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BIFURCATION OF PERIODIC ORBITS OF TIME DEPENDENT HAMILTONIAN SYSTEMS ON SYMPLECTIC MANIFOLDS

Sommario. On a symplectic manifold M, a 1-parameter family of time dependent Hamiltonian vectors fields which possesses a known trivial branch u_λ of 1-periodic solution is considered. If the relative Conley Zehnder index of the monodromy path along $u_\lambda(0)$ is defined and does not vanish it is shown that any neighborhood of the trivial branch contains 1-periodic solutions not in the branch. This result is applied to bifurcation of fixed points of Hamiltonian symplectomorphisms when the first Betty number of M vanishes.

1. Introduction

Existence and multiplicity of periodic trajectories of Hamiltonian vector fields on symplectic manifolds is a traditional field of research, which recently found new input from the work on Arnold's conjecture. Bifurcation of periodic orbits attracted much interest as well. In the fifties much of the research was concerned with the classification of the various bifurcation phenomena for generic families of autonomous vector fields (an excellent review is contained in [1]). Topological methods introduced by Alexander and Yorke [3] in Hopf bifurcation and by Weinstein [20] and Moser [15] in their generalizations of the Liapunov center theorem permitted to drop many of the genericity assumptions that are intrinsic to the approach based on singularity theory. Faddel and Rabinowitz [7] and more recently Bartsch [5] used various index theories based on the S^1 invariance of the action functional in the case of an autonomous vector field. The same property was used by Ize, Dancer and Rybakowski [12], [6] who studied bifurcation of periodic orbits using the S^1-equivariant degree. The methods mentioned above can be applied to the autonomous case only. Bifurcation for time dependent vector fields deserved very little attention. Recently Fitzpatrick, Pejsachowicz and Recht [9] using their results relating the spectral flow to the bifurcation of critical points of strongly indefinite functionals [8] studied bifurcation of periodic solutions of one-parameter families of (time dependent) periodic Hamiltonian systems in $\mathbb{R}^{2n}$. The purpose of this paper is to extend their results to families of time dependent Hamiltonian vector fields acting on symplectic manifolds and to discuss the related problem of bifurcation of fixed points of one parameter families of symplectomorphisms.

Our main result can be stated as follows:

Let $X_\lambda;\ \lambda \in [0,1]$ be a one parameter family of 1-periodic Hamiltonian vector fields on a closed symplectic manifold M. Assume that the family X_λ possesses a known, trivial, branch u_λ of 1-periodic solutions. Let $p_\lambda = u_\lambda(0)$ and let P_λ be the period map of the flow associated to X_λ. If the relative Conley-Zehnder index of the path P_λ along p_λ is defined and does not vanish, then any neighborhood of the trivial branch of periodic solutions contains 1-periodic solutions *not in the branch.*

Notice that the hypothesis in our bifurcation theorem are not of local nature. Most of the difficulties of the extension to manifolds are due to this fact. We first extend to manifolds the concept relative Conley-Zehnder index defined in [9]. An interesting feature of this concept is that in general it depends on the homotopy class of the path and not on the endpoints only. On symplectic manifold one can have closed paths of orbits with nontrivial Conley-Zehnder index. This will be done in section 2. In section 4 following an idea of [18] in the nonparametric case, we construct Darboux coordinates adapted to a given parametrized family of periodic solutions. The new coordinates allow to reduce the proof of bifurcation theorem to the case considered in [9].

In section 3 we state the main theorem and discuss some consequences regarding bifurcation of fixed points of Hamiltonian symplectomorfisms. A symplectomorphism is called Hamiltonian if it can be realized as a time one map of a one-periodic Hamiltonian vector field. Fixed points of a Hamiltonian symplectomorphisms are in one to one corrrespondence with 1-periodic orbits of the corresponding vector field. The Arnold conjecture states that a generic Hamiltonian symplectomorphism has more fixed points that could be predicted from the fixed point index. More precisely, by the fixed point theory a symplectomorphism isotopic to the identity with non-degenerate fixed points must have at least as many fixed points as the Euler-Poincaré characteristic of the manifold. But the number of fixed points of a Hamiltonian symplectomorphism verifying the same non-degeneracy assumptions is bounded bellow by the sum of the Betti numbers. Roughly speaking, this can be explained by the presence of a variational structure in the problem. Fixed points viewed as periodic orbits of the corresponding vector field are critical points of the action functional either if the orbits are contractible or when the symplectic form is exact.

Applied to bifurcation of fixed points of one parameter families of Hamiltonian symplectomorphisms our result shows a similar influence on the presence of a variational structure. In order to illustrate the analogy, let us consider a one parameter family of diffeomorphisms $\psi_\lambda; \lambda \in [0, 1]$ of an oriented manifold M, assuming for simplicity that $\psi_\lambda(p) = p$ and that p is a non degenerate fixed point of $\psi_i; i = 0, 1$. The work of [12] implies that the only homotopy invariant determining the bifurcation of fixed points in terms of the family of linearizations $L \equiv T_p\psi_\lambda$ at p is given by the parity $\pi(L) = \mathrm{sign}(\det T_p\psi_0) \cdot \mathrm{sign}(\det T_p\psi_1) \in \mathbb{Z}_2 = \{1, -1\}$. Here det is the determinant of an endomorphism of the oriented vector space T_pM. In other words bifurcation arise whenever the $\det T_p\psi_\lambda$ change sign at the end points of the interval. Moreover, any family of diffeomorphisms close enough to ψ in the C^1-topology and having p as fixed point undergoes bifurcation as well. On the contrary if both sign coincide one can find a perturbation as above with no bifurcation points at all.

On the other hand, on the base of our theorem, the integer valued Conley-Zehnder index provides a stronger bifurcation invariant for one parameter families of Hamiltonian symplectomorphisms. It forces bifurcation of fixed points whenever $\mathcal{CZ}(L)$ is non zero even when $\pi(L) = 1$. The relation between the two invariants is $\pi(L) = (-1)^{\mathcal{CZ}(L)}$ and it is easy to construct examples with even non-vanishing Conley-Zehnder index. Our assumption that the family of symplectomorphisms is Hamiltonian is due to the method in proof only. It is reasonable to expect that the same is true for an arbitrary path of symplectomorphisms with non-vanishing Conley-Zehnder index. The proof could go either via generating functions or by reduction to the above case, but I was unable to complete all the details involved. Our results in the form treated here do not apply directly to the autonomous case. In that case 1 is always a Floquet multiplier whenever the periodic orbit does not reduce to a singular point. A modification of our arguments using symplectic reduction much as in [14] are quite posible and will be treated separately. Yet another very interesting problem, in the vein of the analogy discussed above, is to obtain multiplicity results for bifurcating fixed points similar to the ones for critical points of functionals obtained by Bartsch [5].

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2. The Maslov index and the Conley Zehnder index

Before going to the manifold setting let us shortly discuss the case of $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ with the standard symplectic form $\omega = \sum dx_i \wedge dy_i$. The group of real $2n \times 2n$ symplectic matrices will be denoted by $\mathbf{Sp}(2n, \mathbb{R})$. The relative Conley-Zehnder index is a homotopy invariant associated to any path $\psi\colon [0, 1] \to \mathbf{Sp}(2n, \mathbb{R})$ of symplectic matrices with no eigenvectors corresponding to the eigenvalue 1 at the end points. This invariant counts algebraically the number of parameters t in the open interval $(0, 1)$ for which $\psi(t)$ has 1 as an eigenvalue. One of the possible constructions uses the Maslov index for non-closed paths. We shall define it along the lines of Arnold [4] for closed paths. For an alternative construction see [16]. A n-dimensional subspace l of $(\mathbb{R}^{2n}, \omega)$ is called *Lagrangian* provided that the restriction of ω to l is 0. The Lagrangian Grassmanian $\Lambda(n)$ consists of all Lagrangian subspaces of $\mathbb{R}^{2n}$ considered as a topological space with the topology it inherits as a subspace of the ordinary Grassmanian of n-planes. Let J be the self adjoint endomorphism representing the form ω with respect to the standard scalar product in $\mathbb{R}^{2n}$. Namely, $\omega(u, v) \equiv < Ju, v >$. Then J is a complex structure, it is indeed the standard one. It coincides with multiplication by i under the isomorphism sending $(x, y) \in \mathbb{R}^{2n}$ into $x + iy$ in $\mathbb{C}^n$. In terms of this representation, a Lagrangian subspace is characterized by $Jl = l^\perp$. Using the above description one can easily identify $\Lambda(n)$ with the homogeneous space $U(n)/O(n)$. The identification is as follows: given any orthonormal basis of a lagrangian subspace l there exist a unique unitary endomorphism $u \in U(n)$ sending the canonical basis of $l_0 = \mathbb{R}^n \times \{0\}$ into the given one and in particular sending l_0 into l. Moreover the isotropy group of l_0 can be easily identified with $O(n)$. Hence we obtain a diffeomorphism between $U(n)/O(n)$ and $\Lambda(n)$ sending $[u]$ into $u(l_0)$. Since the determinant of an element in $O(n)$ is ± 1, the map sending u into the square of the determinant of u factorizes through $\Lambda(n) \equiv U(n)/O(n)$ and hence induces a one form $\Theta \in \Omega^1\big(\Lambda(n)\big)$ given by $\Theta = [det^2]^*\theta$, where $\theta \in \Omega^1(S^1)$ is the standard angular form on the unit circle. This form is called the Keller-Maslov-Arnold form. The Maslov index of a closed path γ in $\Lambda(n)$ is the integer defined by $\mu(\gamma) = \int_\gamma \Theta$. In other words $\mu(\gamma)$ is the winding number of the closed curve $t \to det^2\big(\gamma(t)\big)$. The Maslov index induces an isomorphism between $\pi_1(\Lambda)$ and Z. The construction can be extended to non-closed paths as follows: fix $l \in \Lambda(n)$. If l' is any Lagrangian subspace transverse to l then l' can be easily identified with the graph of a symmetric transformation from Jl into itself. It follows from this that the set T_l of all lagrangian subspaces l' transverse to l is an affine space diffeomorphic to the space of all symmetric forms on $\mathbb{R}^n$ and hence contractible. We shall say that a path in $\Lambda(n)$ is admissible with respect to l if the end points of the path are transverse to l. The Maslov index $\mu(\gamma, l)$ of an admissible path γ with respect to l is defined as follows: take any path δ in T_l joining the end points of γ and define

$$\mu(\gamma; l) \equiv \mu(\gamma') = \int_{\gamma'} \Theta. \tag{1}$$

where γ' is the path γ followed by δ.

Clearly the result is independent of the choice of δ. Moreover, since T_l is contractible, $\mu(\gamma; l)$ is invariant under homotopies keeping the end points in T_l.

Geometricaly, the Maslov index $\mu(\gamma; l)$ can be interpreted as an intersection index of the path γ with the one codimensional analytic set $\Sigma_l = \Lambda(n) - T_l$ (see [17]). From the above definition it is easy to see that the index is additive under concatenation of paths. Namely, given two admissible paths α and β with $\alpha(1) = \beta(0)$

$$\mu(\alpha \star \beta; l) = \mu(\alpha; l) + \mu(\beta; l). \tag{2}$$

Since $\mathbf{Sp}(2n, \mathbb{R})$ is connected it follows easily from the homotopy invariance that $\mu(S\gamma; Sl) = \mu(\gamma; l)$ for any symplectic isomorphism S. This allows to extend the notion of Maslov Index to paths of Lagrangian subspaces in $\Lambda(V)$, where (V, ω) is any finite dimensional symplectic vector space. It is well known that symplectic endomorphisms are characterized by the following property: their graphs are lagrangian subspaces of the symplectic vector space $W = V \times V$ endowed with the symplectic form $\Omega = (-\omega) \times \omega$. Clearly 1 is not an eigenvalue of $P \in \mathbf{Sp}(2n, \mathbb{R})$ if and only if the graph of P is transversal to the diagonal $\Delta \subset V \times V$. A path $\phi\colon [0, 1] \to \mathbf{Sp}(2n, \mathbb{R})$ will be called admissible if 1 is not in the spectrum of its end points. For such a path the *relative Conley-Zehnder index* is defined by

$$\mathcal{CZ}(\phi) = \mu(Graph\phi, \Delta). \tag{3}$$

It follows from the above discussion that $\mathcal{CZ}(\phi)$ is invariant under admissible homotopies and it is additive with respect to the concatenation of paths. Clearly $\mathcal{CZ}(\phi) = 0$ if the fixed point subspace of $\phi(t)$ reduces to $\{0\}$ for all t.

There is one more property of the Conley-Zehnder index that we will need in the sequel. Namely, that for any $\alpha\colon [0, 1] \to \mathbf{Sp}(2n, \mathbb{R})$ and any admissible path ϕ

$$\mathcal{CZ}(\alpha^{-1}\phi\alpha) = \mathcal{CZ}(\phi). \tag{4}$$

This can be proved as follows. Since the spectrum is invariant by conjugation, the homotopy $(t, s) \to \alpha^{-1}(s)\phi(t)\alpha(s)$ shows that $\mathcal{CZ}(\alpha^{-1}\phi\alpha) = \mathcal{CZ}(\alpha^{-1}(0)\phi\alpha(0))$. Now (4) follows by the same argument applied to any path joining $\alpha(0)$ to the identity.

The property (4) allows to associate a Conley-Zehnder index to any admissible symplectic automorphism of a symplectic vector-bundle over an interval. Let I be the interval [0, 1], then any symplectic bundle $\pi\colon E \to I$ over I has a symplectic trivialization (see section 3). If $S\colon E \to E$ is a symplectic endomorphism of E over I well behaved at the end points, then we can define the Conley-Zehnder index of S as follows: if $T\colon E \to I \times \mathbb{R}^{2n}$ is any symplectic trivialization, then $TST^{-1}(t, v)$ has the form $(t, \phi_T(t)v)$ where ϕ_T is an admissible path on $\mathbf{Sp}(2n, \mathbb{R})$. Any change of trivialization induces a change on ϕ_T that has the form of the left hand side in (4) and hence $\mathcal{CZ}(\phi_T)$ is independent of the choice of trivialization. Thus this value $\mathcal{CZ}(S)$ is by definition the Conley-Zehnder index of S.

With this in hand we finally can define the relative Conley-Zehnder index of a path of symplectomorphisms along a path of fixed points. Let M be a closed symplectic manifold and let $Symp(M)$ be the group of all symplectomorphisms endowed with the C^1 topology. Let $\Phi\colon I \to Symp(M)$ be a smooth path of symplectomorphisms of M. Let $\beta\colon I \to M$ be a path in M such that $\beta(t)$ is a fixed point of $\Phi(t)$. Floquet multipliers of $\Phi(t)$ at $\beta(t)$ are by definition the eigenvalues of $S_t = T_{\beta(t)}\Phi(t)\colon T_{\beta(t)}(M) \to T_{\beta(t)}(M)$. A fixed point will be called non degenerate if none of its Floquet multipliers is one. Consistently, we will call the pair (Φ, β) admissible whenever $\beta(i)$ is a non degenerate fixed point of $\Phi(i)$ for $i = 0, 1$.

Let $E = \beta^*[T(M)]$ be the pullback by β of the tangent bundle of M (we use the same notation for the bundle and its total space). The family of tangent maps $S_t = T_{\beta(t)}\Phi(t)$ induces a symplectic automorphism $S\colon E \to E$ over I. By definition the *relative Conley-Zehnder index of* Φ *along* β is

$$\mathcal{CZ}(\Phi;\beta) \equiv \mathcal{CZ}(S). \tag{5}$$

It follows easily from the properties discussed above that $\mathcal{CZ}(\Phi;\beta)$ is invariant by smooth (and even C^1) pairs of homotopies $(\Phi(s,t), \beta(s,t))$ such that $\Phi(s,t)(\beta(s,t)) = \beta(s,t)$ and such that for $i = 0, 1$; $\beta(s,i)$ is a non degenerate fixed point of $\Phi(s,i)$. The index is additive under concatenation. It follows from (4) that it has another interesting property, which for simplicity we state in the case of a constant path $\beta(t) \equiv p$.

If $\Phi, \Psi\colon I \to Symp(M)$ are two admissible paths in the isotropy subgroup of p then

$$\mathcal{CZ}(\Psi \circ \Phi, p) = \mathcal{CZ}(\Phi \circ \Psi, p). \tag{6}$$

In other words $\mathcal{CZ}$ is a "trace".

We close this section with yet another formula that allows to compute the individual contribution of a regular point in the trivial branch to the Conley-Zehnder index. Assume that t_0 is isolated point in the set

$$\Sigma = \{t/\beta(t) \text{is a degenerate fixed point of } \Phi(t)\}.$$

Define $\mathcal{CZ}_{t_0}(\Phi) \equiv \lim_{\epsilon\to 0} \mathcal{CZ}(\Phi;\beta_{|[-\epsilon,\epsilon]})$. The point t_0 will be called regular (cf. [17]) if the quadratic form Q_{t_0} defined on the eigenspace $E_1(S_{t_0}) = Ker(S_{t_0} - Id)$ corresponding to the eigenvalue 1 by $Q_{t_0}(v) = \omega(\dot{S}_{t_0}v, v)$ is nondegenerate. Here $S_t = T_{\beta(t)}\Phi(t)$ as before and $\dot{S}_{t_0}$ denotes the intrinsic derivative of the vector bundle endomorphism S (See [10] chap 1 sect 5). It is easy to see that if t_0 is a regular point then it is an isolated point in Σ and moreover

$$\mathcal{CZ}_{t_0}(\Phi) = -\sigma(Q_{t_0}) \tag{7}$$

where σ denotes the signature of a quadratic form.

We shall not prove this formula here. It follows from the definition of the intrinsic derivative and formula (2.8) in [9].

3. The Main Theorem and Some Consequences

Let M^{2n} be a closed symplectic manifold of dimension $2n$ with symplectic form ϖ. Every smooth time dependent Hamiltonian function $H\colon \mathbb{R} \times M \to M$ gives rise to a time dependent Hamiltonian vector field $X_t\colon \mathbb{R} \times M \to TM$ defined by

$$\varpi(X(t,x), \xi) = d_x H(t,x)\xi$$

for $\xi \in T_x M$. If H is periodic in time with period 1, then so is X. By compactness and periodicity the solutions $u(t)$ of the initial value problem for the Hamiltonian differential equation

$$\begin{cases} \frac{d}{dt}u(t) = X(t, u(t)), \\ u(s) = x \end{cases} \tag{8}$$

are defined for all times t. The flow (or evolution map) associated to each X is the two-parameter family of symplectic diffeomorphisms $\psi : \mathbb{R}^2 \to Symp(M)$ defined by

$$\psi_{s,t}(x) = u(t)$$

where u is the unique solution of (8).

By the uniqueness and smooth dependence on initial value theorems for solutions of differential equations the map $\psi : \mathbb{R}^2 \times M \to M$ is smooth. The diffeomorphisms $\psi_{s,t}$ verify the usual cocycle property of an evolution operator. It follows from this property that for each fixed s, the map sending u into $u(s)$ is a bijection between the set of 1-periodic solutions of the time dependent vector field X and the set of all fixed points of $\psi_{s,s+1}$. Hence in order to find periodic trajectories of (8) we can restrict our attention to the fixed points of $P = \psi_{0,1}$.

The map $P = \psi_{0,1}$ is called the period or Poincaré map of X. A 1-periodic trajectory is called non degenerate* if $p = u(0)$ is a non degenerate fixed point of P, i.e. if the monodromy operator $S_p \equiv T_p P : T_p M \to T_p M$ has no 1 as eigenvalue. Consistently, the eigenvalues of the monodromy operator will be called Floquet multipliers of the periodic trajectory. Although we will not show it here, it is easy to see that the particular choice of $s = 0$ is irrelevant to the property of being non degenerate since the Floquet multipliers do not depend on this choice.

Let us consider now a smooth one parameter family of time dependent Hamiltonian functions $H : I \times \mathbb{R} \times M \to \mathbb{R}$, where $I = [0, 1]$ and each H_λ is one periodic in time. Let $X \equiv \{X_\lambda\}_{\lambda \in [0,1]}$ be the corresponding one parameter family of Hamiltonian vector fields. Then the flows $\psi_{\lambda,s,t}$ associated to each X_λ depend smoothly on the parameter $\lambda \in I$. Suppose also that our 1-parameter family of Hamiltonian vector fields X_λ possesses a known smooth family of 1-periodic solutions u_λ; $u_\lambda(t) = u_\lambda(t + 1)$. Solutions u_λ in this family will be called *trivial* and we will seek for sufficient conditions in order to find nontrivial solutions arbitrarily close to the given family. Identifying $\mathbb{R}/\mathbb{Z}$ with the circle S^1 we will regard the family of trivial solutions either as a path $\tau : I \to C^1(S^1; M)$ defined by $\tau(\lambda) = u_\lambda$ or as a smooth map $u : I \times S^1 \to M$.

A point $\lambda_* \in I$ is called a *bifurcation point* of periodic solutions from the trivial branch u_λ if every neighborhood of $(\lambda_*, u_{\lambda_*})$ in $I \times C^1(S^1; M)$ contain pairs of the form (λ, v_λ) where v_λ is a nontrivial periodic trajectory of X_λ.

It is easy to see that a necessary (but not sufficient) condition for a point λ_* to be of bifurcation is that 1 is a Floquet multiplier of u_{λ_*}. (See for example [4] Proposition 26.1). Thus non degenerate orbits cannot be bifurcation points of the branch. In what follows we will assume that $u(0)$ and $u(1)$ are non degenerate and we will seek for bifurcation points in the open interval $(0, 1)$.

Consider the path $p : I \to M$ given by $p(\lambda) = u_\lambda(0)$. Each $p(\lambda)$ is a fixed point of the symplectomorphism $P_\lambda = \psi_{\lambda,0,1}$. Under our hypothesis, the pair (P, p) is admissible. The number $\mathcal{CZ}(P, p)$ constructed in the previous section will be called the relative Conley-Zehnder index of $X \equiv \{X_\lambda\}_{\lambda \in [0,1]}$ along the trivial family u. We will denote it by $\mathcal{CZ}(X, u)$. It is easy to see that the particular choice of $s = 0$ is unmaterial for the definition of the Conley-Zehnder index.

THEOREM 1.1. *Let $X \equiv \{X_\lambda\}_{\lambda \in [0,1]}$ be a one parameter family of 1-periodic Hamiltonian vector fields on a closed symplectic manifold (M, ϖ). Assume that the family X_λ possesses a known, trivial, branch u_λ of 1-periodic solutions such that $u(0)$ and $u(1)$ are non degenerate.*

*in Abraham Robbins terminology this corrresponds to the concept of a transversal period.

If the relative Conley-Zehnder index $\mathcal{CZ}(X, u) \neq 0$ then the interval I contains at least one bifurcation point for periodic solutions from the trivial branch u.

The proof will be given in the next section.

COROLLARY 1.1. *Let X and u be as before. Let $S_\lambda = T_{p(\lambda)} P_\lambda$ be the monodromy operator at $p(\lambda)$. If $\lambda_* \in I$ is such that 1 is an eigenvalue of S_{λ_*} and the quadratic form $Q_{\lambda_0}(v) = \omega(\dot{S}_{t_0} v, v)$ is nondegenerate on the eigenspace $E_1(S_{\lambda_*})$ and $\sigma(Q_{\lambda_0}) \neq 0$ then λ_* is a bifurcation point for periodic solutions from the trivial branch u.*

This follows from (7).

Let us discuss now the relationship with bifurcation of fixed points of symplectomorphisms. We shall assume here that the first Betti number $\beta_1(M)$ of M vanishes. It is well known that in this case any symplectic diffeomorphism belonging to the connected component of the identity $Symp_0(M)$ of the group of all symplectic diffeomorphisms can be realized as the time one map of a 1-periodic Hamiltonian vector field. Bellow we will sketch the proof of the fact that for smooth paths in $Symp_0(M)$ the corresponding path of Hamiltonian functions can be choosen smoothly.

Given a path Φ in $Symp_0(M)$ let us choose a symplectic isotopy Ψ with $\Psi_1 = Id$ and $\Psi_0 = \Phi(0)$. Define now $\Xi\colon I \times I \to Symp_0(M)$ by

$$\Xi(\lambda, \tau) = \begin{cases} \Phi(\lambda - \tau) & \text{for } \tau \leq \lambda \ \& \ \lambda + \tau \leq 1 \\ \Psi(\tau - \lambda) & \text{for } \lambda \leq \tau \ \& \ \lambda + \tau \leq 1 \\ \Xi(\tau, 1 - \tau) & \text{for } \tau, \lambda \in I \ \& \ \lambda + \tau \geq 1 \end{cases}$$

Then Ξ is continuous on $I \times I$ and moreover $\Xi(\lambda, 0) = \Phi(\lambda)$ and $\Xi(\lambda, 1) = Id$. Let α be any smooth function with $0 \leq \alpha \leq 1$ and such that $\alpha(\tau) \equiv 0$ for $0 \leq \tau \leq 1/3$ and $\alpha(\tau) \equiv 1$ for $2/3 \leq \tau \leq 1$. Then $\tilde{\Xi}(\lambda, \tau) = \Xi(\lambda, \alpha(\tau))$ coincides in a neighborhood of $I \times \{0\}$ with Φ and on a neighborhood of $I \times \{1\}$ with Id.

Using smooth partitions of unity the map $\tilde{\Xi}$ can be approximated by a smooth map $\Theta\colon I \times I \to Symp_0(M)$ having the same property. Now consider the parametrized family of time dependent vector fields $X \equiv X_\lambda;\ \lambda \in I$ defined on $[0, 1] \times M$ by $X_\lambda(t, m) = u'(t)$ where $u(\tau) = \Theta_\lambda(\tau, \Theta_\lambda^{-1}(t, m))$. By construction the map Θ is constant on vertical slices near to the top and bottom side of I^2 and therefore each X_λ vanishes there. This allows to extend X to a parametrized family of 1-periodic vector fields. It is clear from the definition of X that the period map for the field X_λ is precisely Φ_λ.

On the other hand, each $\Theta(\lambda, \tau)$ is symplectic. This implies that for each (λ, t) the one form $\iota(X_{\lambda,t})\varpi$ is closed and therefore exact, since we are assuming that $H^1(M; \mathbb{R}) = 0$. It is easy to see that, fixing the value of the Hamiltonian function at a given point, one can choose a smooth family of one periodic Hamiltonian functions for Ξ_λ.

With this in hand, using Theorem 1.1 and the correspondence between 1-periodic orbits of the Hamiltonian vector field with fixed points of the period map, we obtain:

COROLLARY 1.2. *Assume that $\beta_1(M) = 0$. Let Φ_λ be a path in $Symp_0(M)$ such that $\Phi_\lambda(p) = p$ for all λ and such that as fixed point of Φ_0 and Φ_1, p is non degenerate. Then if $\mathcal{CZ}(\Phi, p) \neq 0$, there exist a $\lambda_* \in (0, 1)$ such that any neighborhood of (λ_*, p) in $I \times M$*

contains a point (λ, q) *such that* q *is a fixed point of* Φ_λ *different from* p *(i.e* λ_* *is a bifurcation point for fixed points of* Φ_λ *from the trivial branch* p*).*

Moreover the same is true for any close enough path in the C^1*-topology lying in the isotropy group of* p*.*

For closed paths we have stronger results.

COROLLARY 1.3. *Let* Φ *be as before but with* $\Phi(0) = \Phi(1)$. *If* $\mathcal{CZ}(\Phi, p) \neq 0$ *then any path* Ψ *in the isotropy group of* p *that is freely* C^1*-homotopic to* Φ *must have nontrivial fixed points arbitrarily close to* p*.*

This follows from the homotopy invariance of the Conley-Zehnder index.

EXAMPLE 1.1. Let M be the symplectic manifold $S^2 = \mathbb{C} \cup \{\infty\}$. Consider the closed path of symplectic maps $\Phi_\theta : S^2 \to S^2$; $\theta \in [0, 1]$ defined by

$$\Phi_\theta(z) = \begin{cases} e^{i2\pi(\theta-1/2)} \cdot z & \text{if } z \in \mathbb{C}, \\ \infty & \text{if } z = \infty. \end{cases} \tag{9}$$

Since Φ_θ is a rotation of angle $\theta - 1/2$ it leaves fixed only the points $z = 0$ and $z = \infty$ except for $\theta = 1/2$. In this case the fixed point set is the sphere S^2.

For each θ the tangent map $T_0\Phi_\theta$ of Φ_θ at the fixed point $z = 0$ equals Φ_θ. The only value of θ for which 1 is an eigenvalue of $T_0\Phi_\theta$ is $\theta = 1/2$ for which the corresponding eigenspace is $\mathbb{C}$. Moreover 0 is a regular degenerate fixed point of $\Phi_{1/2}$ because, as we will see, the quadratic form $Q_{1/2} = \omega(\dot{\Phi}_{1/2}-, -)$ is non degenerate on the eigenspace $E_1(\Phi_{1/2})$.

We will use equation (7) to calculate the relative Conley-Zehnder index of the symplectic isotopy Φ along the constant path of fixed points $p = 0$. Since

$$\dot{\Phi}(1/2) = i2\pi Id$$

it follows that

$$\mathcal{CZ}_0(\Phi; 0) = -\sigma[v \to \omega(\dot{\Phi}(1/2)v, v)] = \sigma[v \to 2\pi < v, v >] = 2.$$

Therefore by Corollary 3.4 any closed path of symplectomorphisms of the sphere keeping 0 fixed and homotopic to Φ has nontrivial fixed points close to zero. Notice that this bifurcation cannot be detected using parity.

4. Darboux Coordinates and the Proof of the Main Theorem

First we prove an elementary lemma about trivializations of symplectic vector bundles.

LEMMA 1.1. *Any symplectic vector bundle* $E \to Z$ *with base space a cylinder* $Z = S^1 \times [0, 1]$ *is symplectically isomorphic to the trivial bundle* $Z \times \mathbb{R}^{2n}$ *endowed with the canonical symplectic form.*

This lemma is a very special case of the Proposition 2.64 in [13] but since the proof in this case is considerable simpler we sketch it here.

Dimostrazione. First of all any symplectic bundle admits a compatible complex structure (see [13]). Thus we have an automorphism $J: E \to E$ such that $J^2 = -Id$ and such that $\omega_z(Jv, v)$ is a positive definite quadratic form on each fiber. This not only makes E into a complex vector bundle, but also endows this complex bundle with a hermitian scalar product defined by $[v, w]_z = \omega_z(Jv, w) + i\omega_z(v, w)$. Let $i: S^1 \to Z$ be the inclusion as $S^1 \times 0$ and let $\pi: Z \to S^1$ be the projection. Since $i \circ \pi$ is homotopic to the identity of Z we have that $\pi^*(i^*E)$ is isomorphic to E as a complex vector bundle (see [11]). Since $U(n)$ is a deformation retract of $Gl(n, \mathbb{C})$ the structure group can be reduced to $U(n)$. It follows from this that we can find an isomorphism preserving the hermitian products and hence the symplectic structure as well. In this way we have reduced the problem to find a unitary trivialization of the bundle $i^*(E)$ over the circle S^1. This can be certainly found over an interval, which is contractible, and then patch the obtained isomorphisms at the end points using the fact that $U(n)$ is connected. (See [19] for an explicit trivialization).

□

We will need also the following parametrized version of Darboux theorem.

LEMMA 1.2. *Let Z be a smooth compact connected manifold (with or without boundary), let U be an open convex neighborhood of 0 in $\mathbb{R}^{2n}$ and let Ω_1 be a closed 2-form on $W = Z \times U$ such that if $i_z: U \to W$ is the inclusion given by $i_z(x) = (z, x)$ then the smooth family of two forms $\omega_z = i_z^*(\Omega_1)$; $z \in Z$ verifies $\omega_z(0) = \omega(0)$, where ω is the standard symplectic form on $\mathbb{R}^{2n}$. Then there exist a smooth family Ψ_z; $z \in Z$ of symplectomorphisms defined on an eventually smaller neighborhood V of 0 such that $\Psi_z(0) = 0$ and $\Psi_z^*(\omega_z)_{|V} = \omega_{|V}$.*

Dimostrazione. We adapt to our situation the Weinstein method in [21]. Let $p: W \to \mathbb{R}^{2n}$ be the projection and $\Omega_0 = p^*(\omega)$. Consider $h: [0, 1] \times W \to W$ defined by $h(t, z, x) = (z, tx)$ then $h_0(W) \subset Z_0 = Z \times \{0\}$, $h_1 = Id_W$ and the restriction of h_t to Z_0 is the identity map of Z_0. Let $X_t: W \to TW$ be the vector field defined by $X_t(z, x) = (z, x, 0, x)$. Thus $X_t(z, x)$ is the tangent vector to the path $h(-, x)$ at the time t.

Applying the Cartan's formula to the form $\Omega = \Omega_1 - \Omega_0$ we get

$$\frac{d}{dt} h_t^* \Omega = h_t^*(\iota(X_t) d\Omega) + d h_t^*(\iota(X_t)\Omega) \tag{10}$$

where $h_t^*(\iota(X_t)\Omega)$ is the one-form on W given at $w = (z, x)$ by

$$h_t^*(\iota(X_t)\Omega)(w)[v] = (\iota(X_t(w))\Omega)(T_w(h_t)v)$$

and $\iota(X_t)$ denotes the interior product or contraction in the first variable.

Integrating the right hand side from zero to one we obtain

$$\Omega - h_0^*\Omega = \int_0^1 h_t^*(\iota(X_t)d\Omega)\, dt + d \int_0^1 h_t^*(\iota(X_t)\Omega)\, dt. \tag{11}$$

If we define for any m-form σ the $m-1$-form $H\sigma = \int_0^1 h_t^*(\iota(X_t)\sigma)\, dt$ then (11) becomes

$$\Omega - h_0^*\Omega = H d\Omega + d H\Omega. \tag{12}$$

Since the restriction of Ω to Z_0 is zero it follows that $h_0^*\Omega = 0$.

On the other hand $d\Omega = 0$ since both Ω_1 and Ω_0 are closed. Putting this information in (10) we get $\Omega = d\Lambda$ where $\Lambda = H\Omega$. From this we obtain that $\omega_z - \omega = i_z^*(\Omega) = i_z^*(d\Lambda) = di_z^*(\Lambda)$. Putting $\lambda_z = i_z^*(\Lambda)$ we obtained a smooth family of one forms on U parametrized by Z such that

$$\omega_z - \omega = d\lambda_z. \tag{13}$$

For $0 \leq s \leq 1$ let us consider $\omega_{s,z} = \omega + s(\omega_z - \omega) = \omega + sd\lambda_z$. Since $\omega_{s,z}(0) = \omega(0)$ is non-degenerate there is a neighborhood V of 0 such that each $\omega_{s,z}$ is non-degenerate on V. Thus we can find a family of time dependent vector fields (here the time is s) $X_{s,z}$ defined in $V \times R$ and smoothly depending on $z \in Z$ such that

$$\iota(X_{s,z})\omega_{s,z} = -\lambda_z. \tag{14}$$

By taking eventually smaller V we can assume that the integral curves of the Cauchy problem

$$\begin{cases} \frac{d}{ds}u(s) = X_{s,z}u(s) \\ u(0) = x \end{cases} \tag{15}$$

are defined for all s in $[0, 1]$. Let us consider the flow $\Phi : Z \times [0, 1] \times V \to V$ be defined by $\Phi(z, s, x) = u(s)$ where u is the unique solution of (14). Using the smooth dependence on parameters of solutions of differential equations and by the usual flow properties we have that $\Phi_{z,s} \colon V \to U$ is a family of diffeomorphisms smoothly depending on (z,s) such that $\Phi_{z,0}x = x$. Now let $\Phi_z = \Phi_{z,1}$ be the time one map, then we have

$$\Phi_z^*\omega_z - \omega = \int_0^1 \frac{d}{ds}\Phi_{z,s}^*\omega_{z,s}\,ds = \int_0^1 \Phi_{z,s}^*(\omega_z - \omega)\,ds + \int_0^1 d(\iota(X_{s,z})\omega_{s,z})\,ds = 0. \tag{16}$$

Here we are using (14) the fact that $d\omega_{s,z} = 0$ and a generalization of (10), proved in [10] (chapter 4 section 1), which states for any one-parameter family of forms ω_s and one parameter family of maps Φ_s that

$$\frac{d}{ds}\Phi_s^*\omega_s = \Phi_s^*\frac{d}{ds}\omega_s + \Phi_s^*\iota(X_s)d\omega_s + d\Phi_s^*\big(\iota(X_s)\omega_s\big)$$

where as before X_s is the tangent vector to the curve $\Phi(-, x)$ at time s. The formula (16) shows that Φ_z is the family we are looking for.

□

REMARK 1.1. In the above Lemma the assumption that Ω_1 is closed can be dropped, the lemma holds for any parametrized family of symplectic forms ω_z. But the proof (along the lines sketched in [13]) is harder since it uses more elaborate tools such as Hodge theory.

LEMMA 1.3. *(compare with Lemma 2.9 in [18]) Let (M, ϖ) be a $2n$-dimensional symplectic manifold. Let $Z = [0, 1] \times S^1$ be a cylinder and let $u \colon Z \to M$ be a map. Then there exist an open neighborhood V of 0 in $\mathbb{R}^{2n}$ and a family of diffeomorphisms $\theta_z \colon V \to M$ smoothly parametrized by $z \in Z$ such that $\theta_z(0) = u(z)$ and $\theta_z^*\varpi = \omega$.*

Dimostrazione. We endow M with an almost complex structure J compatible with ϖ. Then the family of bilinear forms $g(v, w) = \varpi(Jv, w)$ is a Riemannian metric and has an associated exponential diffeomorphism exp_m defined in a neighborhood of zero in the tangent space at $m \in M$. The vector bundle $E = u^*(TM)$ is a symplectic vector bundle over Z and hence, by Lemma 2.1 has a symplectic trivialization. The inverse of this trivialization composed with the natural bundle map from E into TM is a vector bundle homomorphism ψ over u from the trivial symplectic bundle $Z \times \mathbb{R}^{2n}$ into TM such that for each $z \in Z$ and $x, y \in \mathbb{R}^{2n}$

$$\varpi(u(z))(\psi_z x, \psi_z y) = \omega(x, y) .$$

Take O an open neighborhood of the zero section in TM where the exponential map is defined and let U be any neighborhood of 0 in $\mathbb{R}^{2n}$ such that $\psi_z(u) \in O$ for all $z \in Z$. Define $\Psi\colon Z \times U \to M$ by $\Psi(z, x) = exp_{u(z)}\big(\psi_z(x)\big)$. Let $\Omega_1 = \Psi^*\varpi$. Since ϖ is closed it follows that Ω_1 is a closed form on $W = Z \times U$. Moreover with the notations of Lemma 2.2.

$$\omega_z(0) = i_z\Omega_1(0) = \Psi_z^*\varpi(u(z)) = \omega(0)$$

by (4) and the fact that $T_0 exp = Id$. Thus the hypothesis of the lemma 2.2 are verified and hence there exists a neighborhood V of 0 and a smooth family of difeomorphisms Φ_z such that $\Phi_z^*\omega_z = \omega$. Now the family $\theta\colon Z \times V \to M$ defined by $\theta_z = \Psi_z \circ \Phi_z$ satisfies the requirements of the lemma.

□

Proof of Theorem 1.1. Choose local Darboux coordinates $(V, \phi_{\lambda,t})$ on the manifold M near the λ-parameter family $u_\lambda(t)$ of periodic solutions of the Hamiltonian differential equation

$$\begin{cases} \frac{d}{dt}u_\lambda(t) = X_\lambda(t, u_\lambda(t)), \\ u_\lambda(s) = x \end{cases} \tag{17}$$

i.e. V is an open neighborhood of 0 in $\mathbb{R}^{2n}$ and $\phi_{\lambda,t}\colon V \to M$ satisfies $\phi_{\lambda,t}(0) = u_\lambda(t)$ and $\phi_{\lambda,t}^*\varpi = \omega$ on V.

Thus such a solution is sent by the inverse map of $\phi_{\lambda,t}$ into the trivial branch $y_\lambda(t) \equiv 0$ in $\mathbb{R}^{2n}$, solution of the Hamiltonian differential equation

$$\dot{y}_\lambda(t) = Y_\lambda(t, y_\lambda(t)), \tag{18}$$

where Y_λ denotes the Hamiltonian vector field associated to the Hamiltonian function

$$\tilde{H}_{\lambda,t} := H_{\lambda,t} \circ \phi_{\lambda,t} : V \to \mathbb{R} .$$

Because

$$\iota(Y_\lambda)\omega = d\tilde{H}_{\lambda,t} = d(H_{\lambda,t} \circ \phi_{\lambda,t}) = \phi_{\lambda,t}^*(dH_{\lambda,t})$$

and $dH_{\lambda,t} = \iota(X_\lambda)\sigma$ it follows that

$$\begin{aligned} \iota(Y_\lambda)\omega &= \phi_{\lambda,t}^*(\iota(X_\lambda)\varpi) \\ &= \iota(T\phi_{\lambda,t}^{-1}(X_\lambda))\phi_{\lambda,t}^*\varpi \\ &= \iota(T\phi_{\lambda,t}^{-1}(X_\lambda))\omega \end{aligned}$$

hence we get that $T\phi_{\lambda,t}^{-1}(X_\lambda) = Y_\lambda$. This means that the Hamiltonian vector fields Y_λ and X_λ are $\phi_{\lambda,t}$-related, thus so are their integral flows and consequently their respective Poincaré time maps. Therefore we have that

$$\lambda \to \tilde{P}_\lambda = (\phi_{\lambda,1})^{-1} \circ P_\lambda \circ \phi_{\lambda,1}$$

is a symplectic path.

By naturality of the Conley-Zehnder index (Cf. (4)) we conclude that $\mathcal{CZ}(\tilde{P}_\lambda) = \mathcal{CZ}(P_\lambda)$.

On the other hand, writing $TV = V \times \mathbb{R}^{2n}$ the induced vector field $Y_{\lambda,t}$ takes the form $Y_{\lambda,t}(x) = (x, F_{\lambda,t}(x))$ where $F_{\lambda,t}(x) = J_0 \nabla \tilde{H}_\lambda$ is the principal part of the field. Denote with $\Psi_{\lambda,t}$ the flow of $Y_{\lambda,t}$. Let $U_{\lambda,t} = D\Psi_{\lambda,t}(0)$ be the Frechet derivative of $\Psi_{\lambda,t}$ at $0 \in V$. An easy calculation shows that the matrix $U_{\lambda,t}$ is a solution of the initial value problem

$$\begin{cases} U'_{\lambda,t} &= DF_{\lambda,t}(0)U_{\lambda,t} \\ U_{\lambda,0} &= Id. \end{cases} \tag{19}$$

Taking $t = 1$ it follows that $D\tilde{P}_\lambda$ coincide with the path of monodromy matrices M_λ used in [9] in order to define the relative Conley-Zehnder index along a branch of periodic orbits.

In the given trivialization

$$T_0 \tilde{P}_\lambda(v) = D\Psi_{\lambda,1}(0)v = M_\lambda(v)$$

and from this follows that $\mathcal{CZ}(\tilde{P}_\lambda)$ coincide with their definition of the Conley -Zehnder index. Under our hypothesis $\mathcal{CZ}(\tilde{P}_\lambda) \neq 0$ and hence by the main theorem in [9] there are nonzero 1-periodic orbits of (18) that bifurcate from the branch of trivial solutions of the Hamiltonian equation (18).

But nontrivial periodic orbits of the Hamiltonian equation (18) bifurcating from the trivial branch $y_\lambda(t) \equiv 0$ corrrespond by the diffeomorphisms $\phi_{\lambda,t}$ to periodic orbits of the Hamiltonian equation (17) different from u_λ.

This complete the proof of the Theorem.

□

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