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ANISOTROPIC AND DISSIPATIVE FINITE ELASTO-PLASTIC COMPOSITE

Abstract. Here we propose a macroscopic model for elasto-plastic composite, characterized by an initial anisotropy, that can evolve during the large plastic deformation. Application to transversely isotropic and orthotropic composites will be also developed. The paper deals with anisotropic finite elasto-plastic Σ-models, which accounts for the dissipative nature of the plastic flow, within the constructive framework of materials with relaxed configurations in internal variables. Here Σ stands for Mandel's non-symmetric stress tensor,or the quasi-static Eshelby stress-tensor. The appropriate variational inequalities are derived, related rate quasi-static boundary value problem, in our approach to composite materials.

1. Introduction

The continuum approach treats the composites as a single material with different properties in different directions. The macroscopic response will be transversely isotropic about the fiber direction if there exists just one family of reinforced fibres and orthotropic if there are two families. Spencer in [23] formulated yield conditions, flow rules and hardening rules for material reinforced by one and two families of fibres, in small deformations plasticity theory. The yield function is assumed to be not affected by a superposed tension in fibre direction. Spencer in [22] proposed the term of proportional hardening for the corresponding theory of isotropic hardening, for anisotropic plasticity. Rogers in [21] generalized Spencer's results concerning fibre reinforced materials, assuming that the yield condition is unaffected by the superposition of an arbitrary hydrostatic pressure.

Experimental results performed on axially reinforced tubular specimens of boron aluminium composite, under complex loading, reveal the large kinematic hardening effects, see [20]. In [26] the effect of shear on the compressive response and failure was investigated experimentally for an unidirectional composite. Here both axes of loading could be operated in either load or displacement control.

Here we propose a macroscopic model for elasto-plastic composite, characterized by an initial anisotropy, that can evolve during the large plastic deformation. Applications to transversely isotropic and orthotropic composites will be developed, based on the papers [5, 6], which generalized Spencer and Roger's results.

The paper deals with anisotropic finite elasto-plastic $\Sigma-$models, which account for the dissipative nature of the plastic flow, within the constitutive framework of materials with relaxed configurations and internal variables, [1, 2]. Here Σ stands for Mandel's non-symmetric stress tensor, see [15], or the quasi-static Eshelby stress tensor, see [17, 18]. We shown in [9], that

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there exist classes of $\Sigma-$models with hyperelastic properties, for which the dissipation postulate [7] can be equivalently imposed through the normality and convexity properties, despite of the non-injectivity of the function which describes Σ as dependent on elastic strain. Our dissipation postulate extend to anisotropic materials the results obtained by [13, 14, 16, 24].

During the elasto-plastic deformation process, see experimental evidences in [26], the changes in geometry and rotations of material elements cannot be disregarded. Consequently, the field equation and the boundary conditions at time t are properly formulated (see [11]) in terms of the rate of the *nominal* stress. The second objective of the paper is to derive an appropriate variational inequality, related to the rate quasi-static boundary value problem and associated with a generic stage of the process in our approach to composite materials. Only when the dissipative nature of the plastic flow is considered, the variational inequality is caracterized by a bilinear form which becomes symmetric. In a forcomming paper a complete analysis of the bifurcation of the homogeneous deformation will be performed, as in Cleja-Ţigoiu [4], based on the variational inequality, under axial compressive stress. In our analyse it is not necessary to make the assumptions either the fibres are uniformly inclined to the line of the loading by a small angle, or the existence of a sinusoidal imperfection, which is uniformly distributed, as we remark here that the stability can be lost, during plastic deformation.

Further we shall use the following notations:

Lin, Lin^+- the second order tensors and the elements with positive determinant;

$\mathcal{V}-$ the three dimensional vector space;

$Sym, Skew, Sym^+-$ symmetric, skew-symmetric and symmetric and positive definite tensors;

Ort^+- all proper rotation of the orthogonal group Ort;

$\mathbf{A}\cdot\mathbf{B} := \mathrm{tr}\, \mathbf{A}\mathbf{B}^T-$ the scalar product of $\mathbf{A}, \mathbf{B} \in Lin$;

$\mathbf{A}^s = \frac{1}{2}(\mathbf{A}+\mathbf{A}^T)$ and $\mathbf{A}^a = \frac{1}{2}(\mathbf{A}-\mathbf{A}^T)-$ the symmetrical and respectively skew- symmetrical parts of $\mathbf{A} \in Lin$; $\mathbf{I}$ is the identity tensor;

$\mathcal{E}^T-$ the transpose of $\mathcal{E}-$ fourth order tensor, defined for all $\mathbf{A}, \mathbf{B} \in Lin$ by $\mathcal{E}^T\mathbf{A}\cdot\mathbf{B} := \mathbf{A}\cdot\mathcal{E}\mathbf{B}$;

$\dot{\mathbf{u}}-$ represents the derivative with respect to time; $\partial_{\mathbf{G}}\, \varphi(\mathbf{G},\alpha)-$ the partial derivative of the function $\varphi(\mathbf{G},\alpha)$ with respect to $\mathbf{G}$;

$d\,\hat{\Sigma}(\mathbf{G})-$ the differential of the map $\hat{\Sigma}$ at $\mathbf{G}$;

$\mathbf{A}\cdot\mathbf{B} := \mathrm{tr}\, \mathbf{A}\mathbf{B}^T-$ the scalar product of $\mathbf{A}, \mathbf{B} \in Lin$; $\mid \mathbf{A} \mid = \sqrt{\mathbf{A}\cdot\mathbf{A}} \equiv \sqrt{A_{ij}A_{ij}}$ the modulus of the second order tensor and A_{ij} denote its Cartesian components; $|\mathcal{E}|_4 = \sqrt{\sum_{ijkl}\mathcal{E}_{ijkl}^2}$ denotes the modulus of fourth order tensor and $\mathcal{E}_{ijkl}$ are Cartesian components of $\mathcal{E}$;

$< z >= 1/2\,(z+\mid z \mid),\ \forall z \in \mathbf{R}-$ the set of all real numbers;

$\rho_0, \tilde{\rho}, \rho$ are mass densities in initial, relaxed and actual configurations;

$\mathbf{Q}[\alpha] := \mathbf{Q}\alpha\mathbf{Q}^T$ for $\alpha \in Lin$, $\mathbf{Q}[\alpha] := \alpha$ for $\alpha \in R$.

2. Σ-models

We introduce now the constitutive framework of anisotropic elasto-plastic materials, $\Sigma-$models being included, see [8].

We fix a material point $\mathbf{X}$ in the body, considered in the reference configuration k. For an arbitrary given motion χ, defined in a certain neighborhood of $\mathbf{X}$, let consider the deformation gradient $\mathbf{F}(t)$, $\det \mathbf{F}(t) > 0$, $\mathbf{F}(0) = \mathbf{I}$. We assume the *multiplicative decomposition* of the deformation gradient into its *elastic* and *plastic* parts:

$$\mathbf{F}(t) = \mathbf{E}(t)\mathbf{P}(t) \quad \text{where} \quad \mathbf{E}(t) = \nabla\chi(\mathbf{X}, t)\mathbf{K}_t^{-1}, \ \mathbf{P}(t) = \mathbf{K}_t\mathbf{K}_0^{-1} \tag{1}$$

based on the local, current configuration $\mathbf{K}_t$.

We denote by $\mathbf{G} = \mathbf{E}^T\mathbf{E}$ the elastic strain, and by $\mathbf{Y} = (\mathbf{P}^{-1}, \alpha)$ the set of the irreversible variables, where α represent the set of internal variables, scalars and tensors, $\Pi-$ symmetric Piola-Kirchhoff stress tensor $\mathbf{K}_t$, $\mathbf{T}-$ Cauchy stress tensor, related by

$$\frac{\Pi}{\tilde{\rho}} = \mathbf{E}^{-1}\frac{\mathbf{T}}{\rho}\mathbf{E}^{-T}$$

The *elastic type constitutive* in term of Σ is written under the form

$$\begin{array}{l} \Sigma := \hat{\Sigma}(\mathbf{G}, \alpha), \quad \hat{\Sigma}(\mathbf{I}, \alpha) = 0, \\ \mathbf{G}^{-1}\hat{\Sigma}(\mathbf{G}, \alpha) = \hat{\Sigma}^T(\mathbf{G}, \alpha)\mathbf{G}^{-1}, \ \forall\, \mathbf{G} \in Sym^+. \end{array} \tag{2}$$

The value of the tensor function written in $(2)_2$ gives the current value of $\dfrac{\Pi}{\tilde{\rho}}$, taking into account the relation between symmetric Piola-Kirchhoff and Mandel's stress tensors

$$\Sigma = \mathbf{G}\frac{\Pi}{\tilde{\rho}}$$

The *rate independent evolution eqns.* for $\mathbf{P}, \alpha$ are expressed by

$$\begin{array}{l} \dot{\mathbf{P}}\mathbf{P}^{-1} = \mu\,\hat{\mathcal{B}}(\Sigma, \alpha), \quad \dot{\alpha} = \mu\,\hat{\mathbf{m}}(\Sigma, \alpha), \\ \hat{\mathcal{F}}(\cdot, \alpha) : \mathcal{D}_{\hat{\mathcal{F}}} \subset Lin \longrightarrow \mathbf{R}_{\leq 0}, \quad \text{and} \quad \hat{\mathcal{F}}(0, \alpha) < 0, \\ \mu \geq 0, \quad \mu\,\hat{\mathcal{F}} = 0, \quad \text{and} \quad \mu\,\dot{\hat{\mathcal{F}}} = 0. \end{array}$$

Material symmetry requirements (see [1, 3]). We assume that the *preexisting material symmetry* is characterized by *the symmetry group* $g_k \subset Ort^+$, that renders the material functions invariant

$$\hat{\Sigma}(\mathbf{QGQ}^T, \mathbf{Q}[\alpha]) = \mathbf{Q}\hat{\Sigma}(\mathbf{G}, \alpha)\mathbf{Q}^T \quad , \quad \hat{\mathcal{F}}(\mathbf{Q}\Sigma\mathbf{Q}^T, \mathbf{Q}[\alpha]) = \hat{\mathcal{F}}(\Sigma, \alpha),$$

$$\hat{\mathcal{B}}(\mathbf{Q}\Sigma\mathbf{Q}^T, \mathbf{Q}[\alpha]) = \mathbf{Q}\hat{\mathcal{B}}(\Sigma, \alpha)(\mathbf{Q})^T, \quad \hat{\mathbf{m}}(\mathbf{Q}\Sigma\mathbf{Q}^T, \mathbf{Q}[\alpha]) = \mathbf{Q}[\hat{\mathbf{m}}(\mathbf{G}, \alpha)]$$

for every $\mathbf{Q} \in g_k$.

THEOREM 1. *Any $\Sigma-$ model leads to a strain formulation of the elasto- plastic behaviour of the material with respect to the relaxed configuration $\mathbf{K}_t$. Also the material functions are g_k- invariant.*

The appropriate material functions in strain formulations are related to the basic functions from $\Sigma-$models through relationships of the type:

$$\widetilde{\mathcal{F}}(\mathbf{G}, \alpha) = \hat{\mathcal{F}}(\hat{\Sigma}(\mathbf{G}, \alpha), \alpha), \quad \widetilde{\mathcal{B}}(\mathbf{G}, \alpha) = \hat{\mathcal{B}}(\hat{\Sigma}(\mathbf{G}, \alpha), \alpha), \quad \textit{etc.}$$

THEOREM 2 (STRAIN FORMULATION IN THE INITIAL CONFIGURATION).
1. Let $\mathbf{Y} := (\mathbf{P}^{-1}, \alpha)$ characterizes the irreversible behaviour of the body, at the fixed material point. The yield function in the reference configuration associated with the yield function in elastic strain is defined by

$$\mathcal{F}(\mathbf{C}, \mathbf{Y}) := \widetilde{\mathcal{F}}(\mathbf{P}^{-T}\mathbf{C}\mathbf{P}^{-1}, \alpha) \equiv \widetilde{\mathcal{F}}(\mathbf{G}, \alpha) \quad \textit{with}\ \mathbf{Y} \equiv (\mathbf{P}^{-1}, \alpha)$$

as a consequence of (1).

2. The evolution in time of $\mathbf{Y}$ *is governed by the solutions of Cauchy problem (see* [1]*)*

$$
(3)\qquad
\begin{aligned}
&\dot{\mathbf{Y}} = - < \beta(t, \mathbf{Y}) > \ \bar{\mathcal{Y}}(\mathbf{C}(t), \mathbf{Y}) H(\mathcal{F}(\mathbf{C}(t), \mathbf{Y}))\\
&\beta(t, \mathbf{C}) = \partial_{\mathbf{C}} \mathcal{F}(\mathbf{C}(t), \mathbf{Y}) \cdot \dot{\mathbf{C}}(t)\\
&\partial_{\mathbf{Y}} \bar{\mathcal{F}}(\mathbf{C}, \mathbf{Y}) \cdot \bar{\mathcal{Y}}(\mathbf{C}, \mathbf{Y}) = 1 \quad on \quad \mathcal{F}(\mathbf{C}, \mathbf{Y}) = 0\\
&\mathbf{Y}(0) = \mathbf{Y}_0
\end{aligned}
$$

for a given strain history, denoted $\hat{\mathbf{C}} \in \mathcal{G}_s$,

$$
t \in [0, d] \rightarrow \hat{\mathbf{C}}(t) \in Sym^+, \quad with \quad \hat{\mathbf{C}}(t) = \mathbf{C}(t) = \mathbf{F}^T(t)\mathbf{F}(t).
$$

Here H denotes the Heaviside function.
Basic assumptions:

I. There exists an unique solution of the Cauchy problem (3).

II. The smooth yield function $\widetilde{\mathcal{F}}$ is given in such way that

i) $\widetilde{\mathcal{F}} : \mathcal{D}_{\mathcal{F}} \subset Sym^+ \times R^n \longrightarrow R$ is of the class C^1, and $\widetilde{\mathcal{F}}(\mathbf{I}, \alpha) < 0$ for all α;

ii) for all fixed $\alpha \in pr_2\, \mathcal{D}_{\mathcal{F}}-$ the projection on the space of internal variables, the set

$$
\{\mathbf{G} \in Sym^+ \ | \ \widetilde{\mathcal{F}}(\mathbf{G}, \alpha) \leq 0\}
$$

is the closure of a non-empty, connected open set, i.e. if necessary we restrict the yield function to the connected set that contains $\mathbf{I} \in pr_1\, \mathcal{D}_{\mathcal{F}} \subset Sym^+$;

iii) for all $\alpha \in pr_2\mathcal{D}_{\mathcal{F}}$ the set $\{\mathbf{G} \in Sym^+ \ | \ \widetilde{\mathcal{F}}(\mathbf{G}, \alpha) = 0\}$ defines a C^1 differential manifold, called the current yield surface. Hence $\partial_{\mathbf{G}} \widetilde{\mathcal{F}}(\mathbf{G}, \alpha) \neq 0$ on the yield surface.

THEOREM 3. *The dissipation postulate, introduced in* [7] *is equivalent to the existence of the stress potential* (I)*, together with the dissipation inequality* (II)*.*

I. For all $\hat{\mathbf{C}} \in \mathcal{G}_s$ *and for all* $t \in [0, 1)$ *there exist the smooth scalar valued functions,* φ, σ, *related by*

$$
\begin{aligned}
&\sigma(\mathbf{C}, \mathbf{Y}(t)) = \varphi(\mathbf{P}^{-T}(t)\mathbf{C}\mathbf{P}^{-1}(t), \alpha(t)) \quad \forall \mathbf{C} \in \mathcal{U}(\hat{\mathbf{C}}_t) \quad with\\
&\mathcal{U}(\hat{\mathbf{C}}_t) := \{\mathbf{B} \in Sym^+ \mid \mathcal{F}(\mathbf{B}, \mathbf{Y}(t)) \leq 0\}
\end{aligned}
$$

the elastic range, at time t *corresponding to* $\hat{\mathbf{C}} \in \mathcal{G}_s$*. Here* $\hat{\mathbf{C}}_t$ *is the restriction on* $[0, t]$ *of the given history.*
The functions φ, σ, *are stress potentials*

$$
(4)\qquad
\begin{aligned}
&\frac{\Pi(t)}{\tilde{\rho}(t)} = 2\, \partial_{\mathbf{G}} \varphi(\mathbf{G}, \alpha(t)), \quad \frac{\mathbf{T}(t)}{\rho} = 2\, \mathbf{F} \partial_{\mathbf{C}} \sigma(\mathbf{C}(t), \mathbf{Y}(t)) \mathbf{F}^T,\\
&\mathbf{G} = \mathbf{P}^{-T}(t)\mathbf{C}(t)\mathbf{P}^{-1}(t).
\end{aligned}
$$

II. The following equivalent dissipation inequalities

$$
(5)\qquad
\begin{aligned}
&[\partial_{\mathbf{Y}} \sigma(\mathbf{A}, \mathbf{Y}(t)) - \partial_{\mathbf{Y}} \sigma(\mathbf{C}(t), \mathbf{Y}(t))] \cdot \dot{\mathbf{Y}}(t) \geq 0 \quad and\\
&(\Sigma(t) - \Sigma^*) \cdot \dot{\mathbf{P}}(t)\mathbf{P}^{-1}(t) + (\mathbf{a}(t) - \mathbf{a}^*)\dot{\alpha}(t) \geq 0
\end{aligned}
$$

hold for all $\mathbf{G}, \mathbf{G}^*$ *such that* $\widetilde{\mathcal{F}}(\mathbf{G}, \alpha) = 0$, $\widetilde{\mathcal{F}}(\mathbf{G}^*, \alpha) \leq 0$, *when the conjugated forces to internal variables (see [10]) are considered*

$$\mathbf{a}(t) := -\partial_\alpha \varphi(\mathbf{G}(t), \alpha(t)), \quad \mathbf{a}^* = -\partial_\alpha \varphi(\mathbf{G}^*, \alpha(t))$$

Here $\Sigma(t)$, Σ^* are calculated from (2) for the elastic strains $\mathbf{G}(t)$ and $\mathbf{G}^*$.

PROPOSITION 1. *When the dissipation inequality (5)$_2$ is satisfied then modified flow rule*

$$(\partial_{\mathbf{G}} \hat{\Sigma}(\mathbf{G}, \alpha))^T [\dot{\mathbf{P}}\mathbf{P}^{-1}] = \mu \partial_{\mathbf{G}} \widetilde{\mathcal{F}}(\mathbf{G}, \alpha) + \partial^2_{\alpha\,\mathbf{G}} \varphi(\mathbf{G}, \alpha)[\dot{\alpha}] \tag{6}$$

with $\mu \geq 0$, *holds. The dissipation inequality (5)$_1$ imposes that*

$$-\partial_{\mathbf{Y}}[\partial_{\mathbf{C}} \sigma(\mathbf{C}, \mathbf{Y})][\dot{\mathbf{Y}}] = \bar{\mu} \partial_{\mathbf{C}} \mathcal{F}(\mathbf{C}, \mathbf{Y}) \quad \bar{\mu} \geq 0, \tag{7}$$

for all $\mathbf{C} = \mathbf{C}(t)$ *on yield surface* $\mathcal{F}(\mathbf{C}, \mathbf{Y}) = 0$, *for the fixed* $\mathbf{Y} = \mathbf{Y}(t)$, *with* $\bar{\mu} \geq 0$.

To end the discussion about the consequences of the dissipation postulate we recall the basic result, similar to [13]:

THEOREM 4. *1. At any regular point* Σ *of the yield function in stress space* $\hat{\mathcal{F}}(\Sigma, \alpha) = 0$, *but with* $\Sigma = \hat{\Sigma}(\mathbf{G})$, *the appropriate flow rule, i.e. the modified flow rule, takes the form*

$$\begin{array}{l} \mathbf{L}^p \equiv \dot{\mathbf{P}}\mathbf{P}^{-1} = \mu\, \partial_\Sigma \hat{\mathcal{F}}(\Sigma, \alpha) + \mathbf{L}^{p*}; \\ \mathbf{L}^{p*}: \quad (d\, \hat{\Sigma}(\mathbf{G}))^T (\mathbf{L}^{p*}) = 0 \end{array} \tag{8}$$

3. Rate boundary value problem and variational inequalities

We derive the variational inequalities with respect to the actual and respectively initial configurations, related to the rate quasi-static boundary value problem and associated with a generic stage of the process, at the time t. We use an appropriate procedure as in [19, 4] and different motion descriptions that can be found in [25].

The *nominal stress* with respect to the actual configuration at time t, or the *non-symmetric relative* Piola- Kirchhoff, is defined by

$$\mathbf{S}_t(\mathbf{x}, \tau) = (\det \mathbf{F}_t(\mathbf{x}, \tau)) \mathbf{T}(\mathbf{y}, \tau)(\mathbf{F}_t(\mathbf{x}, \tau))^{-T},$$

with

$$\mathbf{F}_t(\mathbf{x}, \tau) = \mathbf{F}(\mathbf{X}, \tau)(\mathbf{F}(\mathbf{X}, t))^{-1}$$

the relative deformation gradient.
Here $\mathbf{x} = \chi(\mathbf{X}, t)$, $\mathbf{y} = \chi(\mathbf{X}, \tau)$, or $\mathbf{y} = \chi_t(\mathbf{x}, \tau) \equiv \chi(\chi^{-1}(\mathbf{x}, t), \tau)-$ the motion in the relative description. At time t we have

$$\begin{array}{lcl} \mathbf{S}_t(\mathbf{x}, t) & = & \mathbf{T}(\mathbf{x}, t) \quad \text{and} \\ \dot{\mathbf{S}}_t(\mathbf{x}, t) & \equiv & \dfrac{\partial}{\partial \tau} \mathbf{S}_t(\mathbf{x}, \tau) \mid_{\tau = t} \\ & = & \rho(\mathbf{x}, t) \dfrac{\partial}{\partial \tau} (\dfrac{\mathbf{T}(\mathbf{y}, \tau)}{\rho(\mathbf{y}, \tau)}) \mid_{\tau = t} - \mathbf{T}(\mathbf{x}, t) \mathbf{L}^T(\mathbf{x}, t). \end{array} \tag{9}$$

Here $\mathbf{L}(\mathbf{x}, t) = \nabla \mathbf{v}(\mathbf{x}, t)$ represents the velocity gradient, in spatial representation.

Let us consider a body, identified with $\Omega \subset \mathbf{R}^3$ in the initial configuration, which undergoes the finite elasto-plastic deformation and occupies the domain $\Omega_\tau = \chi(\Omega, \tau) \subset R^3$, at time τ.

The *equilibrium equation* at time τ, in terms of Cauchy stress tensor $\mathbf{T}(\mathbf{y}, \tau) \in Sym$

$$\operatorname{div} \mathbf{T}(\mathbf{y}, \tau) + \rho(\mathbf{y}, \tau)\mathbf{b}(\mathbf{y}, \tau) = 0, \quad \text{in} \quad \Omega_\tau$$

where $\mathbf{b}$ are the body forces, can be equivalently expressed, with respect to the configuration at time $t-$ taken as the reference configuration

$$\text{(10)} \qquad \begin{aligned} &\operatorname{div} \mathbf{S}_t(\mathbf{x}, \tau) + \rho(\mathbf{x}, t)\mathbf{b}_t(\mathbf{x}, \tau) = 0\,, \quad \text{with} \quad \mathbf{b}_t(\mathbf{x}, \tau) = \mathbf{b}(\chi_t(\mathbf{x}, \tau), \tau) \\ &\mathbf{S}_t(\mathbf{x}, \tau)\mathbf{F}_t^T(\mathbf{x}, \tau) = \mathbf{F}_t(\mathbf{x}, \tau)\mathbf{S}_t^T(\mathbf{x}, \tau) \end{aligned}$$

When the reference configuration is considered to be a natural one, we add the initial conditions

$$\mathbf{S}_0(\mathbf{X}, 0) = 0, \quad \mathbf{F}(\mathbf{X}, 0) = \mathbf{I}, \quad \mathbf{P}(\mathbf{X}, 0) = \mathbf{I}, \quad \alpha(\mathbf{X}, 0) = 0,$$

for every $\mathbf{X} \in \Omega_0$ and the following boundary conditions on $\partial\Omega_t$:

$$\text{(11)} \qquad \mathbf{S}_t(\mathbf{x}, \tau)\mathbf{n}(t)\,|_{\Gamma_{1t}} = \hat{\mathbf{S}}_t(\mathbf{x}, \tau)\,, \quad (\chi_t(\mathbf{x}, \tau) - \mathbf{x})\,|_{\Gamma_{2t}} = \hat{\mathbf{U}}_t(\mathbf{x}, \tau)$$

Here $\partial\Omega_t \equiv \Gamma_{1t} \bigcup \Gamma_{2t}$ denotes the boundary of the thredimensional domain Ω_t, $\mathbf{n}(t)$ is the unit external normal at Γ_{1t}, while $\chi_t(\mathbf{x}, \tau) - \mathbf{x}$ is the displacement vector with respect to the configuration at time t. $\hat{\mathbf{S}}_t$ and $\hat{\mathbf{U}}_t$, the surface loading and the displacement vector are time dependent, τ, prescribed functions, with respect to the fixed at time t configuration.

The rate quasi-static boundary value problem at time t, involves the time differentiation, i.e. with respect to τ, of the equilibrium equations, (10), $\forall\, \mathbf{x} \in \Omega_t$, and of the boundary condition (11), when $\tau = t$

$$\text{(12)} \qquad \begin{aligned} &\operatorname{div} \dot{\mathbf{S}}_t(\mathbf{x}, t) + \rho(\mathbf{x}, t)\dot{\mathbf{b}}_t(\mathbf{x}, t) = 0\,, \\ &\dot{\mathbf{S}}_t(\mathbf{x}, t)\mathbf{n}(t)\,|_{\Gamma_{1t}} = \dot{\hat{\mathbf{S}}}_\mathbf{t}(\mathbf{x}, \mathbf{t}), \\ &\mathbf{v}(\mathbf{x}, t)\,|_{\Gamma_{2t}} = \dot{\hat{\mathbf{U}}}_\mathbf{t}(\mathbf{x}, \mathbf{t}) \end{aligned}$$

using the notation $\dot{\mathbf{b}}_t(\mathbf{x}, t)$ for $\dfrac{\partial}{\partial\tau}\mathbf{b}_t(\mathbf{x}, \tau\,|_{\tau=t}$.

At a generic stage of the process the current values, i.e. at the time t, of $\mathbf{F}$, $\mathbf{T}$, $\mathbf{Y}$, and the set of all material particles, in which the stress reached the current yield surface

$$\Omega_t^p = \chi(\Omega^p, t), \quad \text{with} \quad \Omega^p \equiv \{\mathbf{X} \in \Omega \mid \overline{\mathcal{F}}(\mathbf{C}(\mathbf{X}, t), \mathbf{Y}(\mathbf{X}, t)) = 0\}$$

are known for all $\mathbf{x} \in \Omega_t$, with the current deformed domain Ω_t also determined.

The set of kinematically admissible (at time t) velocity fields is denoted by

$$\mathcal{V}_{ad}(t) \equiv \{\mathbf{v} : \Omega_t \longrightarrow \mathbf{R}^3 \mid \mathbf{v}\,|_{\Gamma_{2t}} = \dot{\hat{\mathbf{U}}}_\mathbf{t}\}.$$

and the set of all admissible plastic multiplier

$$\begin{aligned} \mathcal{M}(t) \equiv \{\delta : \Omega_t \longrightarrow \mathbf{R}_{\geq 0} \mid \quad &\delta(\mathbf{x}, t) = 0, \quad \text{if} \quad \mathbf{x} \in \Omega_t \backslash \Omega_t^p, \\ &\delta(\mathbf{x}, t) \geq 0, \quad \text{if} \quad \mathbf{x} \in \Omega_t^p\}. \end{aligned}$$

THEOREM 5. *At every time t the velocity field,* $\mathbf{v}$, *and the equivalent plastic factor β satisfy the following relationships*

$$\int_{\Omega_t} \rho \big\{ \nabla \mathbf{v} \frac{\mathbf{T}}{\rho} \cdot (\nabla \mathbf{w} - \nabla \mathbf{v}) + 4 \mathbf{F} \partial^2_{\mathbf{CC}} \sigma(\mathbf{C}, \mathbf{Y})[\mathbf{F}^T \{\nabla \mathbf{v}\}^s \mathbf{F}] \mathbf{F}^T \cdot (\{\nabla \mathbf{w}\}^s -$$

$$\{\nabla \mathbf{v}\}^s)\} dx - 2 \int_{\Omega_t^p} \rho \frac{\beta}{h_r} \mathbf{F} \partial_{\mathbf{C}} \mathcal{F}(\mathbf{C}, \mathbf{Y}) \mathbf{F}^T \cdot (\{\nabla \mathbf{w}\}^s - \{\nabla \mathbf{v}\}^s) dx = \tag{13}$$

$$\int_{\Omega_t} \rho \dot{\mathbf{b}} \cdot (\mathbf{w} - \mathbf{v}) dx + \int_{\Gamma_{2t}} \dot{\hat{\mathbf{S}}}_{\mathbf{t}} \cdot (\mathbf{w} - \mathbf{v}) \mathbf{da}$$

and

$$-2 \int_{\Omega_t^p} \frac{\rho}{h_r} (\delta - \beta) \, \mathbf{F} \partial_{\mathbf{C}} \mathcal{F}(\mathbf{C}, \mathbf{Y}) \mathbf{F}^T \cdot (\{\nabla \mathbf{v}\}^s) dx + \int_{\Omega_t^p} \frac{\rho}{h_r} (\delta - \beta) \beta dx \geq 0, \tag{14}$$

which hold for every admissible vector field $\mathbf{w} \in \mathcal{V}_{ad}(t)$, *and for all* $\delta \in \mathcal{M}(t)$.

Proof. In the theorem of virtual power, derived from the rate quasi-static equilibrium equation (12):

$$\int_{\Omega_t} \dot{\mathbf{S}}_t \cdot \nabla \mathbf{w} dx = \int_{\partial \Omega_t} \dot{\mathbf{S}}_t \mathbf{n} \cdot \mathbf{w} da + \int_{\Omega_t} \rho \dot{\mathbf{b}}_t \cdot \mathbf{w} dx, \quad \forall \mathbf{w} \in \mathcal{V}_{ad}(t)$$

we substitute the rate of the nominal stress, at time t, calculated from (9), taking into account the potentiality condition $(4)_2$. First of all we calculate the differential with respect to τ of the right hand side in $(4)_2$, in which we replace $\dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{L}$ and $\dot{\mathbf{C}} = 2\mathbf{FDF}^T$, with $\mathbf{D} = \mathbf{L}^s$. Thus

$$\frac{\partial}{\partial \tau} (\frac{\mathbf{T}}{\rho}) = 2 \mathbf{LF} \partial_{\mathbf{C}} \sigma(\mathbf{C}, \mathbf{Y}) \mathbf{F}^T + 2 \mathbf{F} \partial_{\mathbf{C}} \sigma(\mathbf{C}, \mathbf{Y}) \mathbf{F}^T \mathbf{L}^T +$$

$$2 \mathbf{F} \partial^2_{\mathbf{CC}} \sigma(\mathbf{C}, \mathbf{Y})[2 \mathbf{FDF}^T] \mathbf{F}^T + 2 \mathbf{F} (\partial^2_{\mathbf{YC}} \sigma(\mathbf{C}, \mathbf{Y})[\dot{\mathbf{Y}}]) \mathbf{F}^T \tag{15}$$

in which we introduce the modified flow rule, (7), written under the form (see Remark 2)

$$\partial^2_{\mathbf{YC}} \sigma(\mathbf{C}, \mathbf{Y})[\dot{\mathbf{Y}}] = -\mu \partial_{\mathbf{C}} \mathcal{F}(\mathbf{C}, \mathbf{Y}), \tag{16}$$

Hence the equality (13) follows at once from (9), (15)and (16).

In order to prove (14) we note that $\mu \geq 0$ can be express either by the inequality

$$(\tilde{\mu} - \mu) \, \dot{\hat{\mathcal{F}}} \leq 0, \ \forall \, \tilde{\mu} \geq 0, \quad \text{together with} \quad \mu \, \hat{\mathcal{F}} = 0, \tag{17}$$

or under its explicit dependence on the rate of strain:

$$\mu = \frac{\beta}{h_r}, \quad \text{with} \quad \beta = 2 \partial_\Sigma \hat{\mathcal{F}}(\Sigma, \alpha) \cdot d \hat{\Sigma}(\mathbf{G}, \alpha)[\mathbf{E}^T \mathbf{DE}],$$

$$h_r = 2 \partial_\Sigma \hat{\mathcal{F}}(\Sigma, \alpha) \cdot d \hat{\Sigma}(\mathbf{G}, \alpha)[\{\mathbf{G} \tilde{\mathcal{B}}\}^s] - \partial_\alpha \hat{\mathcal{F}}(\Sigma, \alpha) \cdot \tilde{\mathbf{m}},$$

where the hardening parameter $h_r > 0$. The time derivative of $\hat{\mathcal{F}}(\Sigma, \alpha)$ with (2) is introduced in (17). Consequently, for all $\mathbf{x} \in \Omega_t^p$ we get

$$(\tilde{\mu} - \mu)(-\mu h_r + 2 \partial_\Sigma \hat{\mathcal{F}}(\Sigma, \alpha) \cdot d \hat{\Sigma}(\mathbf{G}, \alpha)[\mathbf{E}^T \mathbf{DE}]) \leq 0. \tag{18}$$

$h_r > 0$. We can substitute μ and $\tilde{\mu}$ by β/h_r and δ/h_r. By integrating on Ω_t^p from (18) the inequality (14) holds, when the equality

$$\partial_{\mathbf{C}}\mathcal{F}\cdot\mathbf{A} = \partial_{\Sigma}\hat{\mathcal{F}}(\Sigma,\alpha)\cdot d\hat{\Sigma}(\mathbf{G},\alpha)[\mathbf{P}^{-T}\mathbf{A}\mathbf{P}^{-1}]) \ , \forall\,\mathbf{A}\in Sym$$

is also used for $\mathbf{A} = \mathbf{F}^T\mathbf{D}\mathbf{F}$.

Let us define the convex set $\widetilde{\mathcal{K}}$ in the appropriate functional space of the solution $\mathbf{H}_{ad}$, by

$$\widetilde{\mathcal{K}} := \{(\mathbf{w},\delta) \mid \ \mathbf{w}\in\mathcal{V}_{ad}(t), \quad \delta:\Omega\longrightarrow\mathbf{R}_{\geq 0}\},$$

and *the bilinear forms*, in the appropriate space $\mathcal{H}_{ab}$:

$$\begin{aligned}
(19) \qquad K[\mathbf{v},\mathbf{w}] &= \int_{\Omega_t}\rho\big(\nabla\mathbf{v}\frac{\mathbf{T}}{\rho}\cdot\nabla\mathbf{w} + 4\mathbf{F}\partial^2_{\mathbf{CC}}\sigma(\mathbf{C},\mathbf{Y})[\mathbf{F}^T\{\nabla\mathbf{v}\}^s\mathbf{F}]\mathbf{F}^T\cdot\{\nabla\mathbf{w}\}^s\big)dx\\
A[\beta,\delta] &= \int_{\Omega_t^p}\frac{\rho}{h_r}\,\beta\,\delta dx\\
B[\delta,\mathbf{v}] &= -2\int_{\Omega_t^p}\frac{\rho}{h_r}\delta\,\mathbf{F}\partial_{\mathbf{C}}\mathcal{F}(\mathbf{C},\mathbf{Y})\mathbf{F}^T\cdot\{\nabla\mathbf{v}\}^s dx
\end{aligned}$$

are defined $\forall\,\mathbf{v},\mathbf{w}\in\mathcal{V}_{ad}(t)$, $\forall\,\delta,\ \beta:\Omega_t\longrightarrow\mathbf{R}_{\geq 0}$.

□

As a consequence of (19) , (13) and (14) the below statement holds:

THEOREM 6. *Find* $\mathbf{U} = (\mathbf{v},\beta)\ \in\widetilde{\mathcal{K}}$, *solution of the variational inequality,* V.I.*:*

$$a[\mathbf{U},\mathbf{V}-\mathbf{U}] \geq f[\mathbf{V}-\mathbf{U}] \quad \forall\,\mathbf{V}\in\widetilde{\mathcal{K}} \tag{20}$$

$a[\cdot,\cdot]$ is the bilinear and symmetric form *defined on* H_{ad}

$$a[\mathbf{V},\mathbf{W}] := K[\mathbf{v},\mathbf{w}] + B[\beta,\mathbf{w}] + B[\delta,\mathbf{v}] + A[\beta,\delta]$$

defined $\forall\,\mathbf{V} = (\mathbf{v},\beta), \mathbf{W} = (\mathbf{w},\delta)$ *and*

$$f[\mathbf{V}] := \int_{\Gamma_{1t}}\dot{\hat{\mathbf{S}}}_{\mathbf{t}}\cdot\mathbf{vda} + \int_{\Omega_t}\rho\dot{\mathbf{b}}_t\cdot\mathbf{vdx}, \quad \Gamma_{1t}\subset\partial\Omega_t. \tag{21}$$

REMARK 1. Under hypotheses: there exists $H_{ad}-$ a Hilbert space, with the scalar product denoted by $\cdot$, the continuity of the bilinear form on H_{ad}, $\mid a[\mathbf{V},\mathbf{U}]\mid\leq c_o\ ||\ V\ ||_H\quad ||\ \mathbf{U}\ ||_H$, and of the linear functional from (21) then the existence of $Q-$ linear operator associated to the bilinear form:

$$a[\mathbf{U},\mathbf{V}] = Q\mathbf{U}\cdot\mathbf{V} \quad \forall\,\mathbf{U},\mathbf{V}\in H_{ad}.$$

The variational problem can be equivalently formulated (see for instance Glowinski, Lions, Trémolières [1976]): Find $\tilde{\mathbf{U}}\in H_{ad}$ such that

$$a[\tilde{\mathbf{U}},\mathbf{U}-\tilde{\mathbf{U}}] - \mathbf{f}\cdot(\mathbf{U}-\tilde{\mathbf{U}}) + \Phi_{\tilde{K}}(\mathbf{U}) - \Phi_{\tilde{K}}(\tilde{\mathbf{U}}) \geq 0 \quad \forall\,\mathbf{U}\in H_{ad}.$$

Here $\Phi_{\tilde{K}}$ - the indicator function of $\tilde{K}$, is zero on $\tilde{K}$, and infinity outside.

By using the subdifferential $\partial\Phi_{\tilde{K}}$ of the function $\Phi_{\tilde{K}}$ the variational inequality becomes

$$-(Q\tilde{\mathbf{U}} - \mathbf{f}) \in \partial\Phi_{\tilde{K}}(\tilde{\mathbf{U}})$$

We recall that the *subdifferential* of $\Phi_{\tilde{K}}$ is defined as the mapping on H_{ad} such that

$$\partial\Phi_{\tilde{K}}(x) = \{\eta \in H \mid \Phi_{\tilde{K}}(y) - \Phi_{\tilde{K}}(x) \geq \eta \cdot (y - x) \quad \forall \quad y \in H\};$$

$\eta \in \partial\Phi_{\tilde{K}}(x)$ are called subgradients of $\Phi_{\tilde{K}}$ (see [18]). The domain of $\partial\Phi_{\tilde{K}}$ coincides with $\tilde{K}$, and $\partial\Phi_{\tilde{K}}(x) = \{0\}$ when x belongs to the interior of $\tilde{K}$.

PROPOSITION 2. *For linear elastic type constitutive equation, in the plastically deformed configuration, the following formula*

$$4\partial^2_{\mathbf{CC}}\sigma(\mathbf{C}, \mathbf{Y})[\mathbf{A}] = \mathbf{P}^{-1}\mathcal{E}[\mathbf{P}^{-T}\mathbf{A}\mathbf{P}^{-1}]\mathbf{P}^{-T}, \quad \forall\, \mathbf{A} \in Sym$$

follows.

In the case of small elastic strains

$$\Delta = \frac{1}{2}(\mathbf{G} - \mathbf{I}) \simeq \epsilon^e \quad \mathbf{E} = \mathbf{R}^e\mathbf{U}^e, \quad \text{where}$$

$$\mathbf{U}^e = \mathbf{I} + \epsilon^e, \quad \mathbf{G} = \mathbf{I} + 2\epsilon^e \quad \text{with} \quad \mid \epsilon^e \mid \leq 1,$$

$\mathbf{R}^e-$ elastic rotation, the following estimations

$$\mid \nabla\mathbf{w}\frac{\mathbf{T}}{\rho} \cdot \nabla\mathbf{w} \mid \leq \mid \nabla\mathbf{w} \mid^2 \mid \mathcal{E} \mid_4 \mid \epsilon^e \mid$$

$$4 \mid \mathbf{F}\partial^2_{\mathbf{CC}}\sigma(\mathbf{C}, \mathbf{Y})[\mathbf{F}^T\{\nabla\mathbf{w}\}^s\mathbf{F}]\mathbf{F}^T \cdot \{\nabla\mathbf{w}\}^s \mid \leq \mid \nabla\mathbf{w} \mid^2 \mid \mathcal{E} \mid_4$$

hold.

In conclusion: in the case of small elastic strains the first terms in the bilinear form $K[\cdot,\cdot]$ can be neglected in the presence of the second one. Moreover, if the behavior of the body, with small elastic strain only, is elastic, which means that $\beta = 0$ in the solution of the variational inequality, (20),then the bilinear form $a[\mathbf{V}, \mathbf{V}]$ for $\mathbf{V} = (\mathbf{v}, 0)$ is symmetric and positive definite.

In a similar manner, but starting from the *equilibrium equation* and the *balance equation of momentum* with respect to the initial configuration, expressed as

$$\text{Div}\,\mathbf{S} + \mathbf{b}_0 = 0, \quad \text{and} \quad \mathbf{S}\mathbf{F}^T = \mathbf{F}\mathbf{S}^T, \quad \text{in} \quad \Omega \quad \text{with}$$

$$\mathbf{S} := (\det\mathbf{F})\mathbf{T}\mathbf{F}^{-T}, \quad \mathbf{S} := \rho_0\mathbf{F}\mathbf{P}^{-1}\frac{\Pi}{\tilde{\rho}}\mathbf{P}^{-T}$$

$\mathbf{S}-$ non-symmetric Piola- Kirchhoff stress tensor, where $\mathbf{b}_0$ are the body forces, we can prove:

THEOREM 7. *The formulation of the rate quasi- static boundary value problem, in the initial configuration leads to the variational inequality:*

Find $(\dot{\mathbf{u}}, \mu) \in \mathcal{V} \times \mathcal{M}$ *, such that* $\forall(\mathbf{v}, \nu) \in \mathcal{V} \times \mathcal{M}$

$$K_0[\dot{\mathbf{u}}, \mathbf{v} - \dot{\mathbf{u}}] + B_0[\mu, \mathbf{v} - \dot{\mathbf{u}}] = \mathcal{R}[\mathbf{v} - \dot{\mathbf{u}}],$$

(22)

$$B_0[\dot{\mathbf{u}}, \nu - \mu] + A_0[\mu, \nu - \mu] \geq 0$$

where K_0, B_0, A_0 *denote the bilinear forms:*

$$\begin{aligned} K_0[\mathbf{v}, \mathbf{w}] &= \int_\Omega \rho_0 \{\nabla \mathbf{v} \mathbf{P}^{-1} \frac{\Pi}{\tilde{\rho}} \mathbf{P}^{-T} \cdot \nabla \mathbf{w} + \overline{\mathcal{E}}^p [\{\mathbf{F}^T \nabla \mathbf{v}\}^s] \cdot \{\mathbf{F}^T \nabla \mathbf{w}\}^s\} d\mathbf{X} \\ B_0[\mu, \mathbf{w}] &= -2 \int_{\Omega^p} \frac{\rho_0}{h_r} \mu \mathbf{P}^{-1} (d\,\hat{\Sigma})^T [\partial_\Sigma \hat{\mathcal{F}}] \mathbf{P}^{-T} \cdot \{\mathbf{F}^T \nabla \mathbf{w}\}^s d\mathbf{X} \\ A_0[\mu, \nu] &= \int_{\Omega^p} \frac{\rho_0}{h_r} \nu \mu d\mathbf{X} \end{aligned}$$

The linear functional

$$\mathcal{R}[\mathbf{v}] = \int_{\Gamma_1} \dot{\mathbf{F}}_0 \cdot \mathbf{v} da + \int_\Omega \rho_0 \dot{\mathbf{b}}_0 \cdot \mathbf{v} dx,$$

represents the virtual power produced by the variation in time of the of the mass force $\mathbf{b}_0$ and of the forces acting on the part Γ_1 of the boundary domain $\partial\Omega$, i.e. $\mathbf{SN} \mid_{\Gamma_1} = \mathbf{F}_0$.

Here we have introduced the elastic tensor with respect to the reference configuration $\overline{\mathcal{E}}^p$

$$\begin{aligned} \overline{\mathcal{E}}^p[\mathbf{A}] : &= 4\,\mathbf{P}^{-1} \partial^2_{\mathbf{GG}} \varphi [\mathbf{P}^{-T} \mathbf{A} \mathbf{P}^{-1}] \mathbf{P}^{-1} \\ \varphi(\mathbf{G}, \alpha) &= \sigma(\mathbf{C}, \mathbf{Y}), \quad \mathbf{G} = \mathbf{P}^{-T} \mathbf{C} \mathbf{P}^{-1}, \quad \mathbf{Y} = (\mathbf{P}^{-1}, \alpha). \end{aligned}$$

Here we denoted by $\mathcal{V} \equiv \{\mathbf{v} \mid \mathbf{v} = \dot{\mathbf{U}}^0 \text{ on } \Gamma_2 \subset \partial\Omega\}$, *the set of admissible displacement rate* (for a given function $\mathbf{U}^0$), and by $\mathcal{M}$, *the set of admissible plastic factors*.

REMARK 2. Note that $\mathbf{F} = \mathbf{I} + \nabla_\mathbf{X} \mathbf{u}$, where $\mathbf{u}(\mathbf{X}, t) = \chi(\mathbf{X}, t) - \mathbf{X}$ represents the displacement vector field and $\dot{\mathbf{F}} = \nabla_\mathbf{X} \dot{\mathbf{u}}$, and the spatial representations of the bilinear form (19) are just represented in (22).

The plastic factor $\mu = \dfrac{\beta}{h_r}$ which enter variational inequality is just the plastic factor which characterizes the evolution of plastic deformation, via the modified flow rule (7). In order to justify the above statement we recall the formula

$$\partial_\mathbf{C} \mathcal{F} = \mathbf{P}^{-1} \partial_\mathbf{G} \tilde{\mathcal{F}} \mathbf{P}^{-T} = \mathbf{P}^{-1} d\hat{\Sigma}^T [\partial_\Sigma \hat{\mathcal{F}}(\Sigma, \alpha)] \mathbf{P}^{-T},$$

and from the modified flow rule (6) we found

$$d\hat{\Sigma}^T [\, \dot{\mathbf{P}} \mathbf{P}^{-1} - \mu \partial_\Sigma \hat{\mathcal{F}}(\Sigma, \alpha)] = 0.$$

On the other hand when we pass to the actual configuration we get

$$B_0[\mu, \dot{\mathbf{u}}] = B[\mu, \mathbf{v}]$$

for

$$\dot{\mathbf{u}} = \frac{\partial \chi}{\partial t}(\mathbf{X}, t) \quad \text{and} \quad \mathbf{v} = \frac{\partial \chi}{\partial t}(\mathbf{X}, t) \mid_{\mathbf{x} = \chi^{-1}(\mathbf{X}, t)}$$

the rate of the displacement vector $\dot{\mathbf{u}}$ and $\mathbf{v}$ represent the velocity at the material point $\mathbf{X}$ in the material and the spatial representation.

4. Composite materials

We describe the composite materials within the framework of Σ-*models, with the potentiality condition and the modified flow rule*.

The macroscopic response will be orthotropic if there are two families reinforced fibres. In our model othotropic symmetry, characterized (see [12]) by the group $g_6 \in Ort$ defined by

$$g_6 := \{\mathbf{Q} \in Ort \mid \quad \mathbf{Q}\mathbf{n}_i = \mathbf{n}_i, \quad \text{or} \quad \mathbf{Q}\mathbf{n}_i = -\mathbf{n}_i, \quad i = 1, 2, 3.\}$$

where $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ is the orthonormal basis of the symmetry directions.

For transverse isotropy we distinguish the *subgroups* g_1, g_4, equivalently described in Liu [1983] by:

$$\begin{array}{lcl} g_1 & \equiv & \{\mathbf{Q} \in Ort \mid \quad \mathbf{Q}\mathbf{n}_1 = \mathbf{n}_1, \quad \mathbf{Q}\mathbf{N}_1\mathbf{Q}^T = \mathbf{N}_1\} \\ g_4 & \equiv & \{\mathbf{Q} \in Ort \mid \quad \mathbf{Q}(\mathbf{n}_1 \otimes \mathbf{n}_1)\mathbf{Q}^T = \mathbf{n}_1 \otimes \mathbf{n}_1\} \end{array}$$

where $\mathbf{N}_1 = \mathbf{n}_2 \otimes \mathbf{n}_3 - \mathbf{n}_3 \otimes \mathbf{n}_2$, for $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ an orthonormal basis, with $\mathbf{n}_1-$ the *symmetry direction*. The general representation theorems of Liu [1983] and Wang [1970] for anisotropic and isotropic functions were consequently employed by [5], to describe the complete set of the constitutive equations under the hypotheses formulated above. Here we give such kind of the model.

The linear g_4-transversely isotropic elastic constitutive equation with five material parameters, in tensorial representation is written with respect to plastically deformed configuration, $\mathbf{K}_t$,

$$\begin{array}{rcl} \dfrac{\Pi}{\tilde{\rho}} = \mathcal{E}(\Delta) & \equiv & [a\Delta\mathbf{n}_1 \cdot \mathbf{n}_1 + c\,\mathrm{tr}\Delta](\mathbf{n}_1 \otimes \mathbf{n}_1) + (c\Delta\mathbf{n}_1 \cdot \mathbf{n}_1 + d\,\mathrm{tr}\Delta)\mathbf{I} + \\ & + & e[(\mathbf{n}_1 \otimes \mathbf{n}_1)\Delta + \Delta(\mathbf{n}_1 \otimes \mathbf{n}_1)] + f\Delta \end{array}$$

The last representation is written in terms of the attached isotropic fourth order elastic tensor, $\widehat{\mathcal{E}}$, such that $\forall\, \mathbf{Q} \in Ort$. Here $\mathcal{E}$ is symmetric and positive definite.

The yield condition is generated via the formula (24) by the function f *orthotropic*, i.e. dependent on fourteen material constant (or scalar functions invariant relative to g_6), such that

$$\begin{array}{rcl} f(\Sigma) & := & \hat{f}(\Sigma^s, \Sigma^a, (\mathbf{n}_1 \otimes \mathbf{n}_1), (\mathbf{n}_2 \otimes \mathbf{n}_2)) \equiv \\ & \equiv & \hat{\mathcal{M}}((\mathbf{n}_1 \otimes \mathbf{n}_1), (\mathbf{n}_2 \otimes \mathbf{n}_2))\Sigma \cdot \Sigma = \\ & = & C_1(\Sigma^s \cdot \mathbf{I})^2 + C_2\Sigma^s \cdot \Sigma^s + C_3(\Sigma^a)^2 \cdot \mathbf{I} + \\ & + & C_4(\Sigma^s \cdot \mathbf{I})(\Sigma^s \cdot (\mathbf{n}_1 \otimes \mathbf{n}_1)) + C_5(\Sigma^s \cdot \mathbf{I})(\Sigma^s \cdot (\mathbf{n}_2 \otimes \mathbf{n}_2)) + \\ & + & C_6\Sigma^s \cdot \{\Sigma^s(\mathbf{n}_1 \otimes \mathbf{n}_1)\}^s + C_7\Sigma^s \cdot \{\Sigma^s(\mathbf{n}_2 \otimes \mathbf{n}_2)\}^s + \\ & + & C_8\Sigma^s \cdot \{(\mathbf{n}_1 \otimes \mathbf{n}_1)\Sigma^a\}^s + C_9\Sigma^s \cdot \{(\mathbf{n}_2 \otimes \mathbf{n}_2)\Sigma^a\}^s \\ & + & C_{10}[\Sigma^s \cdot (\mathbf{n}_1 \otimes \mathbf{n}_1)]^2 + C_{11}[\Sigma^s \cdot (\mathbf{n}_2 \otimes \mathbf{n}_2)]^2 + \\ & + & C_{12}[\Sigma^s \cdot (\mathbf{n}_1 \otimes \mathbf{n}_1)][\Sigma^s \cdot (\mathbf{n}_2 \otimes \mathbf{n}_2)] + C_{13}(\Sigma^a)^2 \cdot (\mathbf{n}_1 \otimes \mathbf{n}_1) + \\ & + & C_{14}(\Sigma^a)^2 \cdot (\mathbf{n}_2 \otimes \mathbf{n}_2) \end{array} \tag{23}$$

REMARK 3. When we consider the symmetrical case, that corresponds to *small elastic strains*, i.e. when $\Sigma^s = \Pi$, $\Sigma^a = 0$ then the yield condition is given from (23) in which $C_3 = C_8 = C_9 = C_{13} = C_{14} = 0$.

The rate evolution equation for plastic deformation expressed by Mandel's nine- dimensional flow rule, i.e. there is a particular representation of the modified flow rule given in (8),

$$\dot{\mathbf{P}}\mathbf{P}^{-1} = \mu \partial_{\Sigma} \mathcal{F}(\Sigma, \alpha, \kappa)$$

is associated to the orthotropic yield function, generated by (23), which describe the proportional and kinematic hardening given by

$$\begin{aligned} &\mathcal{F}(\Sigma, \alpha, \kappa) \equiv f(\overline{\Sigma}, \kappa) - 1 \equiv \\ &\hat{\mathcal{M}}((\mathbf{n}_1 \otimes \mathbf{n}_1), (\mathbf{n}_2 \otimes \mathbf{n}_2), \kappa)\overline{\Sigma} \cdot \overline{\Sigma} - 1 = 0, \quad \overline{\Sigma} = \Sigma - \alpha. \end{aligned} \tag{24}$$

Here we put into evidence the possible dependence on κ of the yield function through the fourth order tensor $\mathcal{M}$.

We provide the constitutive relations for the plastic strain rate, $\mathbf{D}^p$, as well as for the plastic spin $\mathbf{W}^p$, defined by

$$\mathbf{D}^p = 1/2(\mathbf{L}^p + \mathbf{L}^{pT}), \quad \mathbf{W}^p = 1/2(\mathbf{L}^p - \mathbf{L}^{pT}), \quad \text{where} \quad \mathbf{L}^p = \dot{\mathbf{P}}\mathbf{P}^{-1}$$

For orthotropic material the plastic strain rate is given by

$$\mathbf{D}^p = \mu\, \hat{\mathbf{N}}^p(\Sigma, \alpha, \kappa, (\mathbf{n}_1 \otimes \mathbf{n}_1), (\mathbf{n}_2 \otimes \mathbf{n}_2))$$

with

$$\begin{aligned} \hat{\mathbf{N}}^p \; = \; & 2C_1(\overline{\Sigma}^s \cdot \mathbf{I})\mathbf{I} + 2C_2\overline{\Sigma}^s + C_4[(\overline{\Sigma}^s \cdot \mathbf{I})(\mathbf{n}_1 \otimes \mathbf{n}_1) + \\ + \; & (\overline{\Sigma}^s \cdot \mathbf{n}_1 \otimes \mathbf{n}_1)\mathbf{I}] + C_5[(\overline{\Sigma}^s \cdot \mathbf{I})(\mathbf{n}_2 \otimes \mathbf{n}_2) + (\overline{\Sigma}^s \cdot \mathbf{n}_2 \otimes \mathbf{n}_2)\mathbf{I}] + \\ + \; & 2C_6\{\overline{\Sigma}^s(\mathbf{n}_1 \otimes \mathbf{n}_1)\}^s + 2C_7\{\overline{\Sigma}^s(\mathbf{n}_2 \otimes \mathbf{n}_2)\}^s + \\ + \; & C_8\{(\mathbf{n}_1 \otimes \mathbf{n}_1)\overline{\Sigma}^a\}^s + C_9\{(\mathbf{n}_2 \otimes \mathbf{n}_2)\overline{\Sigma}^a\}^s + \\ + \; & 2C_{10}(\overline{\Sigma}^s \cdot (\mathbf{n}_1 \otimes \mathbf{n}_1))(\mathbf{n}_1 \otimes \mathbf{n}_1) + \\ + \; & 2C_{11}(\overline{\Sigma}^s \cdot (\mathbf{n}_2 \otimes \mathbf{n}_2))(\mathbf{n}_2 \otimes \mathbf{n}_2) + \\ + \; & 2C_{12}[(\overline{\Sigma} \cdot (\mathbf{n}_1 \otimes \mathbf{n}_1))(\mathbf{n}_2 \otimes \mathbf{n}_2) + (\overline{\Sigma} \cdot (\mathbf{n}_2 \otimes \mathbf{n}_2))(\mathbf{n}_1 \otimes \mathbf{n}_1)] \end{aligned} \tag{25}$$

and the plastic spin is expressed under the form

$$\begin{aligned} \mathbf{W}^p \; & = \mu\, \hat{\Omega}^p(\Sigma, \alpha, \kappa, \mathbf{n}_1 \otimes \mathbf{n}_1, \mathbf{n}_2 \otimes \mathbf{n}_2) \quad \text{with} \\ \hat{\Omega}^p \; & = -2C_3\overline{\Sigma}^a + C_8\{(\mathbf{n}_1 \otimes \mathbf{n}_1)\overline{\Sigma}^s\}^a + C_9\{(\mathbf{n}_2 \otimes \mathbf{n}_2)\overline{\Sigma}^s\}^a - \\ & -2C_{13}\{\overline{\Sigma}^a(\mathbf{n}_1 \otimes \mathbf{n}_1)\}^a - 2C_{14}\{\overline{\Sigma}^a(\mathbf{n}_2 \otimes \mathbf{n}_2)\}^a \end{aligned} \tag{26}$$

REMARK 4. $\mathbf{W}^p$ involves the terms generated by the symmetric part of $\overline{\Sigma}$, while $\mathbf{D}^p$ contains terms generated by the skew- symmetric part of $\overline{\Sigma}$, with two coupling coefficients C_8, C_9.

REMARK 5. In the case of $\overline{\Sigma} \in Sym$, i.e. for *small elastic strains* and $\alpha \in Sym$, directly from (26) we derive the following expression for *orthotropic plastic spin*

$$\mathbf{W}^p = \mu\, \Omega^p = \mu\, \{C_8\{(\mathbf{n}_1 \otimes \mathbf{n}_1)\overline{\Sigma}^s\}^a + C_9\{(\mathbf{n}_2 \otimes \mathbf{n}_2)\overline{\Sigma}^s\}^a\} \tag{27}$$

But in this case, the yield condition (23) does not depend on the parameters which enter the expression (27) of the plastic spin.

PROPOSITION 3. *From the orthotropic Mandel's flow rule (26) the flow rule characterizing the g_4- transversely isotropic material is derived when $C_5 = C_7 = C_9 = C_{11} = C_{12} = 0$, i.e. dependent on six material constants. The plastic spin is given by (25), in which $C_9 = C_{14} = 0$, i.e. dependent on three constant only.*

Evolution equation for internal variable can be described, see [6], by some new generalization to finite deformation of Armstrong- Frederick hardening rule.

From the orthotrop representation g_4- transversely isotropic case only can be obtained. Thus for plastically *incompressible* material, i.e. $\tilde{\rho} = \rho_0$, the representatio from [21] can be obtained by taking into account small deformation theory. The fibre-inextensible case given in [22] can be also derived from our general representation, when the appropriate yield constant is much grater then the others.

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MICROSTRUCTURE DESCRIBED BY HIERARCHICAL INTERNAL VARIABLES

Abstract. In this paper a clear distinction is made between the different scales and the different processes in the microstructure which influence the dynamics at the macrolevel. In the first case the governing equation for wave propagation is represented by a hierarchy of waves. In the second case it has been shown, how useful the concept of internal variables is. The different processes can be best described by a hierarchy of internal variables. An example of cardiac muscle contraction is briefly described, demonstrating the dependence of the active stress on sliding the molecules and ion concentration involving the corresponding internal variables.

1. Introduction

Continuum mechanics is usually based on macroscopic concepts and quantities, such as energy density, stress, strain, etc. However, materials (whatever their origin is) have usually a microstructure because of inhomogeneities, pores, embedded layers, reinforcements, etc.. This list can be prolonged but one is clear - the description of the behaviour of many materials should take into account both the macroscopic and microscopic properties, occuring at different length scales and involving different physical effects. Within the framework of continuum mechanics, such a behaviour is best described by distinguishing macro stresses and microstresses with interactive microforces ([1], [2]). We feel however, that for materials with complicated properties indicated above, one should start distinguishing clearly the observable and internal variables ([10], [13]). Although the formalism of internal variables is well known ([10], [13]), for the clarity sake we repeat here some basic concepts.

The observable variables are the usual macroscopic field quantities such as elastic strain, for example. These variables are governed by conservation laws and possess inertia. The internal structure of the material (body, tissue, composite, etc.) is supposed to be described by internal variables which are not observable and do not possess inertia. They should compensate our lack of knowledge of the precise description of the microstructure. The formalism of internal variables involves constructing of a dissipation potential $\mathcal{D}$ in parallel to the Lagrangian $\mathcal{L}$ for the observable variable. However, the governing equations of internal variables are kinetic equations (not hyperbolic) – see [10], [13].

The idea of using internal variables for describing dynamical processes in microstructured materials has earlier been presented in [12], [4]. The problems become more complicated when either the scales or possible processes in materials are different and form a certain hierarchy. This brings us directly to the idea of hierarchical internal variables that certainly need generalization

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