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ON BIFURCATIONS FROM NORMAL SOLUTIONS TO SUPERCONDUCTING STATES

Abstract. Motivated by the paper by J. Berger and J. Rubinstein [3] and other recent studies [10], [15], [16], we analyze the Ginzburg-Landau functional in an open bounded set Ω. We mainly discuss the bifurcation problem whose analysis was initiated in [17] and show how some of the techniques developed by the first author in the case of Abrikosov's superconductors [7] can be applied in this context. In the case of non simply connected domains, we come back to [3] and [13], [14] for giving the analysis of the structure of the nodal sets for the bifurcating solutions.

1. Introduction

1.1. Our model

Following the paper by Berger-Rubinstein [3], we would like to understand the minima (or more generally the extrema) of the following Ginzburg-Landau functional. In a bounded, connected, regular[1], open set $\Omega \subset \mathbb{R}^2$ and, for any $\lambda > 0$ and $\kappa > 0$, this functional $G_{\lambda,\kappa}$ is defined, for $u \in H^1(\Omega;\mathbb{C})$ and $A \in H^1_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ such that $\operatorname{curl} A \in L^2$, by

$$(1)\qquad \begin{aligned} G_{\lambda,\kappa}(u,A) = &\int_\Omega \left(\lambda\left(-|u|^2+\frac{1}{2}|u|^4\right)+|(\nabla - iA)u|^2\right)dx_1\cdot dx_2 \\ &+\kappa^2\lambda^{-1}\int_{\mathbb{R}^2}|\operatorname{curl} A - H_e|^2 dx_1\cdot dx_2\,. \end{aligned}$$

Here, for $A=(A_1,A_2)$, $\operatorname{curl} A = \partial_{x_1}A_2 - \partial_{x_2}A_1$, $\operatorname{div} A = \partial_{x_1}A_1 + \partial_{x_2}A_2$ and H_e is a C_0^∞ function on $\mathbb{R}^2$ (or more generally some function in $L^2(\mathbb{R}^2)$). Physically H_e represents the exterior magnetic field.
Let A_e be a solution of

$$(2)\qquad \begin{aligned} \operatorname{curl} A_e &= H_e \\ \operatorname{div} A_e &= 0\,. \end{aligned}$$

It is easy to verify that such a solution exists by looking for A_e in the form $A_e = \big(-\partial_{x_2}\psi_e, \partial_{x_1}\psi_e\big)$. We have then to solve $\Delta\psi_e = H_e$ and it is known to be solvable in $\mathcal{S}'(\mathbb{R}^2)\cap C^\infty(\mathbb{R}^2)$ (or in $\mathcal{S}'(\mathbb{R}^2)\cap H^2_{loc}(\mathbb{R}^2)$ if $H_e \in L^2(\mathbb{R}^2)$). Of course A_e is not unique but we shall discuss about uniqueness modulo gauge transform later and at the end this is mainly the restriction of A_e to Ω which will be considered.
We shall sometimes use the identification between vector fields A and 1-forms ω_A.

[1] with C^∞ boundary

When analyzing the extrema of the GL-functional, it is natural to first consider the corresponding Euler-Lagrange equations (called in our context Ginzburg-Landau equations). This is a system of two equations (with a boundary equation):

$$
\begin{aligned}
&(GL)_1 \quad -(\nabla - iA)^2 u + \lambda u\big(|u|^2 - 1\big) = 0\,, \quad \text{in } \Omega\,, \\
&(GL)_2 \quad \text{curl}^*(\text{curl}\, A - H_e) = \lambda\kappa^{-2}\text{Im}\,[\bar{u}\cdot(\nabla - iA)u]\cdot 1_\Omega\,, \\
&(GL)_3 \quad (\nabla - iA)u\cdot\nu = 0\,, \quad \text{in } \partial\Omega\,.
\end{aligned} \tag{3}
$$

Here ν is a unit exterior normal to $\partial\Omega$. The operator curl* is defined by $\text{curl}^* f := (\partial_{x_2} f, -\partial_{x_1} f)$. Moreover, without loss of generality in our problem, we shall add the condition

$$
(GL)_4 \quad \text{div}\, A = 0 \ \text{ in } \Omega\,. \tag{4}
$$

One can also assume if necessary that the vector potential satisfies

$$
A\cdot\vec{\nu} = 0\,, \tag{5}
$$

on the boundary of Ω, where ν is a normal unit vector to $\partial\Omega$.
Let us briefly recall the argument. One would like to find θ in $C^\infty\big(\overline{\Omega}\big)$ such that $\widetilde{A} = A + d\theta$ satisfies (4) and (5). One can proceed in two steps. The first step is to find a gauge transformation such that (5) is satisfied. This is immediate if the boundary is regular.
We now assume this condition.
The second step consists in solving

$$
\begin{aligned}
&\Delta\theta = -\,\text{div}\, A \ \text{ in } \Omega\,, \\
&\frac{\partial\theta}{\partial\nu} = 0\,, \quad \text{on } \partial\Omega\,.
\end{aligned}
$$

This is a Neumann problem, which is solvable if and only if the right hand-side is orthogonal to the first eigenfunction of the Neumann realization of the Laplacian, that is the constant function $x \mapsto 1$. We have only to observe that $\int_\Omega \text{div}\, A\, dx = 0$ if (5) is satisfied.

An important remark is that the pair $(0, A_e)$ is a solution of the system. This solution is called the normal solution. Of course, any solution of the form $(0, A_e + \nabla\phi)$ with ϕ harmonic is also a solution.

REMARK 1. Note also that the normalization of the functional leads to the property that

$$
G_{\lambda,\kappa}(0, A_e) = 0\,. \tag{6}
$$

The first proposition is standard.

PROPOSITION 1. *If Ω is bounded, the functional $G_{\lambda,\kappa}$ admits a global minimizer which is a solution of the equation.*

We refer to [8], for a proof together with the discussion of the next subsection.

1.2. Comparison with other models

Let us observe that there is another natural problem which may be considered. This is the problem of minimizing, for $(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$, the functional $G^{\Omega}_{\lambda,\kappa}$ defined by

$$\begin{aligned} G^{\Omega}_{\lambda,\kappa}(u, A) = &\int_{\Omega} \left(\lambda \left(-|u|^2 + \frac{1}{2}|u|^4\right) + |(\nabla - iA)u|^2\right) dx_1 \cdot dx_2 \\ &+ \kappa^2\lambda^{-1} \int_{\Omega} |\mathrm{curl}\, A - H_e|^2 \, dx_1 \cdot dx_2 \,. \end{aligned} \tag{7}$$

This may lead to a different result in the case when Ω is not simply connected. According to discussions with Akkermans, this is the first problem which is the most physical (see also the discussion in the appendix).
A comparison between $G_{\lambda,\kappa}$ and $G^{\Omega,D}_{\lambda,\kappa}$ where D is a ball containing Ω and $G^{\Omega,D}_{\lambda,\kappa}$ is defined by

$$\begin{aligned} G^{\Omega,D}_{\lambda,\kappa}(u, A) = &\int_{\Omega} \left(\lambda \left(-|u|^2 + \frac{1}{2}|u|^4\right) + |(\nabla - iA)u|^2\right) dx_1 \cdot dx_2 \\ &+ \kappa^2\lambda^{-1} \int_{D} |\mathrm{curl}\, A - H_e|^2 \, dx_1 \cdot dx_2 \,. \end{aligned} \tag{8}$$

is useful. If b is given with support outside of the ball D, it is easy to see (assuming that b is regular) that there exists a with support outside D such that $\mathrm{curl}\, a = b$. It is indeed sufficient to take the usual transversal gauge

$$a_1 = -x_2 \int_0^1 sb(sx)\, ds\,, \quad a_2 = x_1 \int_0^1 sb(sx)\, ds\,. \tag{9}$$

This shows that, for any D containing Ω, we have

$$\inf G_{\lambda,\kappa}(u, A) = \inf G^{\Omega,D}_{\lambda,\kappa}(u, A)\,. \tag{10}$$

In particular it is enough to consider minimizing sequences $(u_n, A_e + a_n)$ where $\mathrm{supp}\, a_n \subset D$ and D is a ball containing Ω. The proof of the existence of minimizers is then greatly simplified. Finally, it is natural[2] to think that one can replace D by

$$\widetilde{\Omega} := \Omega \cup (\cup_i \mathcal{O}_i)\,, \tag{11}$$

where the $\mathcal{O}_i$ are the holes, that are the bounded connected components of $\mathbb{R}^2 \setminus \Omega$. A proof can be obtained by analyzing the Ginzburg-Landau equations satisfied by a minimizer of $G^{\Omega,D}_{\lambda,\kappa}$. We finally get:

$$\inf G_{\lambda,\kappa}(u, A) = \inf G^{\Omega,\widetilde{\Omega}}_{\lambda,\kappa}(u, A)\,. \tag{12}$$

REMARK 2. If $(u, A_e + a)$ is a solution of the GL-equation then $\mathrm{curl}\, a = 0$ in the unbounded component of $\mathbb{R}^2 \setminus \Omega$ and $\mathrm{curl}\, a = $ const. in each hole (see Lemma 2.1 in [10]). It would be interesting to discuss the possible values of these constants.

[2] This is at least clear when $\widetilde{\Omega}$ is a star-shaped domain by the previous proof. See Section 3, in the proof of Proposition 8 for a complementary argument.

1.3. Standard results

The second proposition which is also quite standard (see for example [8]) is

PROPOSITION 2. *If u is a solution of the first GL-equation with the Neumann boundary condition then*

$$|u(x)| \leq 1\,, \quad \forall x \in \Omega\,. \tag{13}$$

We note also for further use that the solutions of the GL-system are in $C^\infty(\overline{\Omega})$ under the assumption that Ω is regular.

2. Is the normal state a minimizer?

The aim of this section is to give a proof of a result suggested in [3] who said "We expect the normal state to be a stable solution for small λ...".
Although, this result is probably known as folk theorem, we think it is useful to give a proof (following considerations by M. Dutour in a near context [7]) of this property.
Note that connected results are obtained in [10] and more recently in [15], [16].
Before stating the theorem, let us recall that we have called *normal state* a pair (u, A) of the form:

$$(u, A) = (0, A_e)\,, \tag{14}$$

where A_e is any solution of (1).
As already observed, A_e is well defined up to gauge transformation and $(0, A_e)$ is a solution of the GL-system.
So it is effectively natural to ask if $(0, A_e)$ is a global minimum. The first result in this direction is the following easy proposition about the normal state. But let us first introduce:

DEFINITION 1. *We denote by $\lambda^{(1)}$ the lowest eigenvalue of the Neumann realization in Ω of*

$$-\Delta_{A_e} := -(\nabla - iA_e)^2\,.$$

We shall frequently use the assumption

$$\lambda^{(1)} > 0\,. \tag{15}$$

Note the following necessary and sufficient condition for this property (cf. [12]).

PROPOSITION 3. *The condition (15) is satisfied if and only if one of the two following conditions is satisfied:*

1. *H_e is not identically zero in Ω;*
2. *H_e is identically zero in Ω but there exists a closed path γ in Ω such that $\frac{1}{2\pi}\int_\gamma \omega_{A_e} \notin \mathbb{Z}$.*

Let us observe that the second case can only occur when Ω is non simply connected.

PROPOSITION 4. *Under condition (15) and if $\lambda \in \,]0, \lambda^{(1)}[$, the pair $(0, A_e)$ is a non-degenerate (up to gauge transforms) local minimum of $G_{\lambda,\kappa}$.*

The Hessian at $(0, A_e)$ of the GL-functional is indeed the map

$$(\delta u, \delta a) \mapsto ((-\Delta_{A_e} - \lambda)\, \delta u\,,\ \mathrm{curl}^*\, \mathrm{curl}\, \delta a)\,,$$

where we assume that $\mathrm{div}\, \delta a = 0$ and $\delta a \cdot \nu = 0$ at the boundary of $\widetilde{\Omega}$.
Note that this proof gives also:

PROPOSITION 5. *If $\lambda > \lambda^{(1)}$, the pair $(0, A_e)$ is not a local minimum of $G_{\lambda,\kappa}$.*

We refer to [15] for a connected result. Proposition 5 does not answer completely to the question about global minimizers. The next theorem gives a complementary information.

THEOREM 1. *Under assumption (15), then, for any $\kappa > 0$, there exists $\lambda_0(\kappa) > 0$ such that, for $\lambda \in]0, \lambda_0(\kappa)]$, $G_{\lambda,\kappa}$ has only normal solutions as global minimizers.*

REMARK 3. By a variant of the techniques used in [7] in a similar context, one can actually show that, for any $\kappa > 0$, there exists $\lambda_1(\kappa) > 0$ such that, for $\lambda \in]0, \lambda_1(\kappa)]$, all the solutions of the Ginzburg-Landau equations are normal solutions. This will be analyzed in Section 5.

Proof of Theorem 1. Let $(u, A) := (u, A_e + a)$ be a minimizer of the (GL) functional. So it is a solution[3] of (GL) and moreover we have, using (6), the following property:

$$G_{\lambda,\kappa}(u, A) \leq 0\,. \tag{16}$$

Using the inequality $-|u|^2 \geq -\frac{1}{2}|u|^4 - \frac{1}{2}$ and (16), we first get, with $b = \mathrm{curl}\, a$:

$$\frac{\kappa^2}{\lambda} \int_{\mathbb{R}^2} b^2\, dx \leq \frac{\lambda}{2}|\Omega|\,, \tag{17}$$

where $|\Omega|$ is the area of Ω.

We now discuss the link between b and a in $\widetilde{\Omega}$. So we shall only use from (17):

$$\frac{\kappa^2}{\lambda} \int_{\widetilde{\Omega}} b^2\, dx \leq \frac{\lambda}{2}|\Omega|\,, \tag{18}$$

Let us now consider in $\widetilde{\Omega}$, $\tilde{a}$ the problem of finding a solution of

$$\begin{aligned} &\mathrm{curl}\, \tilde{a} = b\,, \quad \mathrm{div}\, \tilde{a} = 0\,, \\ &\tilde{a} \cdot \nu = 0\,, \quad \text{on } \partial\widetilde{\Omega}\,. \end{aligned} \tag{19}$$

We have the following standard proposition (see Lemma 2.3 in [10]).

PROPOSITION 6. *The problem (19) admits, for any $b \in L^2(\widetilde{\Omega})$, a unique solution $\tilde{a}$ in $H^1(\widetilde{\Omega})$. Moreover, there exists a constant C such that*

$$\|\tilde{a}\|_{H^1(\widetilde{\Omega})} \leq C\|b\|_{L^2(\widetilde{\Omega})}\,, \quad \forall b \in L^2\,. \tag{20}$$

[3] We actually do not use this property in the proof.

Proof. Following a suggestion of F. Bethuel, we look for a solution in the form: $\tilde{a} = \operatorname{curl}^* \psi$. We then solve the Dirichlet problem $-\Delta\psi = b$ in $\widetilde{\Omega}$. This gives a solution with the right regularity. For the uniqueness, we observe that $\widetilde{\Omega}$ being connected and simply connected a solution of $\operatorname{curl} \hat{a} = 0$ is of the form $\hat{a} = d\theta$ $\big($with $\theta \in H^2(\widetilde{\Omega})\big)$, and if $\operatorname{div} \hat{a} = 0$ and $\hat{a} \cdot \nu$ on $\partial\widetilde{\Omega}$, we get the equations $\Delta\theta = 0$ and $\nabla\theta \cdot \nu = 0$ on $\partial\widetilde{\Omega}$, which implies $\theta =$ const. and consequently $\hat{a} = 0$. □

We can now use the Sobolev estimates in order to get

$$\|a\|_{L^4(\widetilde{\Omega})} \leq C_1 \|a\|_{H^1(\widetilde{\Omega})} . \tag{21}$$

From (18), (20) and (21), we get the existence of a constant C_2 such that

$$\|a\|_{L^4(\widetilde{\Omega})} \leq C_2 \frac{\lambda}{\kappa} . \tag{22}$$

The second point is to observe, that, for any $\epsilon \in]0, 1[$, we have the inequality

$$\int_\Omega |(\nabla - iA)u|^2 \, dx \geq (1-\epsilon)\|(\nabla - iA_e)u\|^2_{L^2(\Omega)} - \frac{(1-\epsilon)}{\epsilon}\|au\|^2_{L^2(\Omega)} . \tag{23}$$

Taking $\epsilon = \frac{1}{4}$ and using Hölder's inequality, we get

$$\int_\Omega |(\nabla - iA)u|^2 \, dx \geq \frac{3}{4}\|(\nabla - iA_e)u\|^2_{L^2(\Omega)} - 3\|a\|^2_{L^4(\Omega)}\|u\|^2_{L^4(\Omega)} . \tag{24}$$

Using now the ellipticity of $-\Delta_{A_e}$ in the form of the existence of a constant C_1

$$\|u\|^2_{H^1(\Omega)} \leq C_1 \left(\|(\nabla - iA_e)u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \right), \tag{25}$$

and again the Sobolev inequality, we then obtain the existence of a constant C_2 such that

$$\begin{aligned} \int_\Omega |(\nabla - iA)u|^2 \, dx \geq \left(\frac{3}{4} - C_2\|a\|^2_{L^4(\Omega)} \right) \|(\nabla - iA_e)u\|^2_{L^2(\Omega)} \\ - C_2\|a\|^2_{L^4(\Omega)}\|u\|^2_{L^2(\Omega)} . \end{aligned} \tag{26}$$

We get then from (16) and (22), and for a suitable new constant C (depending only on Ω and H_e),

$$\left[\frac{3}{4}\lambda^{(1)} - C\frac{\lambda^2}{\kappa^2} - \lambda \right] \|u\|^2_{L^2(\Omega)} \leq 0 . \tag{27}$$

Using the assumption (15), this gives $u = 0$ for λ small enough and the proof of Theorem 1. □

REMARK 4. Note that with a small improvement of the method, it is possible $\big($taking $\epsilon = \frac{1}{\kappa}$ in (23)$\big)$ to show that one can choose, in the limit $\kappa \to +\infty$, $\lambda_0(\kappa)$ satisfying:

$$\lambda_0(\kappa) \geq \lambda^{(1)} - \mathcal{O}\left(\frac{1}{\kappa}\right) . \tag{28}$$

This will be developed in Section 4.

REMARK 5. Observing that $\lambda \mapsto \frac{1}{\lambda} G_{\lambda,\kappa}(u, A)$ is monotonically decreasing, one easily obtains that the set of λ's such that $(0, A_e)$ is a global minimum is an interval of the form $\big]0, \lambda_0^{opt}(\kappa)\big]$. Inequality (28) implies:

$$\lambda_0^{opt}(\kappa) \geq \lambda^{(1)} - \mathcal{O}\left(\frac{1}{\kappa}\right). \tag{29}$$

Similar arguments are used in [7] for the Abrikosov's case. We recall that, in this case, the domain Ω, is replaced by a torus $\mathbb{R}^2 \setminus \mathcal{L}$ where $\mathcal{L}$ is the lattice generated over $\mathbb{Z}^2$ by two independent vectors of $\mathbb{R}^2$.
Observing now that $\kappa \mapsto G_{\lambda,\kappa}(u, A)$ is monotonically increasing, one easily obtains that the map $\kappa \mapsto \lambda_0^{opt}(\kappa)$ is increasing. Using (29) and Proposition 7, one gets that λ_0^{opt} is increasing from 0 to $\lambda^{(1)}$ for $\kappa \in]0, +\infty[$.

3. Estimates in the case κ small

We have already shown in Proposition 5 that, if $\lambda > \lambda^{(1)}$, then the normal state is not a minimizer. In other words (see Remark 5), under condition (15), we have:

$$0 < \lambda_0^{opt}(\kappa) \leq \lambda^{(1)}. \tag{30}$$

If we come back to the formula (27), one immediately obtains the following first result:

PROPOSITION 7. *There exist constants $\mu_0 \in \big]0, \lambda^{(1)}\big]$ and $\alpha_0 > 0$ such that, for $\lambda \in]0, \mu_0]$ satisfying*

$$\lambda \leq \alpha_0 \kappa, \tag{31}$$

the minimizer is necessarily the normal solution.

In order to get complementary results, it is also interesting to compute the energy of the pair $(u, A) = (1, 0)$. This will give, in some asymptotic regime, some information about the possibility for the normal solution (or later for a bifurcating solution) to correspond to a global minimum of the functional. An immediate computation gives:

$$G_{\lambda,\kappa}(1, 0) = -\frac{\lambda}{2}|\Omega| + \frac{\kappa^2}{\lambda}\int_{\mathbb{R}^2} H_e^2\, dx. \tag{32}$$

We see in particular that when $\frac{\kappa}{\lambda}$ is small, the normal solution cannot be a global minimizer of $G_{\lambda,\kappa}$.
As already observed in Subsection 1.2, what is more relevant is probably the integral $\int_{\widetilde{\Omega}} H_e^2\, dx$ instead of $\int_{\mathbb{R}^2} H_e^2\, dx$ in (32). Note also that it would be quite interesting to determine the minimizers in the limit $\kappa \to 0$. We note indeed that $(1, 0)$ is not a solution of the GL-system, unless H_e is identically zero in Ω. Let us show the following proposition.

PROPOSITION 8. *If*

$$\kappa < \lambda \cdot \left(\frac{|\Omega|}{2\int_\Omega H_e^2\, dx}\right)^{\frac{1}{2}}, \tag{33}$$

and if Ω is simply connected, then the normal solution is not a global minimum.

Proof. Let ψ_n be a sequence of C^∞ functions such that

- $0 \le \psi_n \le 1$;
- $\psi_n = 0$ in a neighborhood of $\overline{\Omega}$;
- $\psi_n(x) \to 1, \forall x \notin \overline{\Omega}$;

We observe that

$$\int_{\mathbb{R}^2} ((1-\psi_n)H_e)^2\, dx \longrightarrow \int_\Omega H_e^2\, dx\,. \tag{34}$$

We can consequently choose n such that:

$$\kappa < \lambda \cdot \left(\frac{|\Omega|}{2\int_{\mathbb{R}^2}((1-\psi_n)H_e)^2\, dx} \right)^{\frac{1}{2}}\,, \tag{35}$$

We now try to find A_n such that

- $\operatorname{curl} A_n = \psi_n H_e$;
- $\operatorname{supp} A_n \cap \Omega = \emptyset$.

We have already shown how to proceed when Ω is starshaped. In the general case, we first choose $\widetilde{A}_n$ such that: $\operatorname{curl} \widetilde{A}_n = \psi_n H_e$, without the condition of support (see (2) for the argument).
We now observe that $\operatorname{curl} \widetilde{A}_n = 0$ in Ω. Using the simple connexity, we can find ϕ_n in $C^\infty(\overline{\Omega})$ such that $\widetilde{A}_n = \nabla\phi_n$. We can now extend ϕ_n outside Ω as a compactly supported C^∞ function in $\mathbb{R}^2$ $\widetilde{\phi}_n$. We then take $A_n = \widetilde{A}_n - \nabla\widetilde{\phi}_n$.
It remains to compute the energy of the pair $(1, A_n)$ (which is strictly negative) in order to achieve the proof of the proposition. □

REMARK 6. In the case when Ω is not simply connected, Proposition 8 remains true, if we replace Ω by $\widetilde{\Omega}$, where $\widetilde{\Omega}$ is the smallest simply connected open set containing Ω.

REMARK 7. It would be interesting to see how one can use the techniques of [1] for analyzing the properties of the zeros of the minimizers, when they are not normal solutions. The link between the two papers is given by the relation $\lambda = (\kappa d)^2$.

In conclusion, we have obtained, the following theorem:

THEOREM 2. *Under condition (15), there exists $\alpha_0 > 0$, such that:*

$$\left(\frac{|\Omega|}{2\int_{\widetilde{\Omega}} H_e^2\, dx} \right)^{\frac{1}{2}} \le \frac{\lambda_0^{opt}(\kappa)}{\kappa} \le \inf\left(\alpha_0, \frac{\lambda^{(1)}}{\kappa}\right). \tag{36}$$

4. Localization of pairs with small energy, in the case κ large

When κ is large and $\lambda - \lambda^{(1)}$ is small enough, we will show as in [7] that all the solutions of non positive energy of the GL-systems are in a suitable neighborhood of $(0, A_e)$ independent of $\kappa \ge \kappa_0 > 0$. This suggests that in this limiting regime these solutions of the GL-equations (if there exist and if they appear as local minima) will furnish global minimizers. Let us show this localization statement. The proof is quite similar to the proof of Theorem 1. We recall that we have (17)-(23). Now we add the condition that, for some $\eta > 0$,

$$\lambda \leq \lambda^{(1)} + \eta \,. \tag{37}$$

Note that we have already solved the problem when $\lambda \leq \lambda^{(1)} - \frac{C}{\kappa}$, so we are mainly interested in the λ's in an interval of the form $\left[\lambda^{(1)} - \frac{C}{\kappa}, \lambda^{(1)} + \eta\right]$.
The second assumption is that we consider only pairs $(u, A) \in H^1(\Omega) \times H^1_{loc}(\mathbb{R}^2)$ such that

$$G_\lambda(u, A) \leq 0\,. \tag{38}$$

We improve (23) into

$$\|(\nabla - iA)u\|^2_{L^2(\Omega)} \geq \left(\left(1 - \epsilon - \frac{C}{\epsilon}\|a\|^2_{L^4(\Omega)}\right)_+ \lambda^{(1)} - \frac{C}{\epsilon}\|a\|^2_{L^4(\Omega)}\right)\|u\|^2_{L^2(\Omega)}\,. \tag{39}$$

Taking $\epsilon = \frac{1}{\kappa}$, we get, using also (22), the existence of κ_0 and C such that, for $\lambda \in \left[0, \lambda^{(1)} + \eta\right]$ and for $\kappa \geq \kappa_0$,

$$\|(\nabla - iA)u\|^2 \geq \left(\left(1 - \frac{C}{\kappa}\right)\lambda^{(1)} - \frac{C}{\kappa}\right)\|u\|^2\,, \tag{40}$$

for any (u, A) such that $G_{\lambda,\kappa}(u, A) \leq 0$.

Coming back to (1), and, using again the negativity of the energy $G_{\lambda,\kappa}(u, A)$ of the pair (u, A), we get

$$\lambda \int_\Omega |u|^4\, dx \leq \left(\eta + \frac{C}{\kappa}\right)\|u\|^2_{L^2(\Omega)}\,. \tag{41}$$

But by Cauchy-Schwarz, we have

$$\int_\Omega |u|^2\, dx \leq |\Omega|^{\frac{1}{2}}\left(\int_\Omega |u|^4\, dx\right)^{\frac{1}{2}}\,. \tag{42}$$

So we get

$$\|u\|_{L^2(\Omega)} \leq \left(\frac{|\Omega|}{\lambda}\right)^{\frac{1}{2}}\left(\eta + \frac{C}{\kappa}\right)^{\frac{1}{2}} \tag{43}$$

We see that this becomes small with η and $\frac{1}{\kappa}$. It is then also easy to control the norm of u in $H^1(\Omega)$. We can indeed use successively (25), (26), (38) and the trivial inequality:

$$\|(\nabla - iA)u\|^2_{L^2(\Omega)} \leq \lambda\|u\|^2_{L^2(\Omega)} + G_\lambda(u, A)\,. \tag{44}$$

The control of $(A - A_e)$ in the suitable choice of gauge is also easy through (17) and (20).
Note also that if $\lambda < \lambda^{(1)}$, we obtain the better

$$\|u\|_{L^2(\Omega)} \leq \frac{C}{\kappa\lambda^{\frac{1}{2}}}\,. \tag{45}$$

So we have shown in this section the following theorem:

THEOREM 3. *There exists $\eta_0 > 0$ such that, for $0 < \eta < \eta_0$ and for $\lambda \leq \lambda^{(1)} + \eta$, then there exists κ_0 such that, for $\kappa \geq \kappa_0$, all the pairs (u, A) with negative energy are in a suitable neighborhood $\mathcal{O}\big(\eta, \frac{1}{\kappa}\big)$ of the normal solution in $H^1(\Omega, \mathbb{C}) \times H^1\big(\Omega, \mathbb{R}^2\big)$ whose size tends to 0 with η and $\frac{1}{\kappa}$.*

REMARK 8. Using the same techniques as in [7], one can also show that there are no solutions of the Ginzburg-Landau equations outside this neighborhood. This is discussed in Section 5.

5. A priori localization for solutions of Ginzburg-Landau equations

In this section, we give the proof of Remarks 3 and 8. The proof is adapted from Subsection 4.4 in [7] which analyzes the Abrikosov situation. Similar estimates can also be found in [10] (or in [1]) but in a different asymptotical regime.

We assume that (u, A) is a pair of solutions of the Ginzburg-Landau equations (3) and rewrite the second Ginzburg-Landau equation, with $A = A_e + a$ in the form:

$$La = \frac{\lambda}{\kappa^2} \operatorname{Im}\left(\bar{u} \cdot (\nabla - i(A_e + a))u\right) . \tag{46}$$

Here L is the operator defined on the space $E^2(\Omega)$, where, for $k \in \mathbb{N}^*$,

$$E^k(\Omega) := \left\{ a \in H^k\left(\Omega; \mathbb{R}^2\right) \;\middle|\; \operatorname{div} a = 0\,,\ a \cdot \nu_{/\partial\Omega} = 0 \right\} , \tag{47}$$

by

$$L = \operatorname{curl}^* \operatorname{curl} = -\Delta\,. \tag{48}$$

One can easily verify that L is an isomorphism from $E^2(\Omega)$ onto $L^2(\Omega)$. One first gets the following

LEMMA 1. *If $(u, A_e + a)$ is a solution of the GL-system (3) for some $\lambda > 0$, then we have:*

$$\|La\| \leq \frac{|\Omega|^{\frac{1}{2}} \lambda^{\frac{3}{2}}}{\kappa^2} . \tag{49}$$

Proof. We start from (46) and using Proposition 2, we obtain:

$$\|La\|^2 \leq \frac{\lambda^2}{\kappa^4} \|(\nabla - iA)u\|^2 . \tag{50}$$

Using the first GL-equation, we obtain:

$$\|La\|^2 \leq \frac{\lambda^3}{\kappa^4} \int_\Omega |u|^2 \left(1 - |u|^2\right) dx\,. \tag{51}$$

Using again Proposition 2, we obtain the lemma. □

So Lemma 1 shows, together with the properties of L, that there exists a constant C_Ω such that

$$\|a\|_{H^2(\Omega)} \leq C_\Omega \frac{\lambda^{\frac{3}{2}}}{\kappa^2} . \tag{52}$$

This permits to control the size of a when λ is small or κ is large. In particular, using Sobolev's injection Theorem, we get the existence of a constant C'_Ω such that:

$$\|a\|_{L^\infty(\Omega)} \leq C'_\Omega \frac{\lambda^{\frac{3}{2}}}{\kappa^2} . \tag{53}$$

The second step consists in coming back to our solution (u, A) of the Ginzburg-Landau equations. Let us rewrite the first one in the form:

$$-\Delta_{A_e} u = \lambda u \left(1 - |u|^2\right) - 2ia \cdot (\nabla - iA_e)u - |a|^2 u . \tag{54}$$

Taking the scalar product with u in $L^2(\Omega)$, we obtain:

$$\begin{aligned} \lambda \big\| |u|^2 \big\|^2 + \langle -\Delta_{A_e} u, u\rangle &\leq \lambda \|u\|^2 + 2\|a\|_{L^\infty} \|u\| \sqrt{\langle -\Delta_{A_e} u, u\rangle} + \|a\|^2_{L^\infty} \|u\|^2 \\ &\leq \left(\lambda + \left(1 + \frac{1}{\epsilon}\right) \|a\|^2_{L^\infty}\right) \|u\|^2 + \epsilon \langle -\Delta_{A_e} u, u\rangle . \end{aligned}$$

We have finally obtained, for any $\epsilon \in]0, 1[$ and any pair (u, A) solution of the GL-equations, the following inequality:

$$\lambda \int_\Omega |u(x)|^4 \, dx + \langle -\Delta_{A_e} u, u\rangle \leq \frac{1}{1-\epsilon} \cdot \left(\lambda + \left(1 + \frac{1}{\epsilon}\right) \|a\|^2_{L^\infty}\right) \|u\|^2 . \tag{55}$$

Forgetting first the first term of the left hand side in (55), we get the following alternative:

- Either $u = 0$,
- or
$$\lambda^{(1)} \leq \frac{1}{1-\epsilon} \cdot \left(\lambda + \left(1 + \frac{1}{\epsilon}\right) \|a\|^2_{L^\infty}\right) .$$

If we are in the first case, we obtain immediately (see (46)), the equation $La = 0$ and consequently $a = 0$. So we have obtained that (u, A) is the normal solution.

The analysis of the occurence or not of the second case depends on the assumptions done in the two remarks, through (53) and for a suitable choice of ϵ $\left(\epsilon = \frac{1}{k}\right)$. So we get immediately the existence of $\lambda_1(\kappa)$ and its estimate when $\kappa \to +\infty$. If we now assume (see (37)) that $\lambda \in \left]\lambda^{(1)} - \eta, \lambda^{(1)} + \eta\right[$, we come back to (55) and write:

$$\lambda \int_\Omega |u(x)|^4 \, dx \leq \left(\frac{1}{1-\epsilon} \cdot \left(\lambda + \left(1 + \frac{1}{\epsilon}\right) \|a\|^2_{L^\infty}\right) - \lambda^1\right) \|u\|^2 .$$

Using (42), this leads to

$$\lambda \|u\|^2 \leq \left(\frac{1}{1-\epsilon} \cdot \left(\lambda + \left(1 + \frac{1}{\epsilon}\right) \|a\|^2_{L^\infty}\right) - \lambda^{(1)}\right)_+ |\Omega| . \tag{56}$$

This shows, as in (43), that u is small in L^2 with η and $\frac{1}{\kappa}$.
We can then conclude as in the proof of Theorem 3. The control of u in H^1 is obtained through (55).

THEOREM 4. *There exists $\eta_0 > 0$ such that, for $0 < \eta < \eta_0$ and for $\lambda \leq \lambda^{(1)} + \eta$, then there exists κ_0 such that for $\kappa \geq \kappa_0$, all the solutions (u, A) of the GL-equations are in a suitable neighborhood $\mathcal{O}\big(\eta, \frac{1}{\kappa}\big)$ of the normal solution in $H^1(\Omega, \mathbb{C}) \times H^1\big(\Omega, \mathbb{R}^2\big)$ whose size tends to 0 with η and $\frac{1}{\kappa}$.*

6. About bifurcations and stability

6.1. Preliminaries

Starting from one normal solution, a natural idea is to see if, when increasing λ from 0, one can bifurcate for a specific value of λ. Proposition 4 shows that it is impossible before $\lambda^{(1)}$. A necessary condition is actually that λ becomes an eigenvalue of the Neumann realization of $-\Delta_{A_e}$ in Ω. So we shall consider what is going on at $\lambda^{(1)}$.
Note here that there is an intrinsic degeneracy to the problem related to the existence of a S^1 action. We have indeed the trivial lemma

LEMMA 2. *If (u, A) is a solution, then $(\exp i\theta u, A)$ is a solution.*

In order to go further, we add the assumption

$$\lambda^{(1)} \text{ is a simple eigenvalue.} \tag{57}$$

In this case, we denote by u_1 a corresponding normalized eigenvector.

Now, one can try to apply the general bifurcation theory due to Crandall-Rabinowitz. Note that, although, the eigenvalue is assumed to be simple, it is not exactly a simple eigenvalue in the sense of Crandall-Rabinowitz which are working with real spaces. Actually, this is only simple modulo this S^1-action. We are not aware of a general theory dealing with this situation in full generality (see however [11]) but special cases involving Schrödinger operators with magnetic field are treated in [17], [2] and [7]. The article [2] is devoted to the case of the disk and [17] (more recently [7]) to the case of Abrikosov's states.
All the considered operators are (relatively to the wave function or order parameter) suitable realizations of operators of the type

$$u \mapsto -\Delta_A u - \lambda f\big(|u|^2\big)u\,,$$

with $f(0) = 1$.

The main theorem is the following:

THEOREM 5. *Under the assumptions (15) and (57), there exist ϵ_0 and a bifurcating family of solutions $(u(\cdot\,;\alpha), A(\cdot\,;\alpha), \lambda(\alpha))$ in $H^1(\Omega,\mathbb{C}) \times E^1(\Omega) \times \mathbb{R}^+$, with $\alpha \in D(0,\epsilon_0) \subset \mathbb{C}$ for the Ginzburg-Landau equations such that*

$$\begin{aligned} u(\cdot\,;\alpha) &= \alpha u_1 + \alpha|\alpha|^2 u^{(3)}(\cdot\,;\alpha)\,, \qquad \textit{with } \big\langle u_1, u^{(3)}\big\rangle = 0\,,\\ A(\cdot\,,\alpha) &= A_e + |\alpha|^2 a_2 + |\alpha|^4 a^{(4)}(\cdot\,;\alpha)\,,\\ \lambda(\alpha) &= \lambda^{(1)} + c(\kappa)|\alpha|^2 + \mathcal{O}\big(|\alpha|^4\big)\,. \end{aligned} \tag{58}$$

Here $u^{(3)}(\cdot\,;\alpha)$ and $a^{(4)}(\cdot\,;\alpha)$ are bounded in H^1.
This solution satisfies, $\forall s \in \mathbb{C}$, $|s| = 1$:

$$u(\cdot\,;s\,\alpha) = s\,u(\cdot\,;\alpha)\,, \quad A(\cdot\,;s\,\alpha) = A(\cdot\,;\alpha)\,. \tag{59}$$

Moreover, if $c(\kappa) \neq 0$, all the solutions (u, A, λ) of the Ginzburg-Landau equations lying in a sufficiently small neighborhood in $H^1 \times E^1 \times \mathbb{R}^+$ of $\big(0, A_e, \lambda^{(1)}\big)$ are described by the normal solutions $(0, A_e, \lambda)$ and the bifurcating solutions.

The constant $c(\kappa)$ will be explicited in the next subsection.

6.2. About the proof, construction of formal solutions

The starting point is the GL-system written in the form

$$\begin{aligned}\big(-\Delta_{A_e}-\lambda^{(1)}\big)u &= \big(\lambda-\lambda^{(1)}\big)u-\lambda u|u|^2-2ia\cdot(\nabla-iA_e)u-\|a\|^2u\\ La &= \frac{\lambda}{\kappa^2}\mathrm{Im}\,(\bar u\cdot(\nabla-iA)u)\end{aligned} \tag{60}$$

We then use the standard method. We look for a solution in the form

$$\begin{aligned}u &= \alpha u_1+\alpha|\alpha|^2u_3+\mathcal{O}\big(|\alpha|^5\big)\,,\\ a &= |\alpha|^2\,a_2+\mathcal{O}\big(|\alpha|^4\big)\end{aligned}$$

and

$$\lambda(\alpha)=\lambda^{(1)}+c(\kappa)\,|\alpha|^2+\mathcal{O}\big(|\alpha|^4\big)\,.$$

We can eliminate the S^1-degeneracy by imposing α real (keeping only the parity). We refer to [7] for details and just detail the beginning of the formal proof which gives the main conditions. We first obtain, using the second equation,

$$a_2=\frac{\lambda^{(1)}}{\kappa^2}b_2\,, \tag{61}$$

with

$$b_2:=L^{-1}\mathrm{Im}\,(\bar u_1\cdot(\nabla-iA_e)u_1)\,. \tag{62}$$

Taking then the scalar product in L^2 with u_1, in the first equation, we get that

$$c(\kappa)=\lambda^{(1)}\left(I_0-\frac{2}{\kappa^2}K_0\right), \tag{63}$$

with

$$I_0:=\int_\Omega|u_1(x)|^4\,dx\,, \tag{64}$$

and

$$K_0=-\langle ib_2\cdot(\nabla-iA_e)u_1,u_1\rangle\,. \tag{65}$$

REMARK 9. From (63), we immediately see that there exists κ_1 such that, for $\kappa\geq\kappa_1$, $c(\kappa)>0$. Moreover, the uniqueness statement in Theorem 5 is true in a neighborhood which can be chosen independently of $\kappa\in[\kappa_1,+\infty[$.

Let us now observe, that, b_2 being divergence free, it is immediate by integration by part that K_0 is real. Computing $\mathrm{Re}\,K_0$, we immediately obtain:

$$K_0=\mathrm{Re}\,K_0=\langle L^{-1}J_1,J_1\rangle\,, \tag{66}$$

where J_1 is the current:

$$J_1:=\mathrm{Im}\,(\bar u_1\cdot(\nabla-iA_e)u_1)\,. \tag{67}$$

We observe that $K_0 > 0$ if and only if J_1 is not identically 0. In the non simply connected case, we shall find a case when $J_1 = 0$. (See Lemma 5).
Following the argument of [7] (Lemme 3.4.9), let us analyze the consequences of $J_1 = 0$. By assumption u_1 does not vanish identically. If $u_1(x_0) \neq 0$, then we can perform in a sufficiently small ball $B(x_0, r_0)$ centered at x_0, the following computation in polar coordinates. We write $u_1 = r(x) \exp i\theta(x)$ and get $J_1 = r(x)^2(A_e - \nabla\theta) = 0$. So $A_e = \nabla\theta$ in this ball and this implies $H_e = 0$ in the same ball. Using the properties of the zero set of u_1 in Ω [9] and the continuity of H_e, we then obtain $H_e = 0$ in Ω. But we know that, if Ω is simply connected, then this implies $\lambda^{(1)} = 0$. So we have the following lemma

LEMMA 3. *If Ω is simply connected and $\lambda^{(1)} > 0$, then $K_0 > 0$.*

Coming back to the first equation and projecting on the orthogonal space to u_1 in $L^2(\Omega)$ $u_1^\perp$, we get:

$$u_3 = R_0 v_3 \,, \tag{68}$$

where v_3 is orthogonal to u_1 and given by:

$$v_3 := 2a_2 \cdot ((\nabla - iA_e)u_1) \,, \tag{69}$$

and R_0 is the inverse of $\big(- \Delta_{A_e} - \lambda^{(1)}\big)$ on the space $u_1^\perp$ and satisfies

$$R_0 u_1 = 0 \,.$$

We emphasize that all this construction is uniform with the parameter $\beta = \frac{1}{\kappa}$ in $]0, \beta_0]$. One can actually extend analytically the equation in order to have a well defined problem in $[-\beta_0, \beta_0]$.

6.3. About the energy along the bifurcating solution

The proof is an adaptation of [7]. Let us just present here the computation of the value of the GL-functional along the bifurcating curve. Although it is not the proof, this gives the right condition for the stability. For this, we observe that if $(u, A_e + a)$ is a solution of the GL-system, then we have:

$$G_{\lambda,\kappa}(u, A) = -\frac{\lambda}{2} \int_\Omega |u|^4 + \frac{\kappa^2}{\lambda} \int_\Omega |\mathrm{curl}\, a|^2 \, dx \,. \tag{70}$$

It is then easy to get the main term of the energy of the function for $(u, A_e + a)$ with $a(\,\cdot\,; \alpha) = |\alpha|^2 a_2(\,\cdot\,) + \mathcal{O}\big(|\alpha|^4\big)$ and $u(\,\cdot\,; \alpha) = \alpha u_1(\,\cdot\,) + \mathcal{O}\big(|\alpha|^3\big)$.

$$G_{\lambda,\kappa}(u(\cdot;\alpha), A(\cdot;\alpha)) = |\alpha|^4 \left(-\frac{\lambda^{(1)}}{2} \int_\Omega |u_1|^4 + \frac{\kappa^2}{\lambda^{(1)}} \int_\Omega |\mathrm{curl}\, a_2|^2 \, dx \right) + \mathcal{O}\big(|\alpha|^6\big) \,. \tag{71}$$

Let us first analyze the structure of the term:

$$K_1 := \frac{\kappa^2}{\lambda^{(1)}} \int_\Omega |\mathrm{curl}\, a_2|^2 \, dx = \frac{\kappa^2}{\lambda^{(1)}} \langle L a_2, a_2 \rangle \,. \tag{72}$$

But we have:

$$K_1 := \frac{\lambda^{(1)}}{\kappa^2}\langle Lb_2, b_2\rangle = \frac{\lambda^{(1)}}{\kappa^2}\langle L^{-1} J_1, J_1\rangle = \frac{\lambda^{(1)}}{\kappa^2} K_0\,. \tag{73}$$

With these expressions, we get

$$G_{\lambda(\alpha),\kappa}(u(\,\cdot\,;\alpha), A(\,\cdot\,;\alpha)) = -|\alpha|^4 \cdot \frac{\lambda^{(1)}}{2}\left(I_0 - \frac{2}{\kappa^2} K_0\right) + \mathcal{O}\big(|\alpha|^6\big)\,. \tag{74}$$

So we get that the energy becomes negative along the bifurcating solution for $0 < |\alpha| \leq \rho_0$, if the following condition is satisfied:

$$\kappa^2 > 2\frac{K_0}{I_0}\,. \tag{75}$$

Another way of writing the result is:

PROPOSITION 9. *Under conditions (15) and (57), then, if*

$$\kappa^2 \neq 2\frac{K_0}{I_0}\,, \tag{76}$$

there exists $\alpha_0 > 0$ *such that, for all* α *satisfying* $0 < |\alpha| \leq \alpha_0$,

$$\big(\lambda(\alpha) - \lambda^{(1)}\big) G_{\lambda(\alpha),\kappa}(u(\,\cdot\,;\alpha), A(\,\cdot\,;\alpha)) < 0\,. \tag{77}$$

In particular, we have shown, in conjonction with Theorem 3, the following theorem:

THEOREM 6. *There exists* $\eta > 0$ *and* κ_0*, such that, for* $\kappa > \kappa_0$ *and* $\lambda \leq \lambda^{(1)} + \eta$*, the global minimum of* $G_{\lambda,\kappa}$ *is realized by the normal solution for* $\lambda \in \big]0, \lambda^{(1)}\big]$ *and by the bifurcating solution for* $\lambda \in \big]\lambda^{(1)}, \lambda^{(1)} + \eta\big]$.

In particular, and taking account of Remark 5, we have:

COROLLARY 1. *There exists* κ_c *such that the map* $\kappa \mapsto \lambda_0^{opt}(\kappa)$ *is an increasing function from* 0 *to* $\lambda^{(1)}$ *for* $\kappa \in [0, \kappa_c]$ *and is constant and equal to* $\lambda^{(1)}$ *for* $\kappa \geq \kappa_c$.

REMARK 10. Note that Theorem 4 gives an additional information. For η small enough and κ large enough, there are actually no other solutions of the GL-equation.

6.4. Stability

The last point is to discuss the stability of the bifurcating solution. We expect that the bifurcating solution gives a local minimum of the GL-functional for κ large enough, and more precisely under condition (75). The relevant notion is here the notion of strict stability. Following [2], we say that (u, A) (with u not identically 0) is strictly stable for $G_{\lambda,\kappa}$ if it is a critical point, if its Hessian is positive and if its kernel in $H^1 \times E^1$ is the one dimensional space $\mathbb{R}(iu, 0)$.
We then have the following theorem:

THEOREM 7. *Under conditions (15), (57), and if (75) is satisfied, then there exists* $\epsilon_0 > 0$*, such that, for* $0 < |\alpha| \leq \epsilon_0$*, the solution* $(u(\,\cdot\,;\alpha), A(\,\cdot\,;\alpha))$ *is strictly stable.*

We refer to [7] for the detailed proof.

7. Bifurcation from normal solutions: special case of non simply connected models

7.1. Introduction

In this section, we revisit the bifurcation problem in the case when Ω is not simply connected and when the external field vanishes inside Ω. In this very particular situation which was considered by J. Berger and J. Rubinstein in [3] (and later in [13], [14]), it is interesting to make a deeper analysis leading for example to the description of the nodal sets of the bifurcating solution. The situation is indeed quite different of the results obtained by [9] in a near context (but with a simply connected Ω). We mainly follow here the presentation in [14] (for which we refer for other results or points of view) but emphasize on the link with the previous section.

7.2. The operator K

We shall now consider the specific problem introduced by [3] and consider the case

$$\operatorname{supp} H_e \cap \overline{\Omega} = \emptyset\,, \tag{78}$$

and, in any hole $\mathcal{O}_i$ of Ω, the flux of H_e satisfies

$$\frac{1}{2\pi}\int_{\mathcal{O}_i} H_e \in \mathbb{Z} + \frac{1}{2}\,. \tag{79}$$

We recall in this context, what was introduced in [13]. We observe that under conditions (78) and (79), there exists a multivalued function ϕ such that $\exp i\phi \in C^\infty(\overline{\Omega})$ and

$$d\phi = 2\omega_A\,, \tag{80}$$

where ω_A is the 1-form naturally attached to the vector A.
We also observe that, for the complex conjugation operator Γ

$$\Gamma u = \bar{u}\,, \tag{81}$$

we have the general property

$$\Gamma \Delta_A = \Delta_{-A}\Gamma\,. \tag{82}$$

Combining (80) and (82), we obtain, for the operator

$$K := (\exp -i\phi)\,\Gamma\,, \tag{83}$$

which satisfies

$$K^2 = Id\,, \tag{84}$$

the following commutation relation

$$K\,\Delta_A = \Delta_A\,K\,. \tag{85}$$

Let us also observe that the Neumann condition is respected by K.
As a corollary, we get

LEMMA 4. *If v is an eigenvector of Δ_A^N, then Kv has the same property.*

This shows that one can always choose an orthonormal basis of eigenvectors u_j such that $Ku_j = u_j$.

7.3. Bifurcation inside special classes

Following [3] (but inside our point of view), we look for solution of the GL-equation in the form (u, A_e) with $Ku = u$. Let us observe that

$$L^2_K(\Omega;\mathbb{C}) := \left\{u \in L^2(\Omega;\mathbb{C}) \;\middle|\; Ku = u\right\}, \tag{86}$$

is a real Hilbert subspace of $L^2(\Omega;\mathbb{C})$.
We denote by H^m_K the corresponding Sobolev spaces:

$$H^m_K(\Omega;\mathbb{C}) = H^m(\Omega;\mathbb{C}) \cap L^2_K\,. \tag{87}$$

We now observe the

LEMMA 5. *If $u \in H^1_K$, then* $\operatorname{Im}(\bar{u}\cdot(\nabla - iA_e)u) = 0$ *almost everywhere.*

Proof. Let us consider a point where $u \neq 0$. Then we have $u = \rho \exp i\theta$ with $2\theta = \phi$ modulo $2\pi\mathbb{Z}$. Remembering that $A_e = \frac{1}{2}\nabla\phi$, it is easy to get the property. □

Once this lemma is proved, one immediately sees that (u, A_e) (with $Ku = u$) is a solution of the GL-system if and only if $u \in H^1_K$ and

$$\begin{aligned} &-\Delta_{A_e} u - \lambda u\left(1 - |u|^2\right) = 0\,,\\ &(\nabla - iA_e)u\cdot\nu = 0\,, \quad \text{on } \partial\Omega\,. \end{aligned} \tag{88}$$

We shall call this new system the reduced GL-equation. But now we can apply the theorem by Crandall-Rabinowitz [6]. By assumption (57), the kernel of $\left(-\Delta_{A_e} - \lambda^{(1)}\right)$ is now a one-dimensional real subspace in L^2_K. Let us denote by u_1 a normalized "real" eigenvector. Note that u_1 is unique up to multiplication by ± 1. Therefore, we have the

THEOREM 8. *Under assumptions (57), (78) and (79), there exists a bifurcating family of solutions $(u(\,\cdot\,;\alpha), \lambda(\alpha))$ in $H^1_K \times \mathbb{R}^+$ with $\alpha \in]-\epsilon_0, +\epsilon_0[$, for the reduced GL-equation such that*

$$\begin{aligned} &u(\alpha) = \alpha\, u_1 + \alpha^3\, v(\alpha)\,,\\ &\langle u_1, v(\alpha)\rangle_{L^2} = 0\,,\\ &\|v(\alpha)\|_{H^2(\Omega)} = \mathcal{O}(1)\,, \end{aligned} \tag{89}$$

$$\lambda(\alpha) = \lambda^{(1)} + c\alpha^2 + \mathcal{O}\left(\alpha^4\right), \tag{90}$$

with

$$c = \lambda^{(1)}\cdot \int_\Omega |u_1|^4\, dx\,. \tag{91}$$

Moreover

$$u(-\alpha) = -u(\alpha)\,, \quad \lambda(-\alpha) = \lambda(\alpha)\,. \tag{92}$$

REMARK 11. Note that the property (92) is what remains of the S^1-invariance when one considers only "real" solutions.

Let us give here the formal computations of the main terms. If we denote by L_0 the operator $L_0 := -\Delta_{A_e} - \lambda^{(1)}$, writing $v(\alpha) = u_3 + \mathcal{O}(\alpha)$, we get:

$$\big(L_0 - c\alpha^2\big)\big(\alpha u_1 + \alpha^3 u_3\big) + \big(\lambda^{(1)}\big)\alpha^3 u_1 |u_1|^2 = \mathcal{O}\big(\alpha^4\big)\,.$$

Projecting on u_1, we get (91). Projecting on $u_1^{\perp}$ and denoting by R_0 the operator equal to the inverse of L_0 on this subspace and to 0 on Ker L_0, we get

$$u_3 = -\lambda^{(1)} R_0\big(u_1|u_1|^2\big) = -\lambda^{(1)} R_0\big(u_1|u_1|^2 - cu_1\big)\,. \tag{93}$$

REMARK 12. By the uniqueness part in Theorem 5, we see that the solution $(u(\,\cdot\,;\alpha), A_e)$ is actually the solution given in this theorem.

Another remark is that

$$G_{\lambda(\alpha),\kappa}(u(\alpha), A_e) = -\frac{\lambda^{(1)}}{2}\cdot\alpha^4\left(\int_\Omega |u_1(x)|^4\,dx\right) + \mathcal{O}\big(|\alpha|^6\big)\,, \tag{94}$$

so that when $\alpha \neq 0$ the energy is decreasing. This is of course to compare with (72) (note that we have $K_0 = 0$). Once we have observed this last property, the local stability of the bifurcated solution near the bifurcation is clear.

The second result we would like to mention concerns the nodal sets. In the case when Ω is simply connected, the analysis of the nodal set of u when (u, A) is a minimizer of the GL-functional is done in [9], using the analyticity of the solutions of the GL-equation and techniques of Courant.

In the non simply connected case, very few results are known. The following theorem is true [3], [14]:

THEOREM 9. *Under assumptions (22), (78) and (79), there exists $\epsilon_1 > 0$ such that, for any $\alpha\, \in]0, \epsilon_1]$, the nodal set of $u(\alpha)$ in H^1_K slits $\overline{\Omega}$ in the sense of [13]. In particular, if there is only one hole, then the nodal set of $u(\alpha)$ consists exactly in one line joining the interior boundary and the exterior boundary.*

An elegant way to recover these results (see [13], [14]) is to lift the situation to a suitable two-fold covering $\Omega^{\mathcal{R}}$.

8. Appendix: Analysis of the various scalings

When considering asymptotical regimes, it is perhaps useful to have an interpretation in terms of the initial variables. According to the statistical interpretation of the Ginzburg-Landau functional (see for example [4]), the starting point is the functional $\big(\tilde{v}, \widetilde{A}\big) \mapsto \mathcal{F}\big(\tilde{v}, \widetilde{A}\big)$ with:

$$\begin{aligned}
\mathcal{F}\big(\tilde{v}, \widetilde{A}\big) \quad := \quad & \frac{1}{8\pi}\int_{\mathbb{R}^2} \big|\text{curl}\,\widetilde{A} - \widetilde{H}_e\big|^2\,d\tilde{x} \\
& + \int_\Omega \frac{\hbar^2}{4m}\left|\left(\nabla - i\frac{2e}{c}\widetilde{A}\right)\tilde{u}\right|^2 d\tilde{x} \\
& + \int_\Omega \left(a|\tilde{u}|^2 + \frac{b}{2}|\tilde{u}|^4\right) d\tilde{x}\,.
\end{aligned}$$

Here a is a parameter which is proportional to $(T - T_c)$ (we are only interested in the case $a < 0$) and b is essentially independent of the temperature. The other parameters are standard: $\hbar = \frac{h}{2\pi}$, h is the Planck constant, e is the charge of the electron and m is the mass of the electron. With $u = \frac{b}{|a|}\tilde{u}$ and $A = \frac{2e}{c}\widetilde{A}$, we obtain:

$$\mathcal{F}\left(\tilde{v}, \widetilde{A}\right) = \frac{|a|\hbar^2}{4mb} G_{\lambda,\kappa}(u, A)\,,$$

with $H_e = \frac{2e}{\hbar c}\widetilde{H}_e$, $\lambda = \frac{4m|a|}{\hbar^2}$ and $\kappa = \frac{mc}{e\hbar}\left(\frac{b}{8\pi}\right)^{\frac{1}{2}}$. Here we emphasize that between the two functionals, no change of space variables is involved.

Let now compare with another standard representation of the Ginzburg-Landau functional. We make this time the change of variables $x = \frac{\kappa}{\sqrt{\lambda}}\hat{x}$ and if we change u and the $1-$form corresponding to A accordingly, we obtain the standard functional:

$$\mathcal{E}\left(\hat{u}, \widehat{A}\right) = G_{\lambda,\kappa}(u, A)\,,$$

with

$$\begin{aligned}\mathcal{E}\left(\hat{u}, \widehat{A}\right) \quad = \quad & \kappa^2 \int_{\widehat{\Omega}} \left(-|\hat{u}|^2 + \frac{1}{2}|\hat{u}|^4\right) d\hat{x} \\ & + \int_{\widehat{\Omega}} \left|\left(\nabla - i\widehat{A}\right)\hat{u}\right|^2 d\hat{x} \\ & + \int_{\widehat{\Omega}} \left|\operatorname{curl}\widehat{A} - \widehat{H}_e\right|^2 d\hat{x}\,,\end{aligned}$$

with

$$\begin{aligned}\widehat{H}_e &= \frac{\kappa^2}{\lambda} H_e\,,\\ \widehat{\Omega} &= \frac{\sqrt{\lambda}}{\kappa}\Omega\,.\end{aligned}$$

Here we observe that the open set Ω is not conserved in the transformation. We have to keep this in mind when comparing in the limit $\kappa \to +\infty$ the contributions of Sandier and Serfaty [19] or [15] with the results presented in this paper.

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