Rend. Sem. Mat. Univ. Pol. Torino
Vol. 57, 2 (1999)

**W. Dambrosio**

# A TIME-MAP APPROACH FOR NON-HOMOGENEOUS STURM-LIOUVILLE PROBLEMS*

**Abstract.** By means of a time-map approach, we study the existence of multiple solutions to the boundary value problem

$$\begin{cases} u'' + f(u) = 0 \\ u(0) = sA, \ u(\pi) = sB. \end{cases}$$

The results depend on the values of the real numbers $s$, $A$ and $B$, and on the behaviour of the ratio $f(u)/u$ for $u$ near zero and near infinity. Both the asymptotically linear and superlinear asymmetric growth conditions at infinity are considered.

## 1. Introduction

This paper is concerned with the existence of multiple solutions to a non-homogeneous Dirichlet problem of the form

$$\begin{cases} u'' + f(u) = 0 \\ u(0) = sA, \ u(\pi) = sB, \end{cases} \tag{1}$$

$A$, $B$ and $s$ being real numbers and $f : \mathbb{R} \longrightarrow \mathbb{R}$ being a continuous function; we define the potential $F(x) = \int_0^x f(t)\, dt$ and we assume that

$$f(x)x > 0 \quad \text{for all} \quad x \neq 0 \tag{2}$$

and

$$\lim_{|x| \to +\infty} F(x) = +\infty\,. \tag{3}$$

It is well-known (see e.g. [1, 2, 5]) that in general the number of solutions to boundary value problems associated to an equation as

$$u'' + f(u) = 0 \tag{4}$$

strongly depends on the behaviour of the ratio $f(u)/u$ for $u \to 0$ and $u \to \infty$. In this article, we deal with two situations which are rather classical in literature: the *asymptotically linear* case, characterized by

$$\lim_{|u| \to +\infty} \frac{f(u)}{u} = \beta \geq 0\,, \tag{5}$$

*Work performed under the auspices of GNAFA, C.N.R. - Italy.

and the *superlinear asymmetric* case, for which we suppose

$$\lim_{u\to+\infty}\frac{f(u)}{u}=+\infty\,,\qquad \lim_{u\to-\infty}\frac{f(u)}{u}=\gamma\geq 0\,; \tag{6}$$

moreover, we shall always assume that

$$\lim_{u\to 0}\frac{f(u)}{u}=h\geq 0\,. \tag{7}$$

When (5) is assumed, then multiplicity results for various boundary value problems have been obtained; more precisely, in [1, 9, 10, 11] the authors prove the existence of multiple solutions for

$$u''+f(u)=q(t)\,, \tag{8}$$

with

$$u(0)=0=u(\pi)\,, \tag{9}$$

for large positive forcing terms $q$. More recently, in [7, 8], it was shown that for every continuous function $h$ there are one, two or three solutions to (8) together with

$$u(0)=\sigma_1,\ u(\pi)=\sigma_2 \tag{10}$$

or

$$u'(0)=\sigma_1,\ u'(\pi)=\sigma_2\,, \tag{11}$$

depending on the position of $(\sigma_1,\sigma_2)$ with respect to the classical Fučik spectrum.

If, in addition to (5), also (7) is considered, then the existence of solutions to (4)–(9) can be proved by studying the "gap" between the numbers $\beta$ and $h$. Indeed, see e.g. [5], the number of solutions of (4)–(9) coincides with the number of eigenvalues of the $u\mapsto -u''$ operator (with boundary conditions (9)) which fall between $\beta$ and $h$ (or viceversa): this means that the number of solutions depends on the number of eigenvalues crossed by the nonlinearity $f$ passing from zero to infinity. For a similar discussion, relative to the more general case of the Laplacian operator in $\mathbb{R}^n$, we refer to [4].

Similar results have been obtained for the superlinear asymmetric case. Indeed, when only (6) is assumed, the existence of multiple solutions to (8)–(9) for large $h$ has been proved in [13] (in the case $h$ constant) and in [15] (for non constant $h$).

More recently, in [2] a result on the lines of the above quoted paper [5] for (4)–(9), under assumptions (6)–(7), has been obtained.

In this paper we shall prove the existence of a certain number of solutions to (1) when (5)–(7) or (6)–(7) are assumed. More precisely, suppose that $B>A>0$ and that there exist positive integers $l$, $j$ and $p$ such that

$$l^2<\beta<(l+1)^2\,, \tag{12}$$

$$j^2<h<(j+1)^2 \tag{13}$$

and

$$p^2<\gamma<(p+1)^2\,; \tag{14}$$

moreover, we consider the intervals $I_1 = \left[j + \frac{5}{2} + \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\frac{j+1}{j}\frac{1}{\pi}, l - 1\right]$, $I_2 = \left[l + 2, j - \frac{3}{2} - \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\frac{j+1}{j}\frac{1}{\pi}\right]$, $I_3 = \left[j + \frac{5}{2} + \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\frac{j+1}{j}\frac{1}{\pi}, 2p - 2\right]$ and $I_4 = \left[2p + 4, j - \frac{3}{2} - \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\frac{j+1}{j}\frac{1}{\pi}\right]$ and we denote by $N$ ($N \in \mathbb{N}$) the number of the integers $m$ such that $2m \in I_1$ or $2m \in I_2$ and by $M$ the number of the integers $m$ such that $2m \in I_3$ or $2m \in I_4$. We observe that $N$ and $M$, as well as the intervals $I_k$ ($k = 1, \ldots, 4$) depend on the assumptions on $\beta$, $h$ and $\gamma$, i.e. on the behaviour of the nonlinearity at infinity and in zero. It could also happen that there are no integers $m$ satisfying one of the previous conditions: in this case we take $N = 0$ or $M = 0$ and no existence and multiplicity results can be obtained (indeed, also in the case $A = B = 0$ it can be shown that there are problems without nontrivial solutions).

Then, we will prove (see Theorem 3 and Theorem 4):

THEOREM 1. *Assume (5) and (7); moreover, suppose (12) and (13) are satisfied. Then, for every $B > A > 0$ there exist $s_k > 0$ ($k = 0, \ldots, 2N - 2$) such that for every $s \in (s_{2i-1}, s_{2i})$ ($i = 0, \ldots, N - 1$; $s_{-1} := 0$) problem (1) has at least $4(N - i)$ nontrivial solutions.*

THEOREM 2. *Assume (6) and (7); moreover, suppose (14) and (13) are satisfied. Then, for every $B > A > 0$ there exist $s_k > 0$ ($k = 0, \ldots, 2M - 2$) such that for every $s \in (s_{2i-1}, s_{2i})$ ($i = 0, \ldots, M - 1$, $s_{-1} := 0$) problem (1) has at least $4(M - i)$ nontrivial solutions.*

We point out that multiplicity results on the lines of Theorem 1 and Theorem 2 can be obtained also when $A \geq B$. Moreover, the cases when (5) or (7) are replaced by

$$\lim_{u \to \pm\infty} \frac{f(u)}{u} = \beta^{\pm}$$

or

$$\lim_{u \to 0^{\pm}} \frac{f(u)}{u} = h^{\pm}$$

can be considered as well. Similarly, multiple solutions can be obtained also when (12) (or (13) or (14)) is not satisfied, i.e. when there exists an integer $l_*$ such that $\beta = l_*^2$. We remark that in this case we are dealing with a resonant situation.

The proofs of Theorem 1 and Theorem 2 are based on the time-map technique introduced in [3]. More precisely, in order to study (1) we need to introduce *three* time-maps $T_1$, $T_2$ and $T_3$ and a function $T : \mathbb{R}^+ \longrightarrow \mathbb{R}^3$ (whose components are $T_1$, $T_2$ and $T_3$) which describe the solutions of our problem: indeed, there exists a set $S \subset \mathbb{R}^3$ ($S$ consists of four families of planes) such that (1) has a solution if and only if $T(\alpha) \in S$ for some $\alpha > 0$. This set $S$ is a 3-dimensional variant of the classical Fučik spectrum [6].

We refer to the papers [3, 14] for a more complete discussion on the use of the time-map technique for the study of boundary value problems.

The structure of the paper is as follows.
In Section 2 we explain the time-map technique and we introduce the set $S$ which is useful in order to describe the solutions to (1). In Section 3 we prove our main results, both for the asymptotically linear and the superlinear asymmetric case.

The author is grateful to Prof. A. Capietto for her valid support and encouragement.

## 2. Definition and asymptotic properties of the time-maps

In this section we study the following Picard problem

$$\begin{cases} u'' + f(u) = 0 \\ u(0) = sA, \; u(\pi) = sB, \end{cases} \tag{15}$$

$A$, $B$ and $s$ being real numbers and $f : \mathbb{R} \longrightarrow \mathbb{R}$ being a continuous function; we define the potential $F(x) = \int_0^x f(t)\,dt$ and we assume that

$$f(x)x > 0 \quad \text{for all} \quad x \neq 0 \tag{16}$$

and

$$\lim_{|x|\to+\infty} F(x) = +\infty\,. \tag{17}$$

We will give a multiplicity result, depending on the values of $A$, $B$ and $s$, for (15) in the case when $B > A > 0$ and $s > 0$; analogous results can be obtained in the other situations.

We introduce the energy associated to the equation in (15), namely $H(x, y) = \frac{1}{2}y^2 + F(x)$. For $\alpha > 0$ we denote by $F^\alpha$ the sub-levels of energy $F(\alpha)$, i.e.

$$F^\alpha = \{(x, y) \in \mathbb{R}^2 : H(x, y) < F(\alpha)\}$$

(see Figure 1). Let $\Gamma^\alpha$ be the boundary of $F^\alpha$; from now on, we shall assume that $\alpha \geq \alpha_0 = sB$. Therefore, the straight lines of equations $x = sA$ and $x = sB$ intersect, in the phase-plane $(x, y) = (u, u')$, the (closed) curve $\Gamma^\alpha$.

For every $\alpha > 0$, let $-\alpha_1 < 0$ be such that $F(-\alpha_1) = F(\alpha)$ and let us define the following time-maps:

$$\tau^+(\alpha) = \frac{\sqrt{2}}{2}\int_0^\alpha \frac{du}{\sqrt{F(\alpha) - F(u)}} \tag{18}$$

and

$$\tau^-(\alpha) = \frac{\sqrt{2}}{2}\int_{-\alpha_1}^0 \frac{du}{\sqrt{F(\alpha) - F(u)}}\,. \tag{19}$$

It is straightforward to check that $\tau^+(\alpha)$ and $\tau^-(\alpha)$ represent the time needed for a rotation, along $\Gamma^\alpha$, in the upper half-plane or in the lower half-plane, from the point of abscissa 0 to the point of abscissa $\alpha$ and from the point of abscissa $-\alpha_1$ to the point of abscissa 0, respectively.

Following the approach of [3], we define, for each energy level $\Gamma^\alpha$, the following three time-maps, which will enable us to describe the solutions of energy $F(\alpha)$. Indeed, we set:

$$T_1(\alpha) = \frac{\sqrt{2}}{2}\int_{sB}^\alpha \frac{du}{\sqrt{F(\alpha) - F(u)}}\,, \tag{20}$$

$$T_2(\alpha) = \frac{\sqrt{2}}{2}\int_{sA}^{sB} \frac{du}{\sqrt{F(\alpha) - F(u)}} \tag{21}$$

and

$$T_3(\alpha) = \tau^-(\alpha) + \frac{\sqrt{2}}{2}\int_0^{sA} \frac{du}{\sqrt{F(\alpha) - F(u)}}\,. \tag{22}$$

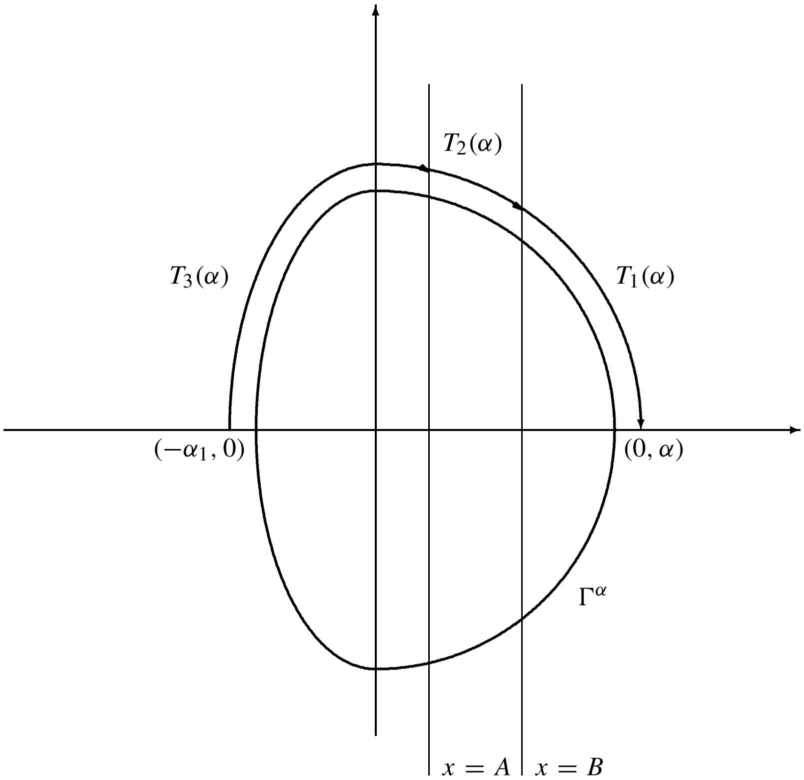


Figure 1: Time-maps for non-homogeneous problems.

As before, $T_1(\alpha)$ is the time needed by a solution of energy $F(\alpha)$ to rotate in the upper half-plane from the point of abscissa $sB$ to the point of abscissa $\alpha$. The quantities $T_2(\alpha)$ and $T_3(\alpha)$ have a similar meaning. We also remark that the symmetry of the orbits with respect to the $x$-axis implies that each $T_i(\alpha)$, $i = 1, 2, 3$, is also the time needed for a rotation between the corresponding points in the half-plane $y < 0$.

First of all, we observe that the functions $T_i$ $(i = 1, 2, 3)$ are continuous (this fact can be easily proved); secondly, we give some asymptotic estimates on $T_i$ $(i = 1, 2, 3)$ when $\alpha$ goes to infinity or to $\alpha_0$. To this aim, let us formally denote $1/0 = \infty$ and $1/\infty = 0$ and let us recall the following celebrated result, due to Z. Opial [12]:

LEMMA 1. *[[12], Corollaire 6] Let $f$ be a continuous function satisfying conditions (16) and (17). Then:*

$$\lim_{x \to +\infty} \frac{f(x)}{x} = k \in [0, +\infty] \quad \Longrightarrow \quad \lim_{\alpha \to +\infty} \tau^+(\alpha) = \frac{\pi}{2\sqrt{k}} \tag{23}$$

*and*

$$\lim_{x\to-\infty}\frac{f(x)}{x}=k\in[0,+\infty] \quad\Longrightarrow\quad \lim_{\alpha\to+\infty}\tau^-(\alpha)=\frac{\pi}{2\sqrt{k}}\,. \tag{24}$$

*Analogously,*

$$\lim_{x\to0^+}\frac{f(x)}{x}=k^+\in[0,+\infty] \quad\Longrightarrow\quad \lim_{\alpha\to0}\tau^+(\alpha)=\frac{\pi}{2\sqrt{k^+}} \tag{25}$$

*and*

$$\lim_{x\to0^-}\frac{f(x)}{x}=k^-\in[0,+\infty] \quad\Longrightarrow\quad \lim_{\alpha\to0}\tau^-(\alpha)=\frac{\pi}{2\sqrt{k^-}}\,. \tag{26}$$

An application of Lemma 1 gives the following proposition, which is a variant of [3, Lemma 3.2]; we point out that the estimates we prove are independent on the parameter $s$ which appears in the boundary conditions:

PROPOSITION 1. *Let us assume that*

$$\lim_{x\to\pm\infty}\frac{f(x)}{x}=\beta_\pm\in[0,+\infty]\,.$$

*Then*

$$\lim_{\alpha\to+\infty}T_1(\alpha) = \frac{\pi}{2\sqrt{\beta_+}}\,, \tag{27}$$

$$\lim_{\alpha\to+\infty}T_2(\alpha) = 0\,, \tag{28}$$

$$\lim_{\alpha\to+\infty}T_3(\alpha) = \frac{\pi}{2\sqrt{\beta_-}}\,. \tag{29}$$

*Proof.* First of all, we observe that the following inequality holds:

$$0\le x\le u\le y<\alpha \quad\Rightarrow\quad \frac{1}{\sqrt{F(\alpha)-F(x)}}\le\frac{1}{\sqrt{F(\alpha)-F(u)}}\le\frac{1}{\sqrt{F(\alpha)-F(y)}}\,. \tag{30}$$

Therefore

$$\int_0^{sB}\frac{du}{\sqrt{F(\alpha)-F(u)}}\le\frac{sB}{\sqrt{F(\alpha)-F(sB)}}\longrightarrow 0 \quad\text{as}\quad \alpha\to+\infty\,. \tag{31}$$

Now, we are in position to obtain the needed estimates: since

$$T_1(\alpha)=\tau^+(\alpha)-\frac{\sqrt{2}}{2}\int_0^{sB}\frac{du}{\sqrt{F(\alpha)-F(u)}}\,,$$

from (31) and (23) we deduce (27).

As far as (28) is concerned, we observe that (30) implies

$$T_2(\alpha)\le\frac{\sqrt{2}}{2}\frac{s(B-A)}{\sqrt{F(\alpha)-F(sB)}}$$

and a trivial application of (31) gives (28).

Finally, since

$$T_3(\alpha)=\tau^-(\alpha)+\frac{\sqrt{2}}{2}\int_0^{sA}\frac{du}{\sqrt{F(\alpha)-F(u)}}\,,$$

again from (30) we obtain (29). □

Now, we will prove some other estimates on the three time-maps introduced above; they are obtained by the study of the function $r : \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined by

$$r(s) = \frac{\sqrt{2}}{2} \int_0^{sA} \frac{du}{\sqrt{F(sB) - F(u)}} . \tag{32}$$

We give the asymptotic estimates for $T_i$ $(i = 1, 2, 3)$ for $\alpha \to \alpha_0$ (we recall that $\alpha_0 = sB$); they are trivial consequences of the continuity of $\tau$:

PROPOSITION 2. *The following estimates hold:*

$$T^0_{1,s} \quad := \quad \lim_{\alpha \to \alpha_0} T_1(\alpha) = 0 , \tag{33}$$

$$T^0_{2,s} \quad := \quad \lim_{\alpha \to \alpha_0} T_2(\alpha) = \tau^+(\alpha_0) - r(s) \tag{34}$$

*and*

$$T^0_{3,s} := \lim_{\alpha \to \alpha_0} T_3(\alpha) = \tau^-(\alpha_0) + r(s) . \tag{35}$$

By Proposition 2, in order to know the values of $T^0_{i,s}$ $(i = 2, 3)$ we must study the function $r$; to this aim, for every $s \in \mathbb{R}^+$, let

$$r_-(s) = \frac{\sqrt{2}}{2} \frac{sA}{\sqrt{F(sB)}} \tag{36}$$

and

$$r_+(s) = \frac{\sqrt{2}}{2} \frac{sA}{\sqrt{F(sB) - F(sA)}} . \tag{37}$$

¿From (30) we immediately obtain that

$$r_-(s) \le r(s) \le r_+(s) \quad \forall s \in \mathbb{R}^+ . \tag{38}$$

Moreover, we can prove the following:

LEMMA 2. *Let us assume that*

$$\lim_{x \to 0} \frac{f(x)}{x} = h . \tag{39}$$

*Then, for the function $r$ defined in (32), we have:*

$$\frac{A}{B} \frac{1}{\sqrt{h}} \le \lim_{s \to 0} r(s) \le \frac{\sqrt{2}}{2} \sqrt{\frac{A}{B - A}} \frac{1}{\sqrt{h}} . \tag{40}$$

*Proof.* Since (38) holds, it is sufficient to prove that

$$\lim_{s \to 0} r_-(s) = \frac{A}{B} \frac{1}{\sqrt{h}} \tag{41}$$

and

$$\lim_{s \to 0} r_+(s) = \frac{\sqrt{2}}{2} \sqrt{\frac{A}{B - A}} \frac{1}{\sqrt{h}} . \tag{42}$$

Indeed, we have

$$\lim_{s\to 0} r_-(s) = \frac{\sqrt{2}}{2}\frac{A}{B}\lim_{s\to 0}\sqrt{\frac{(sB)^2}{F(sB)}}\,;$$

then, an application of De l'Hospital rule, together with (39), gives

$$\lim_{s\to 0} r_-(s) = \frac{\sqrt{2}}{2}\frac{A}{B}\sqrt{2}\frac{1}{\sqrt{h}}\,.$$

Therefore, (41) is fulfilled.

Now, by the mean value theorem we infer that

$$F(sB) - F(sA) = f(s\xi)s(B-A) \tag{43}$$

for some $A < \xi < B$. Therefore,

$$r_+(s) = \frac{\sqrt{2}}{2}\frac{A}{\sqrt{\xi(B-A)}}\sqrt{\frac{s\xi}{f(s\xi)}} \le \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\sqrt{\frac{s\xi}{f(s\xi)}}\,;$$

hence, from (39) we obtain (42). □

Now, let us assume that (39) holds and that there exists an integer $j \in \mathbb{N}$ such that

$$j^2 < h < (j+1)^2\,. \tag{44}$$

We immediately observe that (44) implies

$$\frac{1}{j+1} < \frac{1}{\sqrt{h}} < \frac{1}{j}\,. \tag{45}$$

We are ready to prove the following:

PROPOSITION 3. *Assume (39) and (44); then, there exists* $s_0 > 0$ *such that for every* $s \in (0, s_0)$ *we have*

$$\frac{\pi}{2(j+1)} - \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\frac{1}{j} < T^0_{2,s} < \frac{\pi}{2j} - \frac{A}{B}\frac{1}{j+1} \tag{46}$$

*and*

$$\frac{\pi}{2(j+1)} + \frac{A}{B}\frac{1}{j+1} < T^0_{3,s} < \frac{\pi}{2j} + \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\frac{1}{j}\,. \tag{47}$$

*Proof.* By (25), (40) and (45), we deduce that

$$\lim_{s\to 0} T^0_{2,s} \le \frac{\pi}{2\sqrt{h}} - \frac{A}{B}\frac{1}{\sqrt{h}} < \frac{\pi}{2j} - \frac{A}{B}\frac{1}{j+1}\,.$$

Let $\epsilon > 0$ such that

$$\frac{\pi}{2\sqrt{h}} - \frac{A}{B}\frac{1}{\sqrt{h}} + \epsilon < \frac{\pi}{2j} - \frac{A}{B}\frac{1}{j+1}\,. \tag{48}$$

By the definition of limit, there exists $s_0 > 0$ such that, for every $s \in (0, s_0)$, we have

$$T^0_{2,s} < \frac{\pi}{2\sqrt{h}} - \frac{A}{B}\frac{1}{\sqrt{h}} + \epsilon\,;$$

this relation, together with (48), implies the right inequality in (46). The left inequality in (46) and (47) can be proved in a similar way. □

Now, let us assume that

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = \beta_\pm \,; \tag{49}$$

then, the following analogue of Lemma 2 holds:

LEMMA 3. *Assume (49); then, for the function $r$ defined in (32), we have:*

$$\frac{A}{B} \frac{1}{\sqrt{\beta_+}} \le \lim_{s \to +\infty} r(s) \le \frac{\sqrt{2}}{2} \sqrt{\frac{A}{B-A}} \frac{1}{\sqrt{\beta_+}} \,. \tag{50}$$

The proof of Lemma 3 is a straightforward variant of that of Lemma 2.

Now, suppose that $\beta_\pm = \beta$ and that there exists an integer $l \in \mathbb{N}$ such that

$$l^2 < \beta < (l+1)^2 \,. \tag{51}$$

As before, we observe that (51) implies

$$\frac{1}{l+1} < \frac{1}{\sqrt{\beta}} < \frac{1}{l} \,. \tag{52}$$

The following result can be proved arguing as in Proposition 3:

PROPOSITION 4. *Assume (49) and (51); then, there exists $s^* > 0$ such that for every $s > s^*$ we have*

$$\frac{\pi}{2(l+1)} - \frac{\sqrt{2}}{2} \sqrt{\frac{A}{B-A}} \frac{1}{l} < T^0_{2,s} < \frac{\pi}{2l} - \frac{A}{B} \frac{1}{l+1} \tag{53}$$

*and*

$$\frac{\pi}{2(l+1)} + \frac{A}{B} \frac{1}{l+1} < T^0_{3,s} < \frac{\pi}{2l} + \frac{\sqrt{2}}{2} \sqrt{\frac{A}{B-A}} \frac{1}{l} \,. \tag{54}$$

In the case when

$$\lim_{x \to +\infty} \frac{f(x)}{x} = +\infty \tag{55}$$

and

$$\lim_{x \to -\infty} \frac{f(x)}{x} = \gamma \,, \tag{56}$$

suppose that there exists an integer $l \in \mathbb{N}$ such that

$$l^2 < \gamma < (l+1)^2 \,. \tag{57}$$

Then, from (23), (24), (34), (35) and (50) we obtain

$$\lim_{s \to +\infty} T^0_{2,s} = 0$$

and

$$\lim_{s \to +\infty} T^0_{3,s} = \frac{\pi}{2\sqrt{\gamma}} \,.$$

Hence, we have the following:

PROPOSITION 5. *Assume (55), (56) and (57); then, for every* $\epsilon > 0$ *there exists* $s^* > 0$ *such that for every* $s > s^*$ *we have*

$$T^0_{2,s} < \epsilon \tag{58}$$

*and*

$$\frac{\pi}{2(l+1)} < T^0_{3,s} < \frac{\pi}{2l}\,. \tag{59}$$

Now, we recall from [3] that (15) has a solution of energy $F(\alpha)$ for some $\alpha > 0$ if and only if there exists an integer $m \in \mathbb{N}$ such that

$$2mT_1(\alpha) + (2m-1)T_2(\alpha) + 2mT_3(\alpha) = \pi$$

or

$$2mT_1(\alpha) + (2m-1)T_2(\alpha) + 2(m-1)T_3(\alpha) = \pi$$

or

$$2mT_1(\alpha) + (2m+1)T_2(\alpha) + 2(m+1)T_3(\alpha) = \pi$$

or

$$2mT_1(\alpha) + (2m+1)T_2(\alpha) + 2mT_3(\alpha) = \pi\,.$$

Let us introduce the set $S = S_1 \cup S_2$, where

$$\begin{aligned} S_1 \;=\; & \{(x,y,z) \in \mathbb{R}^3,\ x > 0,\ y \geq 0,\ z > 0 : \text{ there exists } m \in \mathbb{N} \text{ such that} \\ & a_m : 2mx + (2m-1)y + 2mz = \pi \text{ or } b_m : 2mx + (2m-1)y + 2(m-1)z = \pi\} \end{aligned}$$

and

$$\begin{aligned} S_2 \;=\; & \{(x,y,z) \in \mathbb{R}^3,\ x > 0,\ y \geq 0,\ z > 0 : \text{ there exists } m \in \mathbb{N} \text{ such that} \\ & c_m : 2mx + (2m+1)y + 2mz = \pi \text{ or } d_m : 2mx + (2m+1)y + 2(m+1)z = \pi\}\,. \end{aligned}$$

In Figure 2 we have drawn the projection of the set $\mathcal{S}$ on the plane $y = 0$, corresponding to the boundary conditions $A = B$. We note that in this case the set $\mathcal{S}$ reduces to a family of straight lines; moreover, we observe that we find the same "generalized Fučik spectrum" already used e.g. in [2] for the study of homogeneous Dirichlet problems.

¿From [3], we know that problem (15) has a solution of energy $F(\alpha)$ if and only if for the triple $T(\alpha) = (T_1(\alpha), T_2(\alpha), T_3(\alpha))$ we have $T(\alpha) \in S$. This means that there exists a correspondence between the solutions of (15) and the intersections (in $\mathbb{R}^3$) of the set $S$ with the support of the curve $T : \alpha \mapsto T(\alpha)$.

Hence, it is crucial to know the image of $T$ as a set in $\mathbb{R}^3$; more precisely, let us denote by $P_{0,s}$ the point of coordinates $\left(T^0_{1,s}, T^0_{2,s}, T^0_{3,s}\right)$ and by $P_\infty = (x_\infty, y_\infty, z_\infty)$ the point whose coordinates are given by

$$x_\infty = \lim_{\alpha \to +\infty} T_1(\alpha)\,,$$

$$y_\infty = \lim_{\alpha \to +\infty} T_2(\alpha)\,,$$

and

$$z_\infty = \lim_{\alpha \to +\infty} T_3(\alpha)\,.$$

Then, $T$ is a continuous curve connecting the points $P_{0,s}$ and $P_\infty$; moreover, the number of solutions of (15) coincides with the number of planes (belonging to the set $S$) which are crossed by any line from $P_{0,s}$ and $P_\infty$.

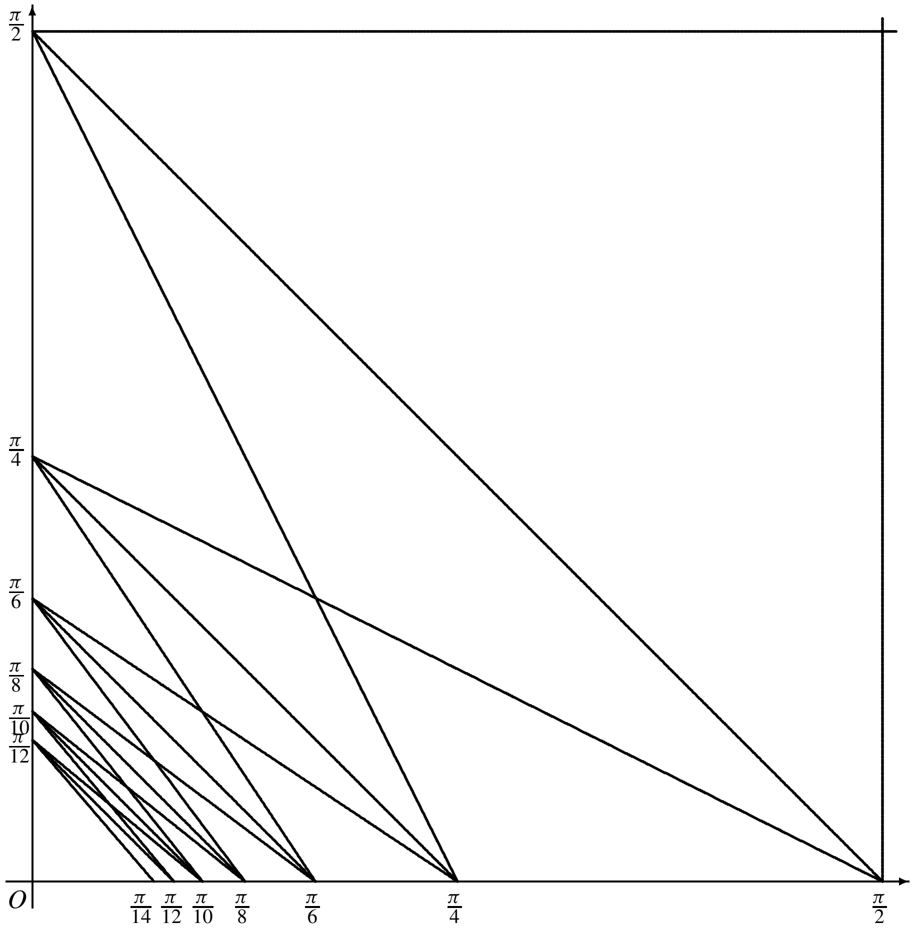


Figure 2: Some of the lines belonging to the set $\mathcal{S}$ for $A = B$.

### 3. The main results

First, we consider the asymptotically linear case, i.e. we assume that

(60) $$\lim_{x \to 0} \frac{f(x)}{x} = h$$

and

(61) $$\lim_{x \to \pm\infty} \frac{f(x)}{x} = \beta\,.$$

Moreover, we suppose that there exist two positive integers $j$ and $l$ such that

(62) $$j^2 < h < (j+1)^2$$

and

$$l^2 < \beta < (l+1)^2 . \tag{63}$$

Finally, let $N$ ($N$ possibly zero) be the number of positive integers $m$ such that $2m \in \Big[ j + \frac{5}{2} + \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\frac{j+1}{j}\frac{1}{\pi}, l-1 \Big]$ or $2m \in \Big[ l+2, j - \frac{3}{2} - \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\frac{j+1}{j}\frac{1}{\pi} \Big]$.

We will prove the following result:

THEOREM 3. *Assume that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function satisfying (16), (60) and (61); moreover, suppose (62) and (63) are satisfied. Then, for every $B > A > 0$ there exist $s_k > 0$ ($k = 0, \dots, 2N-2$) such that for every $s \in (s_{2i-1}, s_{2i})$ ($i = 0, \dots, N-1$; $s_{-1} := 0$) problem (15) has at least $4(N-i)$ nontrivial solutions.*

REMARK 1.

1. A result on the lines of Theorem 3 can be obtained also in the case the ratio $f(x)/x$ has no limit for $x \to 0$ (but there exist the left and the right limit) or when the limits at $\pm\infty$ are different.
2. Multiplicity theorems for (15) can be obtained also when $A$ and $B$ satisfy the condition $A \geq B \geq 0$ or when they are negative. We omit these results, whose proof is a variant of the one of Theorem 3; they are only based on slightly different computations on the time-maps.
3. On the lines of [2], we might state a result similar to Theorem 3 for the case when the numbers $h$ or $\beta$ do not satisfy conditions (62) or (63); indeed, let $j$, $j'$, $l$ and $l'$ be positive integers such that

   $$j^2 < h < (j'+1)^2 \tag{64}$$

   or

   $$l^2 < \beta < (l'+1)^2 ; \tag{65}$$

   then, Theorem 3 holds with $2m \in \Big[ j' + \frac{5}{2} + \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\frac{j'+1}{j'}\frac{1}{\pi}, l-1 \Big]$ or $2m \in \Big[ l' + 2, j - \frac{3}{2} - \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\frac{j+1}{j}\frac{1}{\pi} \Big]$. We also observe that conditions like (64) or (65) mean that the numbers $h$ and $\beta$ can be eigenvalues of the operator $u \mapsto -u''$ with Dirichlet conditions in $(0, \pi)$. For brevity, we omit the details.

Before proving Theorem 3, we recall that, by assumptions (60) and (61), in the present situation we have

$$P_\infty = \left( \frac{\pi}{2\sqrt{\beta}}, 0, \frac{\pi}{2\sqrt{\beta}} \right) \tag{66}$$

and $P_{0,s} = (0, T^0_{2,s}, T^0_{3,s})$, where for every $s \in (0, s_0)$ we have

$$\frac{\pi}{2(j+1)} - \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\frac{1}{j} < T^0_{2,s} < \frac{\pi}{2j} - \frac{A}{B}\frac{1}{j+1} \tag{67}$$

and

$$\frac{\pi}{2(j+1)} + \frac{A}{B}\frac{1}{j+1} < T^0_{3,s} < \frac{\pi}{2j} + \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\frac{1}{j} \tag{68}$$

and for every $s > s^*$

$$\frac{\pi}{2(l+1)} - \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\frac{1}{l} < T^0_{2,s} < \frac{\pi}{2l} - \frac{A}{B}\frac{1}{l+1} \tag{69}$$

and

$$\frac{\pi}{2(l+1)} + \frac{A}{B}\frac{1}{l+1} < T^0_{3,s} < \frac{\pi}{2l} + \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\frac{1}{l}\,. \tag{70}$$

Then, we can prove the following result:

LEMMA 4. *For every $s \in (0, s_0)$, problem (15) has at least $4N$ solutions.*

*Proof.* First of all, we recall that we denote by $N$ the number of integers $m$ such that $2m \in \left[j + \frac{5}{2} + \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\frac{j+1}{j}\frac{1}{\pi}, l-1\right]$ or $2m \in \left[l+2, j - \frac{3}{2} - \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\frac{j+1}{j}\frac{1}{\pi}\right]$. Moreover, let $m_1, \dots, m_N$ be these integers.

Let $s_0 > 0$ be as in Proposition 3 and let us fix $s \in (0, s_0)$. According to the discussion contained in Section 2, the solutions to problem (15) correspond to the intersections of the support of the curve $T$ with the set $\mathcal{S}$; more precisely, problem (15) has a solution of energy $F(\alpha)$ with $T(\alpha) \in a_m$ if the points $P_{0,s}$ and $P_\infty$ belong to the opposite half-spaces generated by the plane $a_m$. Analogous remarks are valid for the planes $b_m$, $c_m$ and $d_m$.

Now, let $a^\infty_m = 2mx_\infty + (2m-1)y_\infty + 2mz_\infty$ and let $a^0_m = 2mT^0_{1,s} + (2m-1)T^0_{2,s} + 2mT^0_{3,s}$. In order to obtain a solution with the prescribed property, it is sufficient that

$$a^\infty_m < \pi < a^0_m \tag{71}$$

or

$$a^0_m < \pi < a^\infty_m\,. \tag{72}$$

Using (63), (67) and (68), we can explicitate (71) and (72); more precisely, we have

$$\frac{2m}{l+1}\pi < a^\infty_m < \frac{2m}{l}\pi \tag{73}$$

and

$$\frac{4m-1}{2(j+1)}\pi + \frac{A}{B}\frac{1}{j+1} < a^0_m < \frac{4m-1}{2j}\pi + \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\frac{1}{j}\,. \tag{74}$$

Therefore, (71) is fulfilled if

$$\begin{cases} \dfrac{4m-1}{2(j+1)}\pi + \dfrac{A}{B}\dfrac{1}{j+1} \geq \pi \\[2ex] \dfrac{2m}{l}\pi \leq \pi\,, \end{cases} \tag{75}$$

while (72) is satisfied if

$$\begin{cases} \dfrac{4m-1}{2j}\pi + \dfrac{\sqrt{2}}{2}\sqrt{\dfrac{A}{B-A}}\dfrac{1}{j} \leq \pi \\[2ex] \dfrac{2m}{l+1}\pi \geq \pi\,. \end{cases} \tag{76}$$

Now, by some easy computations, it is easy to deduce that (76) and (75) are fulfilled for every integer $m_1, \dots, m_N$. This proves that there exist at least $N$ solutions to (15) corresponding to the planes $a_m$.

Analogous computations show that any line between the points $P_{0,s}$ and $P_\infty$ intersects the planes $b_m$, $c_m$ and $d_m$ for $m = m_1, \dots, m_N$. □

*Proof of Theorem 3.* We give the proof for the case

$$j + \frac{5}{2} + \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\frac{j+1}{j}\frac{1}{\pi} \leq l - 1\,,$$

the other case being similar.

From Lemma 4 we infer that for $s \in (0, s_0)$ there are at least $4N$ solutions to (15); more precisely, we can say that for $s \in (0, s_0)$ the point $P_{0,s}$ is "over" all the planes $a_k$, $b_k$, $c_k$ and $d_k$ for $k = m_1, \dots, m_N$, while the point $P_\infty$ is "under" all these planes.

Moreover, using the same argument developed in the proof of Lemma 4 it is easy to prove the following:

**Claim.** *For every $s > s^*$ let $Z_s := [P_\infty, P_{0,s}]$ be the segment from $P_\infty$ and $P_{0,s}$; then for every $k \in \mathbb{N}$ we have*

$$Z_s \cap a_k = Z_s \cap b_k = Z_s \cap c_k = Z_s \cap d_k = \emptyset\,.$$

Roughly speaking, the above Claim means that for large values of $s$ (i.e. for $s > s^*$) there are no planes belonging to the set $S$ between the points $P_{0,s}$ and $P_\infty$.

By the continuity of $T^0_{i,s}$ $(i = 1, 2)$ as functions of $s$, we can deduce that the point $P_{0,s}$, as $s$ increases, crosses the planes $a_k$, $b_k$, $c_k$ and $d_k$ for $k = m_1, \dots, m_N$. Indeed, there are $s_2 > s_1 > s_0$ $(s_2 < s^*)$ such that if $s \in (s_1, s_2)$ the point $P_{0,s}$ is under the planes $a_{m_1}$, $b_{m_1}$, $c_{m_1}$ and $d_{m_1}$, but it is over $a_k$, $b_k$, $c_k$ and $d_k$ for $k = m_2, \dots, m_N$: therefore, for $s \in (s_1, s_2)$, problem (15) has at least $4(N-1)$ solutions.

An inductive argument implies that there exist $s_0 < s_1 < s_2 < \dots < s_{2(n-1)-1} < s_{2(N-1)}$ such that for every $s \in (s_{2i-1}, s_{2i})$ $(i = 0, \dots, N-1,\ s_{-1} := 0)$ the point $P_{0,s}$ is "over" the planes $a_k$, $b_k$, $c_k$ and $d_k$ for $k = m_{i+1}, \dots, m_N$. Therefore, for $s \in (s_{2i-1}, s_{2i})$, the support of any curve connecting $P_{0,s}$ and $P_\infty$ must intersect at least $4((N-i-1)+1)$ planes and this proves the result. □

REMARK 2. Looking at the proof of Lemma 4, we observe that a more precise statement on the number of solutions to (15) could be obtained; indeed, solving the systems (75) and (76), it is possible to compute the exact range of integers $m$ such that (15) has a solution with $T(\alpha) \in a_m$. An analogous remark holds for the planes $b_m$, $c_m$ and $d_m$.

We conclude the paper by considering a superlinear asymmetric situation; more precisely, we assume that conditions

$$\lim_{x\to+\infty} \frac{f(x)}{x} = +\infty \tag{77}$$

and

$$\lim_{x\to-\infty} \frac{f(x)}{x} = \gamma \tag{78}$$

hold. As before, suppose that there exists an integer $l \in \mathbb{N}$ such that

$$l^2 < \gamma < (l+1)^2 . \tag{79}$$

Moreover, let

$$\lim_{x \to 0} \frac{f(x)}{x} = h \tag{80}$$

and let us assume that there exists $j \in \mathbb{N}$ such that

$$j^2 < h < (j+1)^2 . \tag{81}$$

Finally, we denote by $M$ the number of positive integers $m$ such that $2m \in \Big[ j + \frac{5}{2} + \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}} \cdot \cdot \frac{j+1}{j}\frac{1}{\pi}, 2l-2 \Big]$ or $2m \in \Big[ 2l+4, j - \frac{3}{2} - \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\frac{j+1}{j}\frac{1}{\pi} \Big]$.

We can prove the following:

THEOREM 4. *Assume that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function satisfying (16), (77), (78) and (80); moreover, suppose (79) and (81) are satisfied. Then, for every $B > A > 0$ there exist $s_k > 0$ ($k = 0, \dots, 2M-2$) such that for every $s \in (s_{2i-1}, s_{2i})$ ($i = 0, \dots, M-1$, $s_{-1} := 0$) problem (15) has at least $4(M-i)$ nontrivial solutions.*

Some comments analogous to the ones developed in Remark 1 are valid also in the present situation.

The proof of Theorem 4 is exactly the same of the proof of Theorem 3; we only have to give a Lemma which replaces Lemma 4. Indeed, with the same argument developed in the asymptotically linear case, we are able to prove the following:

LEMMA 5. *For every $s \in (0, s_0)$, problem (15) has at least $4M$ solutions.*

*Proof.* The proof follows the same lines of the one of Lemma 4. Indeed, according to the discussion contained in Section 2, problem (15) has a solution of energy $F(\alpha)$ with $T(\alpha) \in a_m$ if the points $P_{0,s}$ and $P_\infty$ belong to the opposite half-spaces generated by the straight line $a_m$. Let $a_m^\infty = 2mx_\infty + (2m-1)y_\infty + 2mz_\infty$ and let $a_m^0 = 2mT^0_{1,s} + (2m-1)T^0_{2,s} + 2mT^0_{3,s}$. Again, as in the asymptotically linear case, in order to obtain a solution with the prescribed property, it is sufficient that

$$a_m^\infty < \pi < a_m^0 \tag{82}$$

or

$$a_m^0 < \pi < a_m^\infty . \tag{83}$$

Now, according to Proposition 1 and to conditions (77), (78) and (79), we obtain that

$$\frac{2m}{2(l+1)}\pi < a_m^\infty < \frac{2m}{2l}\pi . \tag{84}$$

Moreover, as in the proof of Lemma 4, we have

$$\frac{4m-1}{2(j+1)}\pi + \frac{A}{B}\frac{1}{j+1} < a_m^0 < \frac{4m-1}{2j}\pi + \frac{\sqrt{2}}{2}\sqrt{\frac{A}{B-A}}\frac{1}{j} . \tag{85}$$

Therefore, (82) is fulfilled if

$$\begin{cases} \dfrac{4m-1}{2(j+1)}\pi + \dfrac{A}{B}\dfrac{1}{j+1} \geq \pi \\ \\ \dfrac{2m}{2l}\pi \leq \pi, \end{cases} \tag{86}$$

while (83) is satisfied if

$$\begin{cases} \dfrac{4m-1}{2j}\pi + \dfrac{\sqrt{2}}{2}\sqrt{\dfrac{A}{B-A}}\dfrac{1}{j} \leq \pi \\ \\ \dfrac{2m}{2(l+1)}\pi \geq \pi. \end{cases} \tag{87}$$

Now, by some easy computations, it is easy to deduce that (87) and (86) are valid for $m = m_1, \ldots, m_M$. □

## References


[1] BERGER M.S., PODOLAK E., *On the solutions of a nonlinear Dirichlet problem*, Indiana Univ. Math. J. **24** (1975), 837–846.

[2] CAPIETTO A., DAMBROSIO W., *Multiplicity results for some two-point superlinear asymmetric boundary value problem*, Nonlinear Analysis, TMA **38** (1999), 869–896.

[3] DAMBROSIO W., *Time-map techniques for some boundary value problem*, Rocky Mountain J. Math. **28** (1998), 885–926.

[4] DANCER E.N., DU Y., *A note on multiple solutions of some semilinear elliptic problems*, J. Math. Anal. Appl. **211** (1997), 626–640.

[5] DINCA G., SANCHEZ L., *Multiple solutions of boundary value problems: an elementary approach via the shooting method*, NoDEA **1** (1994), 163–178.

[6] FUČIK S., *Solvability of Nonlinear Equations and Boundary Value Problems*, Reidel - Boston 1980.

[7] HARRIS G.A., *The influence of boundary data on the number of solutions of boundary value problems with jumping nonlinearities*, Trans. Amer. Math. Soc. **321** (1990), 417–464.

[8] HARRIS G.A., *On multiple solutions of a nonlinear Neumann problem*, J. Differential Eq. **95** (1992), 75–104.

[9] LAZER A.C., MCKENNA P.J., *On the number of solutions of a nonlinear Dirichlet problem*, J. Math. Anal. Appl. **87** (1981), 282–294.

[10] LAZER A.C., MCKENNA P.J., *On a conjecture related to the number of solutions of a nonlinear Dirichlet problem*, Proc. Roy. Soc. Edinburgh Sect. A **95** (1983), 275–283.

[11] LAZER A.C., MCKENNA P.J., *New multiplicity results for a class of semilinear elliptic and parabolic boundary value problems*, J. Math. Anal. Appl. **107** (1985), 371–395.

[12] OPIAL Z., *Sur les périodes des solutions de l'équation différentielle $x'' + g(x) = 0$*, Ann. Polon. Math. **10** (1961), 49–72.

[13] RUF B., SOLIMINI S., *On a class of superlinear Sturm-Liouville problems with arbitrarily many solutions*, SIAM J. Math. Anal. **17** (1986), 761–771.

[14] ZANOLIN F., *Time-maps and boundary value problems for ordinary differential equations*, Tricomi's ideas and contemporary applied mathematics, 281–302, Atti Convegni Lincei **147** (1998).

[15] ZINNER B., *Multiplicity of solutions for a class of superlinear Sturm-Liouville problems*, J. Math. Anal. Analysis **176** (1993), 282–291.

**AMS Subject Classification: 34B15.**

Walter DAMBROSIO
Dipartimento di Matematica
Università di Torino
Via Carlo Alberto 10
10123 TORINO
e-mail: `dambrosio@dm.unito.it`

*Lavoro pervenuto in redazione il 28.6.1998 e, in forma definitiva, il 12.12.1998.*