}\right)$, so that

$$\frac{(2b)^{\ell+1}}{\alpha+1} \geq \frac{y_\nu - f(x)}{\alpha+1} > F(x, |\xi - x|) \geq |\xi - x|^{\ell+1}\,,$$

so that $|\xi - x| \leq c_1 b$, and hence

$$|\xi - w'| \leq (c_1 + 1)r\,,$$

so that $(\xi, \eta) \in \triangle^{\ell+1}(w, 1, (c_1+1)r)$. By (9),

$$\begin{aligned}\nu &\leq C\left|\log\frac{(2b)^{\ell+1} - \delta(x)}{\delta(x)}\right| = C\left(|\log\delta(x)| - \left|\log\left((2b)^{\ell+1} - \delta(x)\right)\right|\right)\\ &\leq C\left(|\log\delta(x)| - \left|\log(2b)^{\ell+1}\right|\right),\end{aligned}$$

that proves $ii)$.

Finally, we have to show that $iii)$ of Definition 3 holds. Recalling that our metric cubes Q_j have increasing radii and the centers are on a vertical line, if $S = \bigcup_{j=1}^{\nu} Q_j$, then

$$\partial Q_\nu \cap \{y > y_\nu - F(x, r_\nu)\} \subset \partial S\,.$$

Suppose first $\bar{Q}_\nu \subset \big(F_b^{\ell+1}\big)^c$. Then

$$\partial Q_\nu \cap \{y > y_\nu - F(x, r_\nu)\} \subset \partial S \setminus \big(F_b^{\ell+1}\big)^c.$$

By the maximum principle

$$\omega_S^{z_\nu}\big(\partial S \setminus F_b^{\ell+1}\big) \geq \omega_{Q_\nu}^{z_\nu}\,(\partial Q_\nu \cap \{y > y_\nu - F(x, r_\nu)\}) \geq \frac{1}{2}\,,$$

by Lemma 1 and we are done. If the metric cube Q_ν is not completely contained in $\big(F_b^{\ell+1}\big)^c$, it is enough to add N copies of Q_ν centered at the point $(x, y_{\nu+k})$, $y_{\nu+k} = y_\nu + kF(x, \sigma r_\nu)$, $\nu = 1, \cdots, N$ with $\frac{1}{\alpha+1}\big(N\sigma^\beta - \frac{1}{\alpha+1}\big) > 0$. In fact, if $(\xi, \eta) \in Q((x, y_{\nu+N}), r_\nu)$, keeping in mind the definition of r_ν and (8), we have

$$\begin{aligned}
\eta &\geq y_{\nu+N} - F(x, r_\nu) = y_\nu + NF(x, \sigma r_\nu) - F(x, r_\nu)\\
&\geq y_\nu + F(x, r_\nu)\big(N\sigma^\beta - 1\big)\\
&= f(x) + (2b)^{\ell+1} + \frac{(2b)^{\ell+1}}{\alpha+1}\big(N\sigma^\beta - 1\big)\\
&\geq f(\xi) - F(x, |x-\xi|) + (2b)^{\ell+1}\Big(N\sigma^\beta + \frac{\alpha}{\alpha+1}\Big)\\
&\geq f(\xi) - F(x, r_\nu) + (2b)^{\ell+1}\Big(N\sigma^\beta + \frac{\alpha}{\alpha+1}\Big)\\
&\geq f(\xi) + \frac{(2b)^{\ell+1}}{\alpha+1}\Big(N\sigma^\beta - \frac{1}{\alpha+1}\Big)\\
&> f(\xi) + (2b)^{\ell+1}\,,
\end{aligned}$$

so that $(\eta, \xi) \in \big(F_b^{\ell+1}\big)^c$. Clearly, to add N metric cubes does not invalidate our estimate of ν. ☐

COROLLARY 3. *Let $\Omega = \{y > f(x)\} \subset \mathbb{R}^n$ be such that $f \in C^{\ell,1}\big(\mathbb{R}^{n-1}\big)$. If $\frac{\partial^\gamma f}{\partial x^\gamma} = O(|x|)$, as $x \to 0$ when $|\gamma| \leq \ell$, then Ω is a ϕ-Harnack's domain, with $\phi(t) = C|\log t|$.*

REMARK 3. In fact, domains $\Omega = \{y > f(x)\} \subset \mathbb{R}^n$ such that $f \in C^{\ell,1}\big(\mathbb{R}^{n-1}\big)$ are not 'interesting' from our point of view, since it can be proved they are NTA-domains with respect to the Carnot–Carathéodory metric, and then the general results of [4] apply (in other words, they do not have cusps, when we look at them through the Carnot–Carathéodory geometry).

Thus, in the spirit of the present note, we shall not study domains $\Omega = \{y > f(x)\} \subset \mathbb{R}^n$ such that $f \in C^{\ell,1}\big(\mathbb{R}^{n-1}\big)$; let us show now that, on the contrary, if $f \in C^{\ell,\alpha}\big(\mathbb{R}^{n-1}\big)$, with $0 < \alpha < 1$, then Ω can be a ϕ-Harnack's domain for a suitable non–logarithmic function ϕ. More precisely, we have:

THEOREM 5. *Put $\Omega_\alpha = \big\{y > |x|^{\ell+\alpha}\big\}$, $0 < \alpha < 1$; then Ω is a ϕ-Harnack's domain, with*

$$\phi(t) \approx t^{1-\frac{\ell+1}{\ell+\alpha}}\,.$$

Moreover, the above estimate is sharp, in the sense that, if $\theta > 0$ is the constant introduced in the Definition 2 and Remark 1, then there exists c_θ such that the length of any Harnack's chain of parameter θ connecting $(x, y) \in \Omega$ with a point $(\bar{x}, \bar{y}) \in \Omega$ with $\bar{y} > b^{\ell+1} + |\bar{x}|^{\ell+\alpha}$ is bounded from below by $c_\theta \varphi(\delta(x, y))$.

The proof of the above Theorem will be accomplished through several steps. Let us start by proving that the constant θ of (6) can be chosen as small as we need, still taking under control the length of the chain.

LEMMA 4. *Let H be a Harnack's chain of length m and parameter θ in (6); then, for every $\theta' < \theta$ there exists a Harnack's chain of length $M(\theta')m$ and parameter θ', where $M(\theta')$ is a positive number depending only on θ'.*

Proof. Let $Q((x_j, y_j), r_j)$ and $Q((x_{j+1}, y_{j+1}), r_{j+1})$ be two consecutive metric cubes of the given Harnack's chain. We shall prove that for every $\theta' < \theta$ we can connect (x_j, y_j) and (x_{j+1}, y_{j+1}) with a Harnack's chain of length $M(\theta')$ and parameter θ'.

For the sake of simplicity and without loss of generality, we assume $|x_j| \leq |x_{j+1}|$. We put

$$\bar{r}_j = \min\{r_j - \theta \min\{r_j, r_{j+1}\}, r_{j+1} - \theta \min\{r_j, r_{j+1}\}\}.$$

Notice that $\bar{r}_j = (1-\theta)\min\{r_j, r_{j+1}\}$.

Since

$$\bar{r}_j < \min\{r_j, r_{j+1}\} \leq r_k,$$

for $k = j, j+1$, then

$$Q((x_k, y_k), \bar{r}_j) \subset Q((x_k, y_k), \min\{r_j, r_{j+1}\}) \subset Q((x_k, y_k), r_k),$$

for $k = j, j+1$. Let $q \in \mathbb{N}$, $p \leq q$, and consider the points

$$(u_p, y_{j+1}) = \left(\left(1 - \frac{p}{q}\right) x_{j+1} + \frac{p}{q} x_j, y_{j+1}\right).$$

In particular,

$$|u_{p+1} - u_p| = \frac{|x_j - x_{j+1}|}{q}.$$

We hold that $Q((u_p, y_{j+1}), \bar{r}_j) \subset Q((x_{j+1}, y_{j+1}), r_{j+1}) \subseteq \Omega$ for every integer p such that $p \leq q$. Indeed, let us show first that, if ξ is such that $|\xi - u_p| < \bar{r}_j$, then $|\xi - x_{j+1}| < r_{j+1}$. We have

$$|\xi - x_{j+1}| \leq \frac{p}{q}|x_{j+1} - x_j| + \bar{r}_j \leq \theta \min\{r_j, r_{j+1}\} + (1-\theta)\min\{r_j, r_{j+1}\},$$

and the statement is proved. Thus, we need only to prove that, for every $p \leq q$ and for every $\eta \in \mathbb{R}$ such that $\varphi(u_p, |\eta - y_{j+1}|) < \bar{r}_j$, then $\varphi(x_{j+1}, |\eta - y_{j+1}|) < r_{j+1}$; to this end, since $F(u_p, \cdot)$ is increasing, we obtain

$$|\eta - y_{j+1}| < F(u_p, \varphi(u_p, |\eta - y_{j+1}|))$$

thus, the assertion is proved, since $|u_p| < |x_{j+1}|$, and hence

$$|\eta - y_{j+1}| < F(x_{j+1}, \bar{r}_j) \leq F(x_{j+1}, r_{j+1}).$$

We can now give an upper bound of q for a given parameter θ'. Choose $q \in \mathbb{N}$ such that

$$q \geq \frac{|x_{j+1} - x_j|}{\bar{r}_j} \frac{1}{\theta'},$$

and

$$\frac{|x_{j+1}-x_j|}{\bar r_j}\frac{1}{\theta'} \le q \le 1+\frac{\theta}{1-\theta}\frac{1}{\theta'}$$

(keep in mind that $|x_{j+1}-x_j| < \theta \min\{r_{j+1}, r_j\}$). Then

$$|u_{p+1}-u_p| = \frac{|x_{j+1}-x_j|}{q} < \theta' \bar r_j$$

so that condition (6) is satisfied.

In this way, we connected the point (x_{j+1}, y_{j+1}) with (x_j, y_{j+1}); to complete the proof we have now to move vertically from (x_j, y_{j+1}), to (x_j, y_j). We can assume that

$$\begin{aligned}
|x_j - x_{j+1}| &< \theta \min\{r_j, r_{j+1}\}\,,\\
|y_j - y_{j+1}| &< F(x_j, \theta \min\{r_j, r_{j+1}\}) \le F(x_{j+1}, \theta \min\{r_j, r_{j+1}\})\,.
\end{aligned}$$

Let us prove that for any $\tilde y$ lying on the segment between y_j and y_{j+1} we have

$$Q((x_j, \tilde y), \bar r_j) \subseteq Q((x_j, y_j), r_j) \cap Q((x_{j+1}, y_{j+1}), r_{j+1})\,.$$

To this end, take $(\xi, \eta) \in Q((x_j, \tilde y), \bar r_j)$, i.e.

$$|\xi - x_j| < \bar r_j\,, \quad |\eta - \tilde y| < F(x_j, \bar r_j)\,.$$

Clearly, $|\xi - x_j| < r_j$; on the other hand

$$|\xi - x_{j+1}| \le \bar r_j + |x_j - x_{j+1}| < (1-\theta)\min\{r_j, r_{j+1}\} + \theta \min\{r_j, r_{j+1}\} \le r_j\,.$$

Now

$$\begin{aligned}
\max\{|\eta - y_j|, |\eta - y_{j+1}|\} &\le F(x_j, \bar r_j) + \max\{|y_j - \tilde y|, |\tilde y - y_{j+1}|\}\\
&\le F(x_j, \bar r_j) + |y_j - y_{j+1}|\\
&< F(x_j, (1-\theta)\min\{r_j, r_{j+1}\}) + F(x_j, \theta \min\{r_j, r_{j+1}\})\\
&\le (1-\theta)F(x_j, \min\{r_j, r_{j+1}\}) + \theta F(x_j, \min\{r_j, r_{j+1}\})\\
&\le F(x_j, \min\{r_j, r_{j+1}\}) = \min\{F(x_j, r_j), F(x_{j+1}, r_{j+1})\}\,.
\end{aligned}$$

Now we 'climb'. We suppose, without loss of generality, that $y_j < y_{j+1}$ and we define a new finite sequence of points as follows:

$$(x_j, w_k) = \left(x_j, \left(1-\frac{k}{m}\right) y_{j+1} + \frac{k}{m} y_j\right).$$

Arguing as above, choose now $m \in \mathbb{N}$ such that

$$m \ge \frac{|y_{j+1}-y_j|}{F(x_j, \theta' \bar r_j)}\,,$$

and

$$\begin{aligned}
m &\le 1 + \frac{|y_{j+1}-y_j|}{F(x_j, \theta' \bar r_j)}\\
&\le 1 + \frac{F(x_j, \theta \min\{r_j, r_{j+1}\})}{F(x_j, (1-\theta)\theta' \min\{r_j, r_{j+1}\})}\\
&\le 1 + \left(\frac{\theta}{(1-\theta)\theta'}\right)^{\beta},
\end{aligned}$$

by Theorem 3.

As we proved above, the metric cubes $Q((x_j, w_k), \bar{r}_j)$ are contained in

$$Q((x_j, y), r_j) \cap Q((x_{j+1}, y_{j+1}), r_{j+1}) \subseteq \Omega\,,$$

and

$$|w_k - w_{k+1}| = \frac{1}{m}|y_{k+1} - y_k| \le F(x_j, \theta' \bar{r}_j)\,,$$

and then (6) holds with parameter θ'. □

LEMMA 5. *Let Ω_α be defined as in Theorem 5, $\alpha \in (0,1)$. Then for any $\theta \le 1$ there exists a positive constant C_θ such that for every $b, r > 0$, $r, b \le 1$ and for every point $(x, y) \in \triangle^{\ell+1}(0, b/2, r)$, there exists a Harnack's chain $H = \{Q_j = Q_j((x_j, y_j), r_j)\}$, $j = 0, \cdots, \nu$, connecting (x, y) with a point $(\bar{x}, \bar{y})$, $\delta(\bar{x}, \bar{y}) > b^{\ell+1}$, such that $Q_j \subset \triangle^{\ell+1}(0, b, 2r)$, for every $j = 0, \cdots, \nu$, and*

$$\nu \le C_\theta \left(\delta(x, y)^{1 - \frac{\ell+1}{\ell+\alpha}} - b^{(\ell+1)\left(1 - \frac{\ell+1}{\ell+\alpha}\right)} \right).$$

Moreover, without loss of generality, we can assume that the last metric cube of H is contained in $\left(F_{b^{\ell+1}}\right)^c$.

Proof. By Lemma 4 we have only to consider the case $\theta = 1$. We can split the chain we shall build below into two parts. First, if we are very close to the boundary, we can suppose that the largest metric cube contained in Ω and centered at the starting point (x, y) has radius smaller than $|x|/2$. In this case, we move on the segment $[(x, y), (0, y)]$ taking, roughly speaking, metric cubes as large as possible till the radii of metric cubes are larger than $|x|/2$. Then we move vertically, till the center of the last metric cube is above the level $b^{\ell+1}$. If the radius of the first cube is grater than $|x|/2$ we just move vertically as just described.

We recall that for every $(x, y) \in \Omega$ we put, as usual in this note, $\delta = \delta(x, y) = y - |x|^{\ell+\alpha}$. Notice that for every $(x, y) \in \Omega$ the radius r of the largest metric cube contained in Ω satisfies the following equation

$$\delta + |x|^{\ell+\alpha} = r(|x| + r)^\ell + (|x| + r)^{\ell+\alpha}\,. \tag{11}$$

Indeed for every $(u, v) \in Q((x, y), r)$,

$$|u - x| < r\,, \quad \text{and} \quad |v - y| < F(x, r)\,;$$

on the other hand

$$|u|^{\ell+\alpha} \le (|u - x| + |x|)^{\ell+\alpha} \le (r + |x|)^{\ell+\alpha}\,,$$

and recalling (11)

$$\begin{aligned} |u|^{\ell+\alpha} &\le (r + |x|)^{\ell+\alpha} = \delta + |x|^{\ell+\alpha} - r(|x| + r)^\ell = y - F(x, r) \\ &\le v + |v - y| - F(x, r) < v\,, \end{aligned}$$

i.e. $(u, v) \in \Omega$. For such kind of maximal metric cube we give a lower and an upper bound of the radius r depending on $|x|$. From (11) and Lagrange's Theorem, there exists $\eta = \eta(x, r) \in (0, r)$ such that

$$\delta - (\ell + \alpha) r (|x| + \eta)^{\ell+\alpha-1} - r(|x| + r)^\ell = 0\,.$$

In particular we get

$$r = \frac{\delta}{(\ell+\alpha)(|x|+\eta)^{\ell+\alpha-1} + (|x|+r)^{\ell}} \,. \tag{12}$$

Hence, if $|x|/2 > r$, then

$$\left(\frac{2}{3}\right)^{\ell+\alpha-1} \frac{1}{\ell+\alpha+1}\, \frac{\delta}{|x|^{\ell+\alpha-1}} \le r \le \frac{1}{(\ell+\alpha)}\, \frac{\delta}{|x|^{\ell+\alpha-1}} \,, \tag{13}$$

since $\alpha < 1$ and $\eta \in (0, r)$.

Analogously, if $|x|/2 \le r$, from (12) it follows that

$$\left(\frac{1}{3}\right)^{\ell+\alpha-1} \frac{1}{\ell+\alpha+1}\, \frac{\delta}{r^{\ell+\alpha-1}} \le r \,, \tag{14}$$

and from (14) we get

$$\left(\frac{1}{3}\right)^{\frac{\ell+\alpha-1}{\ell+\alpha}} \frac{1}{(\ell+\alpha+1)^{1/(\ell+\alpha)}} \cdot \delta^{1/(\ell+\alpha)} < r < \delta^{1/(\ell+\alpha)} \,, \tag{15}$$

notice that the second inequality follows directly from (11) for any x since

$$\delta + |x|^{\ell+\alpha} \ge (|x|+r)^{\ell+\alpha} \ge |x|^{\ell+\alpha} + r^{\ell+\alpha} \,.$$

For sake of simplicity we divide the proof by stressing the main steps.

First step. Assume that $|x|/2 = |x_0|/2 < r_0$; otherwise we skip all this part of the construction. Let $Q((x_k, y_k), r_k)$, $0 \le k \le k_0$ be the sequence defined as follows: $(x_0, y_0) = (x, y)$, $(x_k, y_k) \in [(x, y), (0, y)]$, $y_k = y$, and

$$|x_{k+1}| = |x_k| - r_k \,, \tag{16}$$

where r_k is the solution in r of the equation

$$\delta_k + |x_k|^{\ell+\alpha} = r(|x_k| + r)^{\ell} + (|x_k| + r)^{\ell+\alpha} \,, \tag{17}$$

$\delta_k = \delta(x_k, y)$ and k_0 is the first integer such that $r_k > |x_k|/2$. Clearly, if for some k we get $r_k > |x_k|$ so that (16) is meaningless, then we stop at the step $k+1$ by taking $x_{k+1} = 0$.

Notice that, as we have proved before, the metric cubes just defined are contained in Ω.

The sequence $(|x_k|)_{0 \le k \le k_0}$ is by definition (16) strictly decreasing, while $(r_k)_{0 \le k \le k_0}$ is strictly increasing; indeed,

$$\begin{aligned} \delta_{k+1} + |x_{k+1}|^{\ell+\alpha} &= y = \delta_k + |x_k|^{\ell+\alpha} \\ &= r_k(|x_k| + r_k)^{\ell} + (|x_k| + r_k)^{\ell+\alpha} \\ &= r_k(|x_{k+1}| + 2r_k)^{\ell} + (|x_{k+1}| + 2r_k)^{\ell+\alpha} \,, \end{aligned}$$

by (16). Now, arguing by contradiction, if $r_k \ge r_{k+1}$, then $2r_k > r_{k+1}$, and by previous equalities, we get

$$\delta_{k+1} + |x_{k+1}|^{\ell+\alpha} > r_{k+1}(|x_{k+1}| + r_{k+1})^{\ell} + (|x_{k+1}| + r_{k+1})^{\ell+\alpha}$$

obtaining a contradiction, since δ_{k+1}, $|x_{k+1}|$, and r_{k+1} satisfy equation (17).

Now, we put

$$\rho_k = c\,\frac{\delta_k}{|x_k|^{\ell+\alpha-1}}\,, \qquad c = \frac{2^{\ell+\alpha-1}}{(\ell+\alpha+1)3^{\ell+\alpha-1}}\,.$$

Notice that, recalling (13), $\rho_k \le r_k$ holds.

Now we are able to give an upper bound for k_0. Indeed for any integer k such that $r_k \le \frac{|x_k|}{2}$ we get

$$|x_{k+1}| = |x_k| - r_k \le |x_k| - c\frac{\delta_k}{|x_k|^{\ell+\alpha-1}}\,. \tag{18}$$

Moreover $(|x_k|)_{k\in\mathbb{N}}$ is decreasing, hence, by (18),

$$\begin{aligned} |x_{k+1}|^{\ell+\alpha} &\le |x_{k+1}||x_k|^{\ell+\alpha-1} \le |x_k|^{\ell+\alpha} + c|x_k|^{\ell+\alpha} - cy \\ &= (1+c)|x_k|^{\ell+\alpha} - cy\,. \end{aligned} \tag{19}$$

In particular we get, by iteration,

$$\begin{aligned} |x_{k+1}|^{\ell+\alpha} &\le (1+c)^{k+1}|x_0|^{\ell+\alpha} - cy\sum_{j=0}^{k}(1+c)^j \\ &= -(1+c)^{k+1}\big(y - |x|^{\ell+\alpha}\big) + y = -(1+c)^{k+1}\delta(x,y) + y\,, \end{aligned}$$

and eventually we obtain

$$\frac{y}{\delta(x,y)} \ge \frac{y - |x_{k+1}|^{\ell+\alpha}}{\delta(x,y)} \ge (1+c)^{k+1}\,.$$

Thus (remember $y \ge \delta$ and $y \le |x|^{\ell+\alpha} + b^{\ell+1}$ since $(x,y) \in \triangle^{\ell+1}(0,b,2r)$)

$$k_0 \le \frac{\log\frac{y}{\delta(x,y)}}{\log(1+c)} \le \frac{\log\frac{\left(b^{\ell+1}+|x|^{\ell+\alpha}\right)}{\delta(x,y)}}{\log(1+c)} \tag{20}$$

holds and the first step is proved.

Second step. Now $r_k > |x_{k_0}|/2$, so that we must 'climb'. To this end we define (x_k, y_k), $0 \le k \le k_1$ as follows: $x_k \equiv x_{k_0}$,

$$y_{k+1} = y_k + F\big(x_{k_0}, r_k\big)\,, \tag{21}$$

where r_k is the solution in r of the equation

$$\delta_k + \left|x_{k_0}\right|^{\ell+\alpha} = r\left(\left|x_{k_0}\right| + r\right)^{\ell} + \left(\left|x_{k_0}\right| + r\right)^{\ell+\alpha}\,,$$

with $\delta_k = \delta(x_{k_0}, y_k)$ and k_1 is the first integer such that $\delta_k > b^{\ell+1}$. We recall that, because of the above choice of r_k, the metric cubes $Q((x_k, y_k), r_k)$ are contained in Ω. Now, in order to give an upper bound for k_1, we put

$$\rho_k = c'\delta_k^{\frac{1}{\ell+\alpha}}\,, \tag{22}$$

where

$$c' = 3^{-\frac{\ell+\alpha-1}{\ell+\alpha}} (\ell + \alpha + 1)^{\frac{-1}{\ell+\alpha}} .$$

By (15) we get $\rho_k \leq r_k$.

Now from (21) it follows that

$$y_{k+1} = y_k + F(x_k, r_k) \geq y_k + F(x_k, \rho_k) .$$

Thus, by (22), we obtain

$$\begin{aligned} y_{k+1} &\geq y_k + c'\delta_k^{1/(\ell+\alpha)} \left(|x_{k_0}| + c'\delta_k^{1/(\ell+\alpha)} \right)^{\ell} \\ &\geq y_k + (c')^{\ell+1} \delta_k^{(\ell+1)/(\ell+\alpha)} . \end{aligned}$$

Replace now the points y_k by the points $\tilde{y}_k$ defined by

$$y_{k+1} = y_k + c\delta_k^{\frac{\ell+1}{\ell+\alpha}} , \tag{23}$$

where $(c')^{\ell+1} = c$. Since $y_{k+1} - y_k \geq \tilde{y}_{k+1} - \tilde{y}_k$, then the first index k such that $\tilde{y}_k > b^{\ell+1}$ will provide an upper bound for k_1 (a sharp one, indeed, since if $x_{k_0} = 0$ then $y_k = \tilde{y}_k$). To avoid cumbersome notations, let us write y_k y_k instead of $\tilde{y}$, y_k being defined by (23).

As a consequence, from (23), it follows that

$$\delta_{k+1} = \delta_k + c\delta_k^{\frac{\ell+1}{\ell+\alpha}} . \tag{24}$$

Notice that

$$\begin{aligned} \delta_k^{1-(\ell+1)/(\ell+\alpha)} - \delta_{k+1}^{1-(\ell+1)/(\ell+\alpha)} &= \left(\frac{\ell+1}{\ell+\alpha} - 1 \right) \int_{\delta_k}^{\delta_{k+1}} t^{-(\ell+1)/(\ell+\alpha)} dt \\ &\geq \left(\frac{\ell+1}{\ell+\alpha} - 1 \right) \delta_{k+1}^{-(\ell+1)/(\ell+\alpha)} (\delta_{k+1} - \delta_k) , \end{aligned} \tag{25}$$

since δ_k is increasing. On the other hand, recalling that we are interested in the case $b < 1$, from (24) it follows that

$$1 - \frac{c\delta_k}{\delta_{k+1}} \leq 1 - \frac{c\delta_k^{\frac{\ell+1}{\ell+\alpha}}}{\delta_{k+1}} = \frac{\delta_k}{\delta_{k+1}} ,$$

since $\frac{\ell+1}{\ell+\alpha} > 1$. Thus

$$\frac{1}{1+c} \leq \frac{\delta_k}{\delta_{k+1}} ,$$

and then, recalling (25) and (24), we get

$$\delta_k^{1-(\ell+1)/(\ell+\alpha)} - \delta_{k+1}^{1-(\ell+1)/(\ell+\alpha)} \geq c \left(\frac{\ell+1}{\ell+\alpha} - 1 \right) \left(\frac{1}{1+c} \right)^{(\ell+1)/(\ell+\alpha)} = \bar{c} .$$

Eventually, summing up from 0 to k, we have

$$\delta_0^{1-(\ell+1)/(\ell+\alpha)} - \delta_{k+1}^{1-(\ell+1)/(\ell+\alpha)} \geq (k+1)\bar{c} ,$$

and in particular

$$\begin{aligned} k_1 &\le C\left(\delta(x,y)^{1-\frac{\ell+1}{\ell+\alpha}} - \delta\left(x_{k_0},y\right)^{1-\frac{\ell+1}{\ell+\alpha}}\right) \\ &\le C\left(\delta(x,y)^{1-\frac{\ell+1}{\ell+\alpha}} - b^{(\ell+1)\left(1-\frac{\ell+1}{\ell+\alpha}\right)}\right) \end{aligned}$$

achieving the proof of the second step. Thus, to prove Lemma 5 we have only to prove that the last metric cube can be chosen in such a way it is fully contained in $\left(F_b^{\ell+1}\right)^c$. To this end, we add to our chain N copies of the last metric cube centered at $y_{k_1} + hF(x_{k_0}, r_{k_1})$, $h = 1, \cdots, N$, with

$$\frac{b^{\ell+1}}{F\left(x_{k_0}, r_{k_1}\right)} \le N \le \frac{b^{\ell+1}}{F\left(x_{k_0}, r_{k_1}\right)} + 1$$

If (ξ, η) belong to N-th metric cube we have to show that $\eta > |\xi|^{\ell+\alpha} + b^{\ell+1}$. Now

$$\begin{aligned} b^{\ell+1} + |\xi|^{\ell+\alpha} &\le b^{\ell+1} + \left(\left|x_{k_0} - \xi\right| + \left|x_{k_0}\right|\right)^{\ell+\alpha} < b^{\ell+1} + \left(r_{k_1} + \left|x_{k_0}\right|\right)^{\ell+\alpha} \\ &\le b^{\ell+1} + y_{k_1} - F\left(x_{k_0}, r_{k_1}\right) \\ &< y_{k_1} + NF\left(x_{k_0}, r_{k_1}\right) - F\left(x_{k_0}, r_{k_1}\right) < \eta\,. \end{aligned}$$

We need now an upper bound for $\frac{b^{\ell+1}}{F(x_{k_0}, r_{k_1})}$. Since $|x_{k_0}| \le 2r_{k_0} < 2r_{k_1}$, we have (by (15))

$$r_{k_1} \ge c\delta_{k_1}^{1/(\ell+\alpha)} \ge cb^{(\ell+1)/(\ell+\alpha)}\,,$$

so that

$$\frac{b^{\ell+1}}{F\left(x_0, r_{k_1}\right)} \le b^{\ell+1-(\ell+1)\frac{\ell+1}{\ell+\alpha}} = b^{(\ell+1)\left(1-\frac{\ell+1}{\ell+\alpha}\right)}\,.$$

On the other hand, since $\delta(x, y) < (b/2)^{\ell+1}$, then

$$\delta(x,y)^{1-\frac{\ell+1}{\ell+\alpha}} - b^{(\ell+1)\left(1-\frac{\ell+1}{\ell+\alpha}\right)} \ge \left(2^{(\ell+1)\left(\frac{\ell+1}{\ell+\alpha}-1\right)} - 1\right) b^{(\ell+1)\left(1-\frac{\ell+1}{\ell+\alpha}\right)} \ge cN\,,$$

and then the estimate of N is similar to that of k_1 and the proof is complete. □

REMARK 4. Let us notice that from Lemma 5 it follows that Ω_α satisfies condition $ii)$ of Definition 3, with $\varphi(t) = Ct^{1-\frac{\ell+1}{\ell+\alpha}}$.

LEMMA 6. *Let Ω_α be the set defined in Theorem 5. For any $\theta \in (0, 1)$, there exists $c_\theta > 0$ such that for any given Harnack's chain, $\{Q_k, k \le \nu\}$, of parameter $theta$ in Ω, connecting a point $(0, y) \in \Omega$ with a point $(\bar{x}, \bar{y})$, $\bar{y} > b^{\ell+1} + |\bar{x}|^{\ell+\alpha}$, there exists another chain of parameter θ, $\left\{\widetilde{Q}_k,\ k \le c\nu\right\}$, connecting $(0, y)$ with $(0, \bar{y})$.*

Proof. By previous Lemma 4, we can replace $\{Q_k : k \le \nu\}$, by a new Harnack's chain $\left\{Q'_k,\ k \le c_\theta \nu\right\}$ with parameter $\theta' = \theta^{\ell+1} < \theta$. If $Q'_k = Q((x_k, y_k), r_k)$, denote now by $\widetilde{Q}_k$ the metric cubes defined by

$$\widetilde{Q}_k = Q\left((0, y_k), \rho_k\right),$$

where $\rho_k^{\ell+1} = r_k(|\xi_k| + r_k)^\ell$, $|\xi_k| = \min\{|x_k|,\ |x_{k+1}|\}$. If $\xi = x_k + r_k \frac{x_k}{|x_k|}$, the point $(\xi, y_k - F(\xi_k, r_k))$ belongs to $Q_k \subset \Omega_\alpha$, since $F(\xi_k, r_k) \le F(x_k, r_k)$, and hence

$$y_k - r_k(|\xi_k| + r_k)^\ell > |\xi|^{\ell+\alpha} = (|\xi_k| + r_k)^{\ell+\alpha} . \tag{26}$$

Hence we can prove that $\widetilde{Q}_k \subset \Omega$. To this end it is enough to verify that

$$y_k - \rho_k^{\ell+1} > \rho_k^{\ell+\alpha} . \tag{27}$$

Indeed, by (26), we get

$$y_k - \rho_k^{\ell+1} = y_k - r_k(|\xi_k| + r_k)^\ell > (|\xi_k| + r_k)^{\ell+\alpha} ; \tag{28}$$

on the other hand, since $\rho_k^{\ell+1} = r_k(|\xi_k| + r_k)^\ell$, we get

$$\rho_k^{\ell+1} \le (|\xi_k| + r_k)^{\ell+1} ,$$

so that $\rho_k \le |\xi_k| + r_k$ and then (27) follows from (28). Since the first metric cube is centred at $(0, y)$ and the last one contains $(0, \bar{y})$, we eventually proved that there exists a chain of metric cubes connecting these metric cubes with length comparable with that of the first Harnack's chain from which we started. Moreover the centers of the new chain have first coordinate zero. It remains to prove that the last chain is a Harnack's chain. Indeed we know that

$$\begin{aligned} |x_k - x_{k+1}| &\le \theta' \min\{r_k, r_{k+1}\} , \\ \varphi(\xi_k, |y_k - y_{k+1}|) &\le \theta' \min\{r_k, r_{k+1}\} . \end{aligned}$$

Let us denote $\tilde{r}_k = \min\{r_k, r_{k+1}\}$. Then

$$|y_k - y_{k+1}| \le F\left(\xi_k, \theta' \tilde{r}_k\right) \le \theta' r_k(|\xi_k| + r_k)^\ell = \theta' \rho_k^{\ell+1} = F(0, \theta\rho_k) .$$

□

Proof of Theorem 5. By Lemma 5, Lemma 6 and Lemma 1 we can state that Ω_α is a $\phi-$Harnack's domain with $\phi(t) = ct^{\frac{\ell+1}{\ell+\alpha}}$.

In order to prove that the estimate is sharp, it will be enough to show that, if $x = 0$, then any Harnack's chain $\{Q_1, \dots, Q_\nu\}$ connecting $(x, y) = (0, y)$ with a point $(\bar{x}, \bar{y})$, $\bar{y} > b^{\ell+1}$ cannot have less than

$$const.\left(y^{1-\frac{\ell+1}{\ell+\alpha}} - b^{\ell\left(1-\frac{\ell+1}{\ell+\alpha}\right)}\right) \tag{29}$$

elements. By Lemma 6 we can assume without loss of generality that the centers of the Q_j's lie on $\{x = 0\}$. Moreover, we can replace it by $\widetilde{Q}_1, \dots, \widetilde{Q}_\nu$, where $\widetilde{Q}_k = Q((0, \tilde{y}_k), \tilde{r}_k)$, where $\tilde{r}_k^{\ell+\alpha} = \tilde{y}_k - \tilde{r}_k^{\ell+1}$ and $\tilde{y}_{k+1} = \tilde{y}_k + (\theta\tilde{r}_k)^{\ell+1}$ (in other words, the chain becomes shorter if we take the metric cubes 'as large as possible'). Now $\tilde{y}_k \ge \tilde{r}_k^{\ell+\alpha}$, so that $\tilde{y}_{k+1} \le \tilde{y}_k + \theta^{\ell+1}\tilde{y}_k^{\frac{\ell+1}{\ell+\alpha}}$, and hence keeping in mind $\tilde{y}_{k+1} \ge \tilde{y}_k$, we get

$$\tilde{y}_k^{1-\frac{\ell+1}{\ell+\alpha}} \le \tilde{y}_{k+1}^{1-\frac{\ell+1}{\ell+\alpha}} + \theta^{\ell+1} ,$$

that yields

$$c_\theta \nu \ge y^{1-\frac{\ell+1}{\ell+\alpha}} - b^{(\ell+1)\left(1-\frac{\ell+1}{\ell+\alpha}\right)} ,$$

and hence (29) follows. □

References

[1] BASS R.F. AND BURDZY K., *A boundary Harnack principle for twisted Hölder domains*, Ann. of Math. **134** (1991), 253–276.

[2] BONY J.M., *Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés*, Ann. Inst. Fourier Grenoble **19** (1969), 277–304.

[3] BONY J.M., *Opérateurs elliptiques degénérés associés aux axiomatiques de la théorie du potentiel*, C.I.M.E. 1 Ciclo Stresa 1969, Potential Theory 1970, 69–119.

[4] CAPOGNA L. AND GAROFALO N., *Boundary behavior of nonnegative solutions of subelliptic equations in NTA domains for Carnot-Carathéodory metrics*, J. Fourier Anal. Appl. **4** (1998), 403–432.

[5] CAFFARELLI L., FABES E., MORTOLA S. AND SALSA S., *Boundary behavior of nonnegative solutions of elliptic operators in divergence form*, Indiana J. Math. **30** (1981), 621–640.

[6] FERRARI F., *On boundary behavior of harmonic functions on Hölder domains*, J. Fourier Anal. Appl. **4** (1998), 447–461.

[7] FERRARI F. AND FRANCHI B., *A local doubling formula for the harmonic measure associated with subelliptic operators*, (2000) preprint.

[8] FRANCHI B., *Weighted Sobolev-Poincaré inequalities and pointwise estimates for a class of degenerate elliptic equations*, Trans. Amer. Math. Soc. **327** (1991), 125–158.

[9] FRANCHI B., GUTIÉRREZ C. AND WHEEDEN R.L., *Weighted Sobolev-Poincaré inequalities for Grushin type operators*, Comm. Partial Differential Equations **19** (1994), 523–604.

[10] FRANCHI B. AND LANCONELLI E., *Hölder regularity theorem for a class of linear nonuniformly elliptic operators with measurable coefficients*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. **10** (1983), 523–541.

[11] GRUSHIN V.V., *On a class of hypoelliptic operators*, Math. USSR, Sbornik **12** (1970-1971), 458–476.

[12] HÖRMANDER L., *Hypoelliptic second-order differential equations*, Acta Math. **119** (1967), 147–171.

[13] HUNT R.R. AND WHEEDEN R.L., *Positive harmonic functions on Lipschitz domains*, Trans. Amer. Mat. Soc. **147** (1970), 507–527.

[14] JERISON D. AND KENIG C., *Boundary behavior of harmonic functions in non-tangentially accessible domains*, Adv. Math. **46** (1982), 80–147.

[15] NAGEL A., STEIN E.M. AND WAINGER S., *Balls and metrics defined by vector fields I: basic properties*, Acta Math. **155** (1985), 103–147.

[16] STEIN E.M., *Harmonic analysis*, Princeton University Press 1993.

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