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LINEARIZATION OF NONHOLONOMIC SYSTEMS AT EQUILIBRIUM POINTS

Sommario. We discuss linearization for nonholonomic dynamics at equilibria, using an approach that although more complicated than the usual should be more convincing and yields more information.

1. Introduction

Linearization of nonholonomic systems at equilibrium positions has been plagued from the very beginning by mistakes and misunderstandings. Bottema [3] corrected Whittaker's originary procedure [8]; and Naimark and Fufaev [7] developed Bottema's idea, deriving a scheme which is the currently accepted one. Something nagging however remains about the derivation of the characteristic equation for a nonholonomic system: the equations of such a system, with the undetermined multipliers thrown in, do *not* form a system of ordinary differential equations, in spite of what is said in [7]; and the multipliers are *not* independent variables. The linearization being essentially an approximation, one can say that it is validated experimentally by the results it gives; but a satisfactory approach should be convincing also from the theoretical viewpoint.

Here we try to do that by making use of the well–known fact that the equation of a nonholonomic system may be put in normal form, and the system then is equivalent to an holonomic system, in which only the initial conditions in phase space which satisfy the constraint are allowed. This fact has recently been used (see [9]) from Gaetano Zampieri, whom I thank for pointing out the question, and for many helpful discussions.

The linearization of the equivalent holonomic system is a well understood procedure; we prove that the characteristic polynomial so obtained factors as $ds^{2m}\beta(s)$, where $\beta(s)$ is the Bottema polynomial, and d is a nonzero constant. Our approach is more complicated, but gives more insight into the question of stability, clarifies the insurgence of vanishing roots and makes possible the discovery of facts not apparent from the easier but more formal usual treatment. In particular, some instability results (see section 6) are consequence of this approach; we also discover that the characteristic equation has at least $2m$ vanishing roots, independently on any manifold structure of the set of equilibria.

Equilibria of a nonholonomic system form, in the generic case, a manifold; but this is not always the case, as we show in section 7; and nonholonomic systems can have isolated equilibria (section 8).

Finally, it is well–known that Whittaker's procedure is acceptable for equilibria of the nonholonomic system which are also equilibria of the unconstrained system; some authors maintain that these are the only acceptable equilibria (see [4]); now, a skier on a slope can remain at rest when his skis are orthogonal to the slope's gradient, even if the component of the gravity tangent to the slope is nonzero; this is obvious, and should end any further speculation on this matter.

2. Reduction to an holonomic problem

We make use of the following well known concept: if E, F, G are normed linear spaces, U open in E, the differential of a map $A : U \to \mathrm{L}(F, G)$ at a point $u \in U$ is identified with a bilinear, in general nonsymmetric, map $A'(u)[\cdot,\cdot] : E \times F \to G$, so that $A(u + \Delta u)[v] = A(u)[v] + A'(u)[\Delta u, v] + o(\Delta u)[v]$, for each $v \in F$, and $\Delta u \in E$ small enough. The linear dependence on some variables is sometimes indicated by enclosing these in square brackets; this is always the case for bilinear maps, less often so for just linear ones. We have an open region Ω of the euclidean $n-$space $X \approx \mathbb{R}^n$, and sufficiently regular functions $K : \Omega \to \mathrm{Sym}(X)$, $Q : \Omega \times X \to X$, $B : \Omega \to \mathrm{L}(X, \Lambda)$, where $\Lambda \approx \mathbb{R}^m$ is another euclidean space, and m, n integers, $0 < m < n$; for every $q \in \Omega$ the rank of $B(q)$ is m, and $K(q)$ is positive definite; $T(q)[\dot{q}, \dot{q}] = \dot{q} \cdot (K(q)\dot{q})$ is the kinetic energy, $Q(q, \dot{q})$ is the generalized force. We have Lagrange's equations with nonholonomic constraints:

$$\left\{\begin{array}{lcl} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}(t, q, \dot{q})\right) - \frac{\partial T}{\partial q} - Q(q,\ \dot{q}) & = & B^*(q)\lambda \\ B(q)\dot{q} & = & 0 \end{array}\right. \tag{1}$$

$B^*(q) \in \mathrm{L}(\Lambda, X)$ is the adjoint of $B(q)$ with respect to the given scalar products. By a solution, or an admissible motion, we naturally mean C^2 functions $q : I \to X$,
$lambda : I \to \Lambda$, where I is an interval of $\mathbb{R}$ (time interval) which substituted into the above make them identically true.

These equations rewrite as

$$K(q)(\ddot{q} + \Gamma(q)[\dot{q}, \dot{q}]) = Q(q, \dot{q}) + B^*(q)\lambda, \qquad B(q)\dot{q} = 0;$$

$q \mapsto \Gamma(q)$ is a map from Ω into the space of symmetric bilinear maps from $X \times X$ into X (Christoffel symbols). Differentiating $B(q)\dot{q} = 0$ with respect to time t,

$$B'(q)[\dot{q}, \dot{q}] + B(q)\ddot{q} = 0$$

and putting equations together:

$$\left\{\begin{array}{lcl} K(q)(\ddot{q} + \Gamma(q)[\dot{q}, \dot{q}]) & = & Q(q, \dot{q}) + B^*(q)\lambda \\ B'(q)[\dot{q}, \dot{q}] + B(q)\ddot{q} & = & 0 \end{array}\right.$$

that is

$$\left\{\begin{array}{lcl} K(q)\ddot{q} - B^*(q)\lambda & = & Q(q, \dot{q}) - K(q)\Gamma(q)[\dot{q}, \dot{q}] \\ -B(q)\ddot{q} & = & B'(q)[\dot{q}, \dot{q}] \end{array}\right. \tag{2}$$

Using a block-defined linear operator of $X \times \Lambda$ into itself the equations may be written (we omit indication of the dependence on $q, \dot{q}$, unless this last is bilinear):

$$\begin{pmatrix} K & -B^* \\ -B & 0 \end{pmatrix} \begin{pmatrix} \ddot{q} \\ \lambda \end{pmatrix} = \begin{pmatrix} Q - K\Gamma[\dot{q}, \dot{q}] \\ B'[\dot{q}, \dot{q}] \end{pmatrix} \tag{3}$$

Recall now that $V = \ker(B)$ has the image of the adjoint B^* as its orthogonal, $V^\perp = \mathrm{Im}(B^*)$, and that $K(V) \cap V^\perp = \{0\}$, since K is definite; this readily implies that $D = BK^{-1}B^*$ is an authomorphism of Λ (we have $\lambda \in \ker(D)$ iff $K^{-1}(B^*\lambda) \in V$, equivalently $B^*\lambda \in K(V)$); it is now a non difficult exercise of linear algebra to verify that

$$\text{(4)}\qquad \begin{pmatrix} K & -B^* \\ -B & 0 \end{pmatrix}^{-1} = \begin{pmatrix} K^{-1}(1_X - B^*D^{-1}BK^{-1}) & -K^{-1}B^*D^{-1} \\ -D^{-1}BK^{-1} & -D^{-1} \end{pmatrix}$$

Put $P = 1_X - B^*D^{-1}BK^{-1}$. We obtain:

$$\text{(5)}\qquad \begin{cases} \ddot{q} &= K^{-1}PQ - K^{-1}PK\Gamma[\dot{q},\dot{q}] - B^*B'[\dot{q},\dot{q}] \\ \lambda &= -D^{-1}BK^{-1}Q + D^{-1}BK^{-1}\Gamma[\dot{q},\dot{q}] - D^{-1}B'[\dot{q},\dot{q}] \end{cases}$$

Assume that the generalized force may be split as $Q(q,\dot{q}) = S(q) - F(q)\dot{q}$, into a positional term $S(q)$ and a dissipative term $-F(q)\dot{q}$, where $F(q)$ is a symmetric positive semidefinite matrix.

This assumption shall hold through all the remaining part of the paper.

The preceding rewrite

$$\text{(6)}\qquad \begin{cases} \ddot{q} &= K^{-1}PS - K^{-1}PF[\dot{q}] - K^{-1}PK\Gamma[\dot{q},\dot{q}] - B^*B'[\dot{q},\dot{q}] \\ \lambda &= -D^{-1}BK^{-1}S + D^{-1}BK^{-1}F[\dot{q}] + D^{-1}BK^{-1}\Gamma[\dot{q},\dot{q}] - D^{-1}B'[\dot{q},\dot{q}] \end{cases}$$

or else, setting

$$f(q) = K^{-1}(q)P(q)S(q); \qquad R(q) = K^{-1}(q)P(q)F(q),$$

we obtain

$$\text{(7)}\qquad \begin{cases} \ddot{q} &= f(q) - R(q)\dot{q} + g(q)[\dot{q},\dot{q}] \\ \lambda &= h(q) + \sigma(q)[\dot{q}] + k(q)[\dot{q},\dot{q}] \end{cases}$$

with obvious definitions for g, h, σ, k; notice in particular that we have

$$\text{(8)}\qquad f(q) = K^{-1}(q)(S(q) + B^*(q)h(q)); \qquad R(q) = K^{-1}(q)(F(q) - B^*(q)\sigma(q)).$$

System (7) is equivalent to the preceding one, and the second equation can be eliminated; it shows that nonholonomic dynamics is reduced to the study of the Cauchy problem

$$\text{(9)}\qquad \ddot{q} = f(q) - R(q)[\dot{q}] + g(q)[\dot{q},\dot{q}] \qquad q(t_0) = q_0; \quad q'(t_0) = \dot{q}_0,$$

where the initial conditions are restricted by $B(q_0)[\dot{q}_0] = 0$. The system is in the subspace of the phase space (q,p) given by $\{(q,p) : B(q)[p] = 0\}$: it is a (locally trivial) vector bundle, whose fiber at q is the linear space ker $B(q)$. Notice that in any case $B(q)\dot{q}$ is a first integral for the preceding second order equation. In phase space the equation writes

$$\text{(10)}\qquad \begin{cases} \dot{q} &= p \\ \dot{p} &= f(q) - R(q)[p] + g(q)[p,p] \end{cases}; \qquad B(q)p = 0.$$

If the positional force is conservative, $S(q) = -\nabla U(q)$ for some $U : \Omega \to \mathbb{R}$, it is not difficult to see that the total energy $E = T(q)[p,p] + U(q)$ is a Liapunov function for the nonholonomic system, and a first integral in the absence of dissipation; in fact one has $\dot{E} = -p \cdot (F(q)p) \leq 0$ (recall that F is positive semidefinite, and that if $(q(t), p(t))$ is a motion, then $p(t) \cdot (B^*(q(t))\lambda) = 0$ for all $\lambda \in \Lambda$).

All of the above is essentially in [9].

3. Equilibrium points

The equilibria of a nonholonomic system correspond to constant solutions, and are to be found at the points $q_0 \in \Omega$ such that for some $\lambda_0 \in \Lambda$ we have

$$-S(q_0) = B^*\lambda_0.$$

Notice that when the system is written in the form (10) the equilibria are exactly the zeroes of f; and $\lambda = h(q)$ is found from the second equation in (7): the set of equilibria is a subset of the graph of the function $h : \Omega \to \Lambda$, an $n-$dimensional submanifold of $X \times \Lambda$ diffeomorphic to Ω. If $S(q_0) = 0$, then the preceding equation is satisfied by $\lambda_0 = 0$; an unconstrained equilibrium remains of course an equilibrium when velocity constraints are added, but there are other equilibria, as remarked in the introduction. In the generic case, the set of equilibria

$$\{(q, \lambda) \in X \times \Lambda : \ Q(q, 0) + B^*(q)\lambda = 0\}$$

will be an $m-$dimensional submanifold of $X \times \Lambda$, contained in the graph of h, which then projects onto an $m-$dimensional submanifold of X. This is certainly the case if the solution set is non–empty, and the linear operator $Q'(q, 0)[\,\cdot\,] + (B^*)'[\,\cdot\,, \lambda] \in \mathrm{L}(X)$ has rank n for every (q, λ) in the solution set; λ acts then as a system of parameters for the manifold. This is the generic situation, but exceptions are not hard to find (sections 7, 8).

4. Linearization at an equilibrium

If $\Phi(q, p) = (p, f(q) - R(q)[p] + g(q)[p, p])$, the differential of Φ at an equilibrium point $(q_0, 0)$ is

$$\Phi'(q_0, 0) = \begin{pmatrix} 0 & 1_X \\ f'(q_0) & -R(q_0) \end{pmatrix}. \tag{11}$$

We want to study $f'(q_0)$ (see section 2 for the definition of f, R, etc.). For this, the following is a crucial result:

PROPOSITION 3.1. *$P(q)$ is a projector onto the space $K(q)(\ker(B(q))$ and $K(q)^{-1}P(q)$ has $\ker(B(q))$ as image, $\ker(B(q))^{\perp}$ as kernel.*

Dimostrazione. For simplicity, omit q from the operators; P is a projector iff it is idempotent, and this is true iff $B^*D^{-1}BK^{-1}$ is idempotent, which is immediate to check:

$$(B^*D^{-1}BK^{-1})(B^*D^{-1}BK^{-1}) = B^*D^{-1}BK^{-1}B^*D^{-1}BK^{-1} = \\ B^*D^{-1}DD^{-1}BK^{-1} = B^*D^{-1}BK^{-1};$$

$B^*D^{-1}BK^{-1}$ is a projector onto the space $B^*(\Lambda)$, with kernel $K(\ker(B))$: all this is immediate. This implies that $1_X - B^*D^{-1}BK^{-1}$ has kernel $B^*(\Lambda) = \ker(B)^{\perp}$, and $K(\ker(B))$ as image.

□

From $f(q) = K^{-1}(q)P(q)S(q)$ we get

$$\begin{aligned} f'(q)[\,\cdot\,] &= -K^{-1}(q)K'[\,\cdot\,, K^{-1}(q)P(q)S(q)] + \\ & \quad K^{-1}(q)\left(P'(q)[\,\cdot\,, S(q)] + P(q)S'(q)[\,\cdot\,]\right) \end{aligned}$$

The idempotence of P implies that

$$P'(q)[\,\cdot\,, S(q)] = P'(q)[\,\cdot\,, P(q)S(q)] + P(q)P'[\,\cdot\,, S(q)];$$

substituting in the above we get

$$\begin{aligned} f'(q)[\,\cdot\,] &= -K^{-1}(q)K'[\,\cdot\,, f(q)] \\ & \quad K^{-1}(q)\left(P'(q)[\,\cdot\,, P(q)S(q)] + P(q)P'[\,\cdot\,, S(q)] + P(q)S'(q)[\,\cdot\,]\right), \end{aligned}$$

and at an equilibrium point q_0 we have $P(q_0)S(q_0) = K(q_0)f(q_0) = 0 = f(q_0)$; at equilibria we are then able to write:

$$f'(q_0)[\,\cdot\,] = K^{-1}(q_0)P(q_0)\left(P'[\,\cdot\,, S(q_0)] + S'(q_0)[\,\cdot\,]\right).$$

COROLLARY 3.1. *At an equilibrium point q_0 the images of $f'(q_0)$ and of $R(q_0)$ are contained in* $\ker(B(q_0))$.

Dimostrazione. See above for $f'(q_0)$; for R, simply recall that $R = K^{-1}PF$.

□

It will also be useful to differentiate $f(q)$ written in the form $f(q) = K^{-1}(q)(S(q) + B^*(q)h(q))$; we get

$$\begin{aligned} f'(q)[\,\cdot\,] &= -K^{-1}(q)K'[\,\cdot\,, K^{-1}(q)(S(q) + B^*(q)h(q))] + \\ & \quad K^{-1}(q)(S'(q)[\,\cdot\,] + B^{*\prime}(q)[\,\cdot\,, h(q)] + B^*(q)h'(q)[\,\cdot\,]), \end{aligned}$$

which at equilibria becomes

$$f'(q_0)[\,\cdot\,] = K^{-1}(q_0)(S'(q_0)[\,\cdot\,] + B^{*\prime}(q_0)[\,\cdot\,, h(q_0)] + B^*(q_0)h'(q_0)[\,\cdot\,]). \tag{12}$$

5. Linear algebra for the characteristic equations

In the sequel, q_0 is an equilibrium point; for simplicity, put $M = f'(q_0))$, $R = R(q_0)$; M, R are both linear operators in X, whose image is contained in $V = \ker B$, an $(n-m)$-dimensional subspace of X; W is the operator

$$\Phi'(q_0, 0) = \begin{pmatrix} 0 & 1_X \\ f'(q_0) & -R(q_0) \end{pmatrix} = \begin{pmatrix} 0 & 1_X \\ M & -R \end{pmatrix}$$

of $X \times X$. We assume orthonormal coordinates, thus identifying X with $\mathbb{R}^n$. We shall speak of the eigenvalues of W and other linear operators; it is understood that we go to the complexifications $X_{\mathbb{C}} \approx \mathbb{C}^n$, or $X_{\mathbb{C}} \times X_{\mathbb{C}} \approx \mathbb{C}^n \times \mathbb{C}^n$ of these spaces when we speak of complex eigenvalues, but we still denote X or $X \times X$ these spaces. The following contains what we are able to say about the characteristic polynomial of W:

PROPOSITION 3.2. *Let α be a complex number, (u, v) a non zero vector in $X \times X$. Then α is an eigenvalue of the $n \times n$ matrix*

$$W = \begin{pmatrix} 0 & 1_X \\ M & -R \end{pmatrix},$$

with (u, v) as an associated eigenvector if and only if $u \in X$ is a non–zero vector in the kernel of $\alpha^2 1_X + \alpha R - M$; and the characteristic polynomial $\chi(s) = \det(s 1_{X\times X} - W)$ of W coincides with $\det(s^2 1_X + sR - M)$. Moreover, assume that the image spaces $M(X)$, $R(X)$ are both contained in a subspace V of X of dimension $n - m$. Then $\dim\ker W \geq m$, $\dim\ker W^2 \geq 2m$, and $\dim\ker(W)$ is strictly smaller than $\dim\ker(W^2)$; hence the characteristic polynomial of W has s^{2m} as a factor, and W is not semisimple.

Dimostrazione. The first statement asserts that $v = \alpha u$, $Mu - Rv = \alpha v$, equivalent to $Mu - \alpha Ru = \alpha^2 u$; and if $u = 0$ then also $v = \alpha u = 0$. To get the remaining statement, consider

$$s 1_{X\times X} - W = \begin{pmatrix} s 1_X & -1_X \\ -M & s 1_X + R \end{pmatrix}.$$

Multiplying the last n columns by s and adding to the first n columns we get the operator

$$\begin{pmatrix} 0 & -1_X \\ -M + sR + s^2 1_X & s 1_X + R \end{pmatrix}. \tag{13}$$

whose determinant $\chi(s) = \det(s^2 1_X + sR - M)$ coincides with $\det(s 1_{X\times X} - W)$.

The assertions on the kernels are immediate: $W(X \times X) \subseteq X \times V$ implies $\dim\ker W \geq m$; and $W^2(X \times X) \subseteq V \times V$ implies $\dim\ker W^2 \geq 2m$; moreover, it is plain that the projection onto the first factor of the image of W is all of X, whereas the image of W^2 projects into V; then the image of W is strictly larger than the image of W^2, which implies the reverse inclusion between the kernels. The generalized kernel of W has then dimension $\geq 2m$, which implies that s^{2m} divides the characteristic polynomial; and the minimum polynomial of W has s^2 as a factor. □

We now clarify the relation between this characteristic equation and that in [7]. Clearly the kernel of (13) in the above proof coincides with the kernel of

$$\begin{pmatrix} s^2 1_X + sR - M & 0 \\ 0 & 1_X \end{pmatrix}. \tag{14}$$

and also the determinants are the same. We now forget in (14) the last $n - m$ rows and columns, obtaining the block matrix

$$\begin{pmatrix} s^2 1_X + sR - M & 0 \\ 0 & 1_m \end{pmatrix}. \tag{15}$$

whose kernel is clearly $\{(u,\lambda) \in \mathbb{R}^n \times \mathbb{R}^m : u \in \ker(s^2 1_X + sR - M),\ \lambda = 0\}$, and whose determinant is $\chi(s)$. Right-multiplying (15) by the invertible $(n+m) \times (n+m)$ matrix used in section 2 we get:

$$(16) \qquad \begin{pmatrix} K & -B^* \\ -B & 0 \end{pmatrix} \begin{pmatrix} s^2 1_X + sR - M & 0 \\ 0 & 1_X \end{pmatrix} = \begin{pmatrix} s^2 K + sPF - N & -B^* \\ -s^2 B & 0 \end{pmatrix}$$

(recall that $BM = BR = 0$, since M, R have images contained in $\ker(B)$); we put $N = KM$; we have $N = S'(q_0) + (B^*)'[\,\cdot\,, h_0] + B^* h'(q_0)$, as from the last statement in section 3. Clearly the determinant of (16) is that of (15) multiplied by a nonzero constant factor d, and coincides with

$$(17) \qquad s^{2m} \det \begin{pmatrix} s^2 K + sPF - N & B^* \\ B & 0 \end{pmatrix}.$$

We want to prove that both the kernel and the determinant of the operators

$$(18) \qquad \begin{pmatrix} s^2 K + sPR - S'(q_0) - (B^*)'[\cdot, h_0] - B^* h'(q_0) & B^* \\ B & 0 \end{pmatrix},$$

$$(19) \qquad \begin{pmatrix} s^2 K + sR - S'(q_0) - (B^*)'[\,\cdot\,, h_0] & B^* \\ B & 0 \end{pmatrix},$$

are equal; the matrix (19) is the one appearing in [7]. The difference of the operators in the left upper corner is $-s(1_X - P)R - B^* h'(q_0)$, an operator whose image is contained in $B^*(\Lambda)$; the conclusion follows immediately from the following easy observation:

Let $H, E \in L(X)$ be linear operators; if the image of E is contained in the image of B^*, then the operators of $L(X \times \Lambda)$ given by

$$\begin{pmatrix} H + E & B^* \\ B & 0 \end{pmatrix} \qquad \begin{pmatrix} H & B^* \\ B & 0 \end{pmatrix}$$

have the same kernel and the same determinant.

Dimostrazione. (u, λ) is in the kernel of the operator on the left if and only if $u \in \ker B$ and $Hu + Eu = -B^*\lambda$; in other words, $u \in X$ is the first component of an element of the kernel of the operator if and only if $Hu + Eu \in B^*(\Lambda)$; and similarly, $Hu \in B^*(\Lambda)$ if and only if u is the first component of an element of the kernel of the operator on the right. But since $Eu \in B^*(\Lambda)$ for every $u \in X$, this subset of vectors of X is the same. Since $B^*(\Lambda)$ has the columns of B^* as a basis, the hypothesis on E is verified if and only if the columns of E are linear combinations of the columns of B^*, and this implies that the matrices are column equivalent, by elementary operations; thus determinants are equal.

□

We have proved:

$$d\chi(s) = s^{2m}\beta(s),$$

where $\beta(s)$ (the "Bottema polynomial") is the determinant described in [7]

$$\beta(s) = \det \begin{pmatrix} s^2K + sR(q_0) - S'(q_0) - (B^*)'[\,\cdot\,, h_0] & B^* \\ B & 0 \end{pmatrix}$$

and $d = \det \begin{pmatrix} K & B^* \\ B & 0 \end{pmatrix}$.

The equation $\beta(s) = 0$ should be called the reduced characteristic equation of the given nonholonomic system, at the equilibrium q_0. Its roots are then exactly the eigenvalues of the linearization of the system, except for $2m$ zero roots; it should be noted that β has degree $2n-2m$, and not necessarily has zero as a root.

6. Instability

Since, in general, equilibria of a nonholonomic system are not isolated it has been suggested that one should not speak of the stability of single equilibria, but of the entire "manifold" of equilibria; a theory has been developed in [7]. The theory uses the unstated assumption that the rank of $S'(q_0) + (B^*)'[\,\cdot\,, h_0]$ is n at all points of the equilibrium set, or at least that the set of equilibria is a manifold. The notion of instability for an equilibrium should then be the following: let Z be the set of equilibria (a closed set, essentially the zero-set of f); a point $(q_0, 0) \in Z$ is $Z-$unstable if there exists an open set $V \supseteq Z$ such that, for every neigborhood U of $(q_0, 0)$ there is a point $(q, p) \in U$ such that the trajectory starting at (q, p) exits V at some future instant.

If the characteristic equation at an equilibrium point has a root with strictly positive real part, is the equilibrium unstable, also in this more general sense? It is stated without proof in [7] that this is the case. Here we prove it (a part of the following argument is due to Gaetano Zampieri), dispensing with any assumptions concerning the manifold structure of the equilibrium set. First, note that if $c > 0$ is smaller than the minimum positive real part of an eigenvalue of W, then there is a solution $\varphi(t) = (q(t), p(t))$ of (10), defined on $]-\infty, 0]$, such that:

$$\varphi(t) \neq (q_0, 0) \quad \text{for all } t \leq 0, \qquad \lim_{t\to-\infty} |\varphi(t) - (q_0, 0)|e^{-ct} = 0$$

(see [H], remark to Corollary 6.1, pag. 243). Such a solution is admissible: $B(q(t))p(t)$ is constant, and has limit $B(q_0)0 = 0$ as $t \to -\infty$, hence it is identically zero.

PROPOSITION 3.3. *If the characteristic equation of a nonholonomic system at an equilibrium point has a root with stricly positive real part, then this equilibrium is unstable.*

Dimostrazione. Using the solution $\varphi :]-\infty, 0] \to \Omega \times X$ as above, pick an open set V containing Z but not $\varphi(0)$ (Z, as above, is the set of equilibria, a closed set); for every neighborhood U of $(q_0, 0)$ there is $\bar{t} < 0$ such that $\varphi(\bar{t}) \in U$; but $\varphi(t + \bar{t})$, at time $t = -\bar{t}$, is outside of V.

□

Recall that, in the absence of dissipation ($F = R = 0$) the eigenvalues of W are exactly the square roots of the eigenvalues of $f'(q_0)$ (Proposition 3.2); since nonzero square roots are in opposite pairs, the only case in which we do not necessarily have instability is that in which all eigenvalues of $f'(q_0)$ are real and negative (≤ 0), which corresponds to all roots of the characteristic equation being purely imaginary (or zero).

As a corollary, we obtain an instability theorem due to Barone (see [1]), which we state in a more general form:

COROLLARY 3.2. *Assume that there is no dissipation. Let* (q_0, λ_0) *be an equilibrium point such that* $K(q_0) = 1_X$*; assume that* $v \mapsto v \cdot \big(S'(q_0)v + B^{*\prime}[v, \lambda_0]\big)$ *is a positive definite form on the space* $\ker(B(q_0))$*. Then* q_0 *is unstable.*

Dimostrazione. We have seen in Corollary 3 that the image of

$$f'(q_0)[\,\cdot\,] = S'(q_0)[\,\cdot\,] + B^{*\prime}(q_0)[\,\cdot\,, h(q_0)] + B^*(q_0)h'(q_0)[\,\cdot\,]$$

is contained in $V = \ker(B(q_0))$; since $B^*(\Lambda) = V^\perp$, we have $(\lambda_0 = h(q_0))$

$$\begin{aligned} v \cdot (f'(q_0)v) &= v \cdot (S'(q_0)v) + v \cdot (B^{*\prime}[v, \lambda_0]) + v \cdot (B^*(q_0)h'(q_0)v) \\ &= v \cdot (S'(q_0)v) + v \cdot (B^{*\prime}[v, \lambda_0]), \end{aligned}$$

that is, the quadratic form of the statement concides with $v \mapsto v \cdot (f'(q_0)v)$. Positivity of this, together with $f'(q_0)(X) \subseteq V$, implies that *all* eigenvalues of $f'(q_0)$, except for m trivial ones, have stricly positive real part.

□

7. An example

We consider: $\Omega = \mathbb{R}^3$; kinetic energy $T(q)[\dot q, \dot q] = |\dot q|^2/2$, potential energy $U(q) = \varepsilon|q|^2/2$, where $\varepsilon \in \mathbb{R}/\{0\}$ and $q = (x, y, z) \in \mathbb{R}^3$, with viscous resistance $R(q)\dot q = -\rho\dot q$, $\rho > 0$ a scalar; the constraint on the velocities is $B(q)\dot q = x\dot x + y\dot y + (1+x-y)\dot z$. The system is nonholonomic ($dB \wedge B = -(x+y)\,dx \wedge dy \wedge dz \neq 0$). Differentiating the constraint with respect to time we get $x\ddot x + y\ddot y + (1+x-y)\ddot z + \dot x^2 + \dot y^2 + (\dot x - \dot y)\dot z = 0$; the equations are

$$\text{(20)} \qquad \begin{cases} \ddot x + \rho\dot x + \varepsilon x &= x\lambda \\ \ddot y + \rho\dot y + \varepsilon y &= y\lambda \\ \ddot z + \rho\dot z + \varepsilon z &= (1+x-y)\lambda \\ x\ddot x + y\ddot y + (1+x-y)\ddot z + \dot x^2 + \dot y^2 + (\dot x - \dot y)\dot z &= 0 \end{cases}$$

Next we look for equilibria, the solutions of the system

$$\varepsilon x = x\lambda; \quad \varepsilon y = y\lambda; \quad \varepsilon z = (1+x-y)\lambda;$$

if $x = y = 0$, the third gives $z = \lambda/\varepsilon$; hence the $z-$axis is made of equilibria; if either x, or y is nonzero we get $\lambda = \varepsilon$; for this λ the first two are true for all x, y, and the third gives $z = (1+x-y)/\varepsilon$, a plane of equilibria. Notice that the matrix $U''(q) - (B^*)'[\,\cdot\,, \lambda]$ is

$$\begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ \lambda & -\lambda & 0 \end{pmatrix} = \begin{pmatrix} \varepsilon-\lambda & 0 & 0 \\ 0 & \varepsilon-\lambda & 0 \\ -\lambda & \lambda & \varepsilon \end{pmatrix},$$

with rank always 3, except for $\lambda = \varepsilon$, which corresponds to the equilibria of the plane $z = (1+x-y)/\varepsilon$. Notice also that at $(0, 0, \varepsilon)$ the set of equilibria does not have any manifold structure.

The characteristic polynomial at the point $(0, 0, 0)$ is

$$\chi(s) = s^2(s^2 + \rho s + \varepsilon)^2.$$

The system has the total energy $E(q, \dot{q}) = (|\dot{q}|^2 + \varepsilon|q|^2)/2$ as a Liapunov function, which is also a first integral if $\rho = 0$: this is true both for the holonomic system obtained forgetting the velocity constraints, and the non–holonomic one given. Assuming $\varepsilon > 0$, the energy is a proper function (it tends to $+\infty$ at infinity) and is positive definite; then the holonomic associated system has the origin as unique point of stable equilibrium, which if $\rho > 0$ is also asymptotically stable and a global attractor. Stability is of course in this case not destroyed by adding the velocity constraint, since E is still a Liapunov function; but asymptotic stability of the origin vanishes. Observe in fact that the first two equations in (20) are the same; the solution with initial value $x_0 = y_0 = u$, $\dot{x}_0 = \dot{y}_0 = v$ will then be the same function $\xi(t)$, for all $t \geq 0$; from the relation $x\dot{x} + y\dot{y} + (1 + x - y)\dot{z} = 0$ we get $2\xi\dot{\xi} + \dot{z} = 0$; we then have $z + \xi^2 = 2u^2 + z_0$ along the solution, and z, ξ cannot both tend to 0 as $t \to \infty$. Asymptotic stability has been destroyed from the constraint on the velocities.

8. An isolated equilibrium

Consider the system

$$\begin{array}{ll} T(q) = |\dot{q}|^2/2; & F(q)\dot{q} = \rho\dot{q} \quad (\rho \in \mathbb{R},\ \rho \geq 0); \\ U(q) = \frac{(x-1)^2+(y-1)^2}{2}; & B(q)\dot{q} = -y\dot{x} + x\dot{y} + ((y-1)^2 + z^2)\dot{z} \end{array}$$

(observe that B has rank 1 for every q, and that $dB(q) \wedge B(q) = (z^2 - 2y + 1)\, dx \wedge dy \wedge dz \neq 0$) whose equilibria are the solution of the system $\nabla U(q) = B^*(q)\lambda$, that is

$$x - 1 = -y\lambda \quad y - 1 = x\lambda \quad 0 = ((y-1)^2 + z^2)\lambda$$

If $\lambda = 0$ we have the solutions $\{(1, 1, z) : z \in \mathbb{R}\}$; it is a line of equilibria; if $\lambda \neq 0$ we get the solution $(0, 1, 0)$, $\lambda = 1$, an isolated equilibrium.

The characteristic polynomial at the point $(0, 1, 0)$ is $\chi(s) = \det(s1_6 - W) = s^3(s+\rho)(1 + \rho s + s^2)$. Of course the energy is a Liapunov function for the system, but this time E is not proper, since it does not depend on z.

9. Nonlinear constraints

We sketch here how the method can be applied also to not necessarily linear nonholonomic constraints (a nice mechanical example is, for instance, in [2]). These are $b(q, \dot{q}) = 0$, with $b : \Omega \times X \to \Lambda$ a sufficiently regular function; we assume that $b(q, 0) = 0$ for every $q \in \Omega$, an hypothesis satisfied by a large class of constraints, e.g. homogeneous ones of positive degree ([6]). We put $A(q, p) = \partial_q b(q, p)$, $B(q, p) = \partial_p b(q, p)$ ($A(q, p)$, $B(q, p)$ are both elements of $\mathrm{L}(X, \Lambda)$); assume that for every $(q, p) \in \Omega \times X$ such that $b(q, p) = 0$ the rank of $B(q, p)$ is m (full rank hypothesis). Differentiating $b(q, p) = 0$ with respect to time t we obtain:

$$A(q, \dot{q})\dot{q} + B(q, \dot{q})\ddot{q} = 0.$$

Observe that since we assumed that $b(q, 0) = 0$ identically, we have $A(q, 0) = 0$ for every $q \in \Omega$. We then proceed exactly as in section 2, with $A(q, \dot{q})[\dot{q}]$ replacing the term $B'(q)[\dot{q}, \dot{q}]$, and $B(q, \dot{q})\ddot{q}$ replacing $B(q)\ddot{q}$. We obtain again (10), except that now B, and hence also D, P, f, h etc. depend also on $p = \dot{q}$, and not only on q.

In phase space the equation writes

$$
(21) \qquad \begin{cases} \dot{q} &= p \\ \dot{p} &= f(q,p) - R(q,p)[p] + g(q,p)[p,p] \end{cases} ; \qquad b(q,p) = 0.
$$

where

$$
\begin{aligned}
f(q,p) &= K^{-1}(q)P(q,p)S(q); \\
R(q,p) &= K^{-1}(q)P(q,p)F(q) - D^{-1}(q,p)A(q,p); \\
g(q,p) &= -K^{-1}(q)P(q,p)K(q); \\
P(q,p) &= 1_X - B^*(q,p)D^{-1}(q,p)B(q,p)K^{-1}(q).
\end{aligned}
$$

Calling again $\Phi(q,p)$ the second member of (21), we have to show that at an equilibrium point $(q_0, 0)$ we still have a structure for $\Phi'(q_0, 0)$ as in section 5, that is

$$
\Phi'(q_0, 0) = \begin{pmatrix} 0 & 1_X \\ M & N \end{pmatrix}
$$

with M, N operators such that $M(X)$, $N(X)$ are contained in $V = \ker(B(q_0, 0))$. It is not difficult to follow the proof given in section 5 to show that this is the case; for N, recall also that $A(q_0, 0) = 0$.

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