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SINGULARITIES OF STABILIZING FEEDBACKS

1. Introduction

This paper is concerned with the stabilization problem for a control system of the form

$$\dot{x} = f(x, u), \qquad u \in K\,, \tag{1}$$

assuming that the set of control values $K \subset \mathbb{R}^m$ is compact and that the map $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ is smooth. It is well known [6] that, even if every initial state $\bar{x} \in \mathbb{R}^n$ can be steered to the origin by an open-loop control $u = u^{\bar{x}}(t)$, there may not exist a continuous feedback control $u = U(x)$ which locally stabilizes the system (1). One is thus forced to look for a stabilizing feedback within a class of discontinuous functions. However, this leads to a theoretical difficulty, because, when the function U is discontinuous, the differential equation

$$\dot{x} = f\,(x, U(x)) \tag{2}$$

may not have any Carathéodory solution. To cope with this problem, two approaches are possible.

I) On one hand, one may choose to work with completely arbitrary feedback controls U. In this case, to make sense of the evolution equation (2), one must introduce a suitable definition of "generalized solution" for discontinuous O.D.E. For such solutions, a general existence theorem should be available.

II) On the other hand, one may try to solve the stabilization problem within a particular class of feedback controls U whose discontinuities are sufficiently tame. In this case, it will suffice to consider solutions of (2) in the usual Carathéodory sense.

The first approach is more in the spirit of [7], while the second was taken in [1]. In the present note we will briefly survey various definitions of generalized solutions found in the literature [2, 11, 12, 13, 14], discussing their possible application to problems of feedback stabilization. In the last sections, we will consider particular classes of discontinuous vector fields which always admit Carathéodory solutions [3, 5, 16], and outline some research directions related to the second approach.

In the following, $\overline{\Omega}$ and $\partial\Omega$ denote the closure and the boundary of a set Ω, while B_ε is the open ball centered at the origin with radius ε. To fix the ideas, two model problems will be considered.

Asymptotic Stabilization (AS). Construct a feedback $u = U(x)$, defined on $\mathbb{R}^n \setminus \{0\}$, such that every trajectory of (2) either tends to the origin as $t \to \infty$ or else reaches the origin in finite time.

Suboptimal Controllability (SOC). Consider the minimum time function

$$T(\bar{x}) \doteq \min\{t : \text{there exists a trajectory of (1) with } x(0) = \bar{x},\ x(t) = 0\}\ . \tag{3}$$

Call $R(\tau) \doteq \{x : T(x) \le \tau\}$ the set of points that can be steered to the origin within time τ. For a given $\varepsilon > 0$, we want to construct a feedback $u = U(x)$, defined on a neighborhood V of $R(\tau)$, with the following property. For every $\bar{x} \in V$, every trajectory of (2) starting at $\bar{x}$ reaches a point inside B_ε within time $T(\bar{x}) + \varepsilon$.

Notice that we are not concerned here with time optimal feedbacks, but only with suboptimal ones. Indeed, already for systems on $\mathbb{R}^2$, an accurate description of all generic singularities of a time optimal feedback involves the classification of a large number of singular points [4, 15]. In higher dimensions, an ever growing number of different singularities can arise, and time optimal feedbacks may exhibit pathological behaviors. A complete classification thus appears to be an enormous task, if at all possible. By working with suboptimal feedbacks, we expect that such bad behaviors can be avoided. One can thus hope to construct suboptimal feedback controls having a much smaller set of singularities.

2. Nonexistence of continous stabilizing feedbacks

The papers [6, 19, 20] provided the first examples of control systems which can be asymptotically stabilized at the origin, but where no continuous feedback control $u = U(x)$ has the property that all trajectories of (2) asymptotically tend to the origin as $t \to \infty$. One such case is the following.

EXAMPLE 1. Consider the control system on $\mathbb{R}^3$

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3) = (u_1, u_2, x_1 u_2 - x_2 u_1) . \tag{4}$$

As control set K one can take here the closed unit ball in $\mathbb{R}^2$. Using Lie-algebraic techniques, it is easy to show that this system is globally controllable to the origin. However, no smooth feedback $u = U(x)$ can achieve this stabilization.

Indeed, the existence of such a feedback would imply the existence of a compact neighborhood V of the origin which is positively invariant for the flow of the smooth vector field $g(x) \doteq f(x, U(x))$. Calling $T_V(x)$ the contingent cone [2, 8] to the set V at the point x, we thus have $g(x) \in T_V(x)$ at each boundary point $x \in \partial V$. Since g cannot vanish outside the origin, by a topological degree argument, there must be a point x^* where the field g is parallel to the x_3-axis: $g(x^*) = (0, 0, y)$ for some $y > 0$. But this is clearly impossible by the definition (4) of the vector field.
Using a mollification procedure, from a continuous stabilizing feedback one could easily construct a smooth one. Therefore, the above argument also rules out the existence of continuous stabilizing feedbacks.

We describe below a simple case where the problem of suboptimal controllability to zero cannot be solved by any continuous feedback.

EXAMPLE 2. Consider the system

$$(\dot{x}_1, \dot{x}_2) = (u, -x_1^2), \qquad u \in [-1, 1] . \tag{5}$$

The set of points that can be steered to the origin within time $\tau = 1$ is found to be

$$R(1) = \left\{ (x_1, x_2) : x_1 \in [-1, 1], \quad \frac{1}{3}|x_1^3| \le x_2 \le \frac{1}{4}\left(\frac{1}{3} + |x_1| + x_1^2 - |x_1^3|\right) \right\} . \tag{6}$$

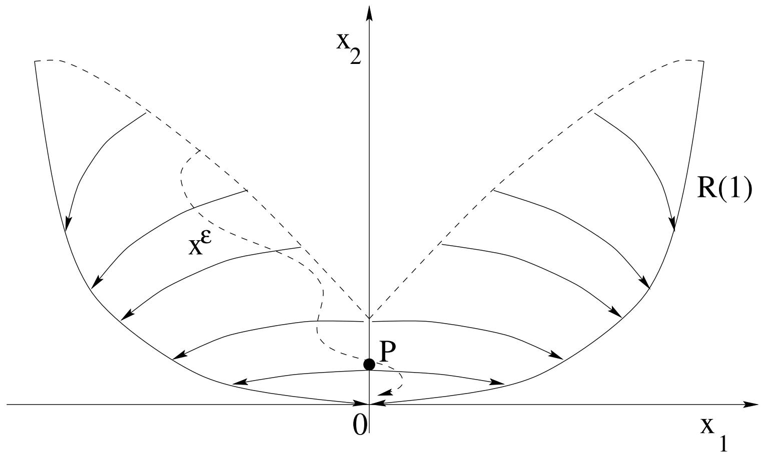

figure 1

Moreover, all time-optimal controls are bang-bang with at most one switching, as shown in fig. 1.

Assume that for every $\varepsilon > 0$ there exists a continuous feedback U_ε such that all trajectories of

$$\dot{x} = (\dot{x}_1, \dot{x}_2) = \left(U_\varepsilon(x), -x_1^2\right)$$

starting at some point $\bar{x} \in R(1)$ reach the ball B_ε within time $T(\bar{x})+\varepsilon$. To derive a contradiction, fix the point $P = (0, 1/24)$. By continuity, for each ε sufficiently small, there will be at least one trajectory $x^\varepsilon(\cdot)$ starting from a point on the upper boundary

$$\partial^+ R(1) \doteq \left\{(x_1, x_2) : x_1 \in [-1, 1], \quad x_2 = \frac{1}{4}\left(\frac{1}{3} + |x_1| + x_1^2 - |x_1^3|\right)\right\} \tag{7}$$

and passing through P before reaching a point in B_ε. By compactness, as $\varepsilon \to 0$ we can take a subsequence of trajectories $x^\varepsilon(\cdot)$ converging to function $x^*(\cdot)$ on [0, 1]. By construction, $x^*(\cdot)$ is then a time optimal trajectory starting from a point on the upper boundary $\partial^+ R(1)$ and reaching the origin in minimum time, passing through the point P at some intermediate time $s \in]0, 1[$. But this is a contradiction because no such trajectory exists.

3. Generalized solutions of a discontinuous O.D.E.

Let g be a bounded, possibly discontinuous vector field on $\mathbb{R}^n$. In connection with the O.D.E.

$$\dot{x} = g(x)\,, \tag{8}$$

various concepts of "generalized" solutions can be found in the literature. We discuss here the two main approaches.

(A) Starting from g, by some regularization procedure, one constructs an upper semicontinuous multifunction G with compact convex values. Every absolutely continuous function which satisfies a.e. the differential inclusion

$$\dot{x} \in G(x) \tag{9}$$

can then be regarded as generalized solutions of (8).

In the case of *Krasovskii solutions*, one takes the multifunction

(10) $$G(x) \doteq \bigcap_{\varepsilon>0} \overline{co}\,\{g(y) : |y-x| < \varepsilon\} \,.$$

Here $\overline{co}\,A$ denotes the closed convex hull of the set A. The *Filippov solutions* are defined similarly, except that one now excludes sets of measure zero from the domain of g. More precisely, calling $\mathcal{N}$ the family of sets $A \subset \mathbb{R}^n$ of measure zero, one defines

(11) $$G(x) \doteq \bigcap_{\varepsilon>0} \bigcap_{A\in\mathcal{N}} \overline{co}\,\{g(y) : |y-x| < \varepsilon,\ y \notin A\} \,.$$

Concerning solutions of the multivalued Cauchy problem

(12) $$x(0) = \bar{x}, \qquad \dot{x}(t) \in G\,(x(t)) \qquad t \in [0, T]\,,$$

one has the following existence result [2].

THEOREM 1. *Let g be a bounded vector field on $\mathbb{R}^n$. Then the multifunction G defined by either (10) or (11) is upper semicontinuous with compact convex values. For every initial data $\bar{x}$, the family $\mathcal{F}^{\bar{x}}$ of Carathéodory solutions of (12) is a nonempty, compact, connected, acyclic subset of $\mathcal{C}\left([0, T]\,;\,\mathbb{R}^n\right)$. The map $\bar{x} \mapsto \mathcal{F}^{\bar{x}}$ is upper semicontinuous. If g is continuous, then $G(x) = \{g(x)\}$ for all x, hence the solutions of (8) and (9) coincide.*

It may appear that the nice properties of Krasovskii or Filippov solutions stated in Theorem 1 make them a very attractive candidate toward a theory of discontinuous feedback control. However, quite the contrary is true. Indeed, by Theorem 1 the solution sets for the multivalued Cauchy problem (12) have the same topological properties as the solution sets for the standard Cauchy problem

(13) $$x(0) = \bar{x}, \qquad \dot{x}(t) = g\,(x(t)) \qquad t \in [0, T]$$

with continuous right hand side. As a result, the same topological obstructions found in Examples 1 and 2 will again be encountered in connection with Krasovskii or Filippov solutions. Namely [10, 17], for the system (4) one can show that for every discontinuous feedback $u = U(x)$ there will be some Filippov solution of the corresponding discontinuous O.D.E. (2) which does not approach the origin as $t \to \infty$. Similarly, for the system (5), when $\varepsilon > 0$ is small enough there exists no feedback $u = U(x)$ such that every Filippov solution of (2) starting from some point $\bar{x} \in R(1)$ reaches the ball B_ε within time $T(\bar{x}) + \varepsilon$.

The above considerations show the necessity of a new definition of "generalized solution" for a discontinuous O.D.E. which will allow the solution set to be possibly disconnected. The next paragraph describes a step in this direction.

(B) Following a second approach, one defines an algorithm which constructs a family of ε-approximate solutions x_ε. Letting the approximation parameter $\varepsilon \to 0$, every uniform limit $x(\cdot) = \lim_{\varepsilon\to 0} x_\varepsilon(\cdot)$ is defined to be a generalized solution of (8).

Of course, there is a wide variety of techniques [8, 13, 14] for constructing approximate solutions to the Cauchy problem (13). We describe here two particularly significant procedures.

Polygonal Approximations. By a general *polygonal ε-approximate* solution of (13) we mean any function $x : [0, T] \mapsto \mathbb{R}^n$ constructed by the following procedure. Consider a partition of the interval $[0, T]$, say $0 = t_0 < t_1 < \cdots < t_m = T$, whose mesh size satisfies

$$\max_i (t_i - t_{i-1}) < \varepsilon\,.$$

For $i = 0, \dots, m-1$, choose arbitrary outer and inner perturbations $e_i, e'_i \in \mathbb{R}^n$, with the only requirement that $|e_i| < \varepsilon$, $|e'_i| < \varepsilon$. By induction on i, determine the values x_i such that

$$|x_0 - \bar{x}| < \varepsilon, \qquad x_{i+1} = x_i + (t_{i+1} - t_i)\big(e_i + g(x_i + e'_i)\big) \tag{14}$$

Finally, define $x(\cdot)$ as the continuous, piecewise affine function such that $x(t_i) = x_i$ for all $i = 0, \dots, m$.

Forward Euler Approximations. By a *forward Euler ε-approximate* solution of (13) we mean any polygonal approximation constructed without taking any inner perturbation, i.e. with $e'_i \equiv 0$ for all i.

In the following, the trajectories of the differential inclusion (12), with G given by (10) or (11) will be called respectively *Krasovskii* or *Filippov solutions* of (13). By a *forward Euler solution* we mean a limit of forward Euler ε-approximate solutions, as $\varepsilon \to 0$. Some relations between these different concepts of solutions are illustrated below.

THEOREM 2. *The set of Krasovskii solutions of (13) coincides with the set of all limits of polygonal ε-approximate solutions, as $\varepsilon \to 0$.*

For a proof, see [2, 9].

EXAMPLE 3. On the real line, consider the vector field (fig. 2)

$$g(x) = \begin{cases} 1 & \text{if } x \geq 0\,, \\ -1 & \text{if } x < 0\,. \end{cases}$$

The corresponding multifunction G, according to both (10) and (11) is

$$G(x) = \begin{cases} \{1\} & \text{if } x > 0\,, \\ [-1, 1] & \text{if } x = 0\,, \\ \{-1\} & \text{if } x < 0\,. \end{cases}$$

The set of Krasovskii (or Filippov) solutions to (13) with initial data $\bar{x} = 0$ thus consists of all functions of the form

$$x(t) = \begin{cases} 0 & \text{if } t \leq \tau\,, \\ t - \tau & \text{if } t > \tau\,, \end{cases}$$

together with all functions of the form

$$x(t) = \begin{cases} 0 & \text{if } t \leq \tau\,, \\ \tau - t & \text{if } t > \tau\,, \end{cases}$$

for any $\tau \geq 0$. On the other hand, there are only two forward Euler solutions:

$$x_1(t) = t, \qquad x_2(t) = -t\,.$$

In particular, this set of limit solutions is not connected.

EXAMPLE 4. On $\mathbb{R}^2$ consider the vector field (fig. 3)

$$g(x_1, x_2) \doteq \begin{cases} (0, -1) & \text{if } x_2 > 0\,, \\ (0, 1) & \text{if } x_2 < 0\,, \\ (1, 0) & \text{if } x_2 = 0\,. \end{cases}$$

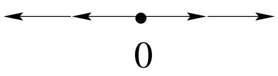

figure 2

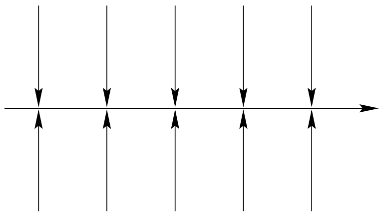

figure 3

The corresponding Krasovskii multivalued regularization (10) is

$$G_K(x_1, x_2) = \begin{cases} \{(0, -1)\} & \text{if } x_2 > 0\,, \\ \{(0, 1)\} & \text{if } x_2 < 0\,, \\ \overline{co}\,\{(0, -1), (0, 1), (1, 0)\} & \text{if } x_2 = 0\,. \end{cases}$$

Given the initial condition $\bar{x} = (0, 0)$, the corresponding Krasovskii solutions are all the functions of the form $t \mapsto (x_1(t), 0)$, with $\dot{x}_1(t) \in [0, 1]$ almost everywhere. These coincide with the limits of forward Euler approximations. On the other hand, since the line $\{x_2 = 0\}$ is a null set, the Filippov multivalued regularization (11) is

$$G_F(x_1, x_2) = \begin{cases} \{(0, -1)\} & \text{if } x_2 > 0\,, \\ \{(0, 1)\} & \text{if } x_2 < 0\,, \\ \overline{co}\,\{(0, -1), (0, 1)\} & \text{if } x_2 = 0\,. \end{cases}$$

Therefore, the only Filippov solution starting from the origin is the function $x(t) \equiv (0, 0)$ for all $t \geq 0$.

4. Patchy vector fields

For a general discontinuous vector field g, the Cauchy problem for the O.D.E.

$$\dot{x} = g(x) \tag{15}$$

may not have any Carathéodory solution. Or else, the solution set may exhibit very wild behavior. It is our purpose to introduce a particular class of discontinuous maps g whose corresponding trajectories are quite well behaved. This is particularly interesting, because it appears that various stabilization problems can be solved by discontinuous feedback controls within this class.

DEFINITION 1. *By a* patch *we mean a pair* (Ω, g) *where* $\Omega \subset \mathbb{R}^n$ *is an open domain with smooth boundary and* g *is a smooth vector field defined on a neighborhood of* $\overline{\Omega}$ *which points strictly inward at each boundary point* $x \in \partial\Omega$.

Calling $\mathbf{n}(x)$ the outer normal at the boundary point x, we thus require

$$\langle g(x), \mathbf{n}(x) \rangle < 0 \qquad \text{for all } x \in \partial\Omega\,. \tag{16}$$

DEFINITION 2. *We say that $g : \Omega \mapsto \mathbb{R}^n$ is a* patchy vector field *on the open domain Ω if there exists a family of patches $\{(\Omega_\alpha, g_\alpha) : \alpha \in \mathcal{A}\}$ such that*
- $\mathcal{A}$ is a totally ordered set of indices,
- the open sets Ω_α form a locally finite covering of Ω,
- the vector field g can be written in the form

$$g(x) = g_\alpha(x) \text{ if } x \in \Omega_\alpha \setminus \bigcup_{\beta>\alpha} \Omega_\beta . \tag{17}$$

By defining

$$\alpha^*(x) \doteq \max \{\alpha \in \mathcal{A} : x \in \Omega_\alpha\} , \tag{18}$$

we can write (17) in the equivalent form

$$g(x) = g_{\alpha^*(x)}(x) \text{ for all } x \in \Omega . \tag{19}$$

We shall occasionally adopt the longer notation $\big(\Omega, g, (\Omega_\alpha, g_\alpha)_{\alpha\in\mathcal{A}}\big)$ to indicate a patchy vector field, specifying both the domain and the single patches. Of course, the patches $(\Omega_\alpha, g_\alpha)$ are not uniquely determined by the vector field g. Indeed, whenever $\alpha < \beta$, by (17) the values of g_α on the set $\Omega_\beta \setminus \Omega_\alpha$ are irrelevant. This is further illustrated by the following lemma.

LEMMA 1. *Assume that the open sets Ω_α form a locally finite covering of Ω and that, for each $\alpha \in \mathcal{A}$, the vector field g_α satisfies the condition (16) at every point $x \in \partial\Omega_\alpha \setminus \cup_{\beta>\alpha}\Omega_\beta$. Then g is again a patchy vector field.*

Proof. To prove the lemma, it suffices to construct vector fields $\tilde{g}_\alpha$ which satisfy the inward pointing property (16) at every point $x \in \partial\Omega_\alpha$ and such that $\tilde{g}_\alpha = g_\alpha$ on $\Omega_\alpha \setminus \cup_{\beta>\alpha}\Omega_\beta$. To accomplish this, for each α we first consider a smooth vector field v_α such that $v_\alpha(x) = -\mathbf{n}(x)$ on $\partial\Omega_\alpha$. The map $\tilde{g}_\alpha$ is then defined as the interpolation

$$\tilde{g}_\alpha(x) \doteq \varphi(x) g_\alpha(x) + (1 - \varphi(x))\, v_\alpha(x) ,$$

where φ is a smooth scalar function such that

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in \Omega_\alpha \setminus \cup_{\beta>\alpha}\Omega_\beta , \\ 0 & \text{if } x \in \partial\Omega_\alpha \text{ and } \langle g(x), \mathbf{n}(x)\rangle \geq 0 . \end{cases}$$

□

The main properties of trajectories of a patchy vector field (fig. 4) are collected below.

THEOREM 3. *Let $\big(\Omega, g, (\Omega_\alpha, g_\alpha)_{\alpha\in\mathcal{A}}\big)$ be a patchy vector field.*

(i) If $t \to x(t)$ is a Carathéodory solution of (15) on an open interval J, then $t \to \dot{x}(t)$ is piecewise smooth and has a finite set of jumps on any compact subinterval $J' \subset J$. The function $t \mapsto \alpha^*(x(t))$ defined by (18) is piecewise constant, left continuous and non-decreasing. Moreover there holds

$$\dot{x}((t-)) = g\,(x(t)) \text{ for all } t \in J . \tag{20}$$

(ii) For each $\bar{x} \in \Omega$, the Cauchy problem for (15) with initial condition $x(0) = \bar{x}$ has at least one local forward Carathéodory solution and at most one backward Carathéodory solution.

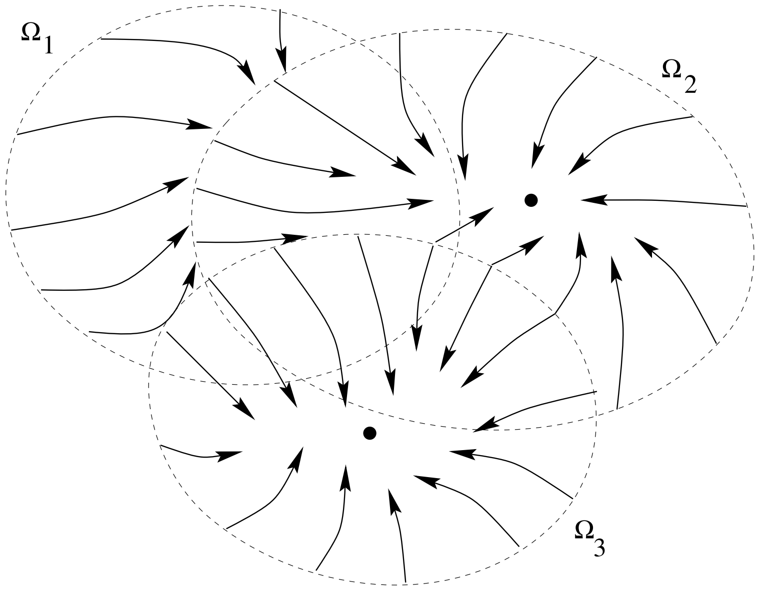

figure 4

(iii) *The set of Carathéodory solutions of (15) is closed. More precisely, assume that* $x_\nu : [a_\nu, b_\nu] \mapsto \Omega$ *is a sequence of solutions and, as* $\nu \to \infty$, *there holds*

$$a_\nu \to a, \qquad b_\nu \to b, \qquad x_\nu(t) \to \hat{x}(t) \text{ for all } t \in]a, b[\,.$$

Then $\hat{x}(\cdot)$ *is itself a Carathéodory solution of (15).*

(iv) *The set of a Carathéodory solutions of the Cauchy problem (13) coincides with the set of forward Euler solutions.*

Proof. We sketch the main arguments in the proof. For details see [1].

To prove (i), observe that on any compact interval $[a, b]$ a solution $x(\cdot)$ can intersect only finitely many domains Ω_α, say those with indices $\alpha_1 < \alpha_2 < \cdots < \alpha_m$. It is now convenient to argue by backward induction. Since Ω_{α_m} is positively invariant for the flow of g_{α_m}, the set of times $\{t \in [a, b] : x(t) \in \Omega_{\alpha_m}\}$ must be a (possibly empty) interval of the form $]t_m, b]$. Similarly, the set $\{t \in [a, b] : x(t) \in \Omega_{\alpha_{m-1}}\}$ is an interval of the form $]t_{m-1}, t_m]$. After m inductive steps we conclude that

$$\dot{x}(t) = g_{\alpha_j}(x(t)) \qquad t \in]t_j, t_{j+1}[$$

for some times t_j with $a = t_1 \leq t_2 \leq \cdots \leq t_{m+1} = b$. All statements in (i) now follow from this fact. In particular, (20) holds because each set Ω_α is open and positively invariant for the flow of the corresponding vector field g_α.

Concerning (ii), to prove the local existence of a forward Carathéodory solution, consider the index

$$\bar{\alpha} \doteq \max\left\{\alpha \in \mathcal{A} : \bar{x} \in \overline{\Omega_\alpha}\right\}.$$

Because of the transversality condition (16), the solution of the Cauchy problem

$$\dot{x} = g_{\bar{\alpha}}(x), \qquad x(0) = \bar{x}$$

remains inside $\Omega_{\bar{\alpha}}$ for all $t \geq 0$. Hence it provides also a solution of (15) on some positive interval $[0, \delta]$.

To show the backward uniqueness property, let $x_1(\cdot)$, $x_2(\cdot)$ be any two Carathéodory solutions to (15) with $x_1(0) = x_2(0) = \bar{x}$. For $i = 1, 2$, call

$$\alpha_i^*(t) \doteq \max\{\alpha \in \mathcal{A} : x_i(t) \in \Omega_\alpha\} .$$

By (i), the maps $t \mapsto \alpha_i^*(t)$ are piecewise constant and left continuous. Hence there exists $\delta > 0$ such that

$$\alpha_1^*(t) = \alpha_2^*(t) = \bar{\alpha} \doteq \max\{\alpha \in \mathcal{A} : \bar{x} \in \Omega_\alpha\} \text{ for all } t \in]-\delta, 0] .$$

The uniqueness of backward solutions is now clear, because on $]-\delta, 0]$ both x_1 and x_2 are solutions of the same Cauchy problem with smooth coefficients

$$\dot{x} = g_{\bar{\alpha}}(x), \qquad x(0) = \bar{x} .$$

Concerning (iii), to prove that $\hat{x}(\cdot)$ is itself a Carathéodory solution, we observe that on any compact subinterval $J \subset]a, b[$ the functions u_ν are uniformly continuous and intersect a finite number of domains Ω_α, say with indices $\alpha_1 < \alpha_2 < \cdots < \alpha_m$. For each ν, the function

$$\alpha_\nu^*(t) \doteq \max\{\alpha \in \mathcal{A} : x_\nu(t) \in \Omega_\alpha\}$$

is non-decreasing and left continuous, hence it can be written in the form

$$\alpha_\nu^*(t) = \alpha_j \text{ if } t \in]t_j^\nu, t_{j+1}^\nu] .$$

By taking a subsequence we can assume that, as $\nu \to \infty$, $t_j^\nu \to \hat{t}_j$ for all j. By a standard convergence result for smooth O.D.E's, the function $\hat{x}$ provides a solution to $\dot{x} = g_{\alpha_j}(x)$ on each open subinterval $I_j \doteq]\hat{t}_j, \hat{t}_{j+1}[$. Since the domains Ω_β are open, there holds

$$\hat{x}(t) \notin \Omega_\beta \text{ for all } \beta > \alpha_j, \quad t \in I_j .$$

On the other hand, since g_{α_j} is inward pointing, a limit of trajectories $\dot{x}_\nu = g_{\alpha_j}(x_\nu)$ taking values within Ω_{α_j} must remain in the interior of Ω_{α_j}. Hence $\alpha^*\big(\hat{x}(t)\big) = \alpha_j$ for all $t \in I_j$, achieving the proof of (iii).

Regarding (iv), let $x_\varepsilon : [0, T] \mapsto \Omega$ be a sequence of forward Euler ε-approximate solutions of (13), converging to $\hat{x}(\cdot)$ as $\varepsilon \to 0$. To show that $\hat{x}$ is a Carathéodory solution, we first observe that, for $\varepsilon > 0$ sufficiently small, the maps $t \mapsto \alpha^*(x_\varepsilon(t))$ are non-decreasing. More precisely, there exist finitely many indices $\alpha_1 < \cdots < \alpha_m$ and times $0 = t_0^\varepsilon \leq t_1^\varepsilon \leq \cdots \leq t_m^\varepsilon = T$ such that

$$\alpha^*(x_\varepsilon(t)) = \alpha_j \qquad t \in]t_{j-1}^\varepsilon, t_j^\varepsilon] .$$

By taking a subsequence, we can assume $t_j^\varepsilon \to \hat{t}_j$ for all j, as $\varepsilon \to 0$. On each open interval $]\hat{t}_{j-1}, \hat{t}_j[$ the trajectory $\hat{x}$ is thus a uniform limit of polygonal approximate solutions of the smooth O.D.E.

$$\dot{x} = g_{\alpha_j}(x) . \tag{21}$$

By standard O.D.E. theory, $\hat{x}$ is itself a solution of (21). As in the proof of part (iii), we conclude observing that $\alpha^*\big(\hat{x}(t)\big) = \alpha_j$ for all $t \in]\hat{t}_{j-1}, \hat{t}_j]$.

To prove the converse, let $x : [0, T] \mapsto \Omega$ be a Carathéodory solution of (13). By (i), there exist indices $\alpha_1 < \cdots < \alpha_m$ and times $0 = t_0 < t_1 < \cdots < t_m = T$ such that $\dot{x}(t) = g_{\alpha_j}(x(t))$

for $t \in]t_{j-1}, t_j[$. For each $n \geq 1$, consider the polygonal map $x_n(\cdot)$ which is piecewise affine on the subintervals $[t_{j,k}, t_{j,k+1}]$, $j = 1, \ldots, m$, $k = 1, \ldots, n$ and takes values $x_n(t_{j,k}) = x_{j,k}$. The times $t_{j,k}$ and the values $x_{j,k}$ are here defined as

$$t_{j,k} \doteq t_{j-1} + \frac{k}{n}(t_j - t_{j-1}), \qquad x_{j,k} \doteq x(t_{j,k} + 2^{-n}).$$

As $n \to \infty$, it is now clear that $x_n \to x$ uniformly on $[0, T]$. On the other hand, for a fixed $\varepsilon > 0$ one can show that the polygonals $x_n(\cdot)$ are forward Euler ε-approximate solutions, for all $n \geq N_\varepsilon$ sufficiently large. This concludes the proof of part (iv).

□

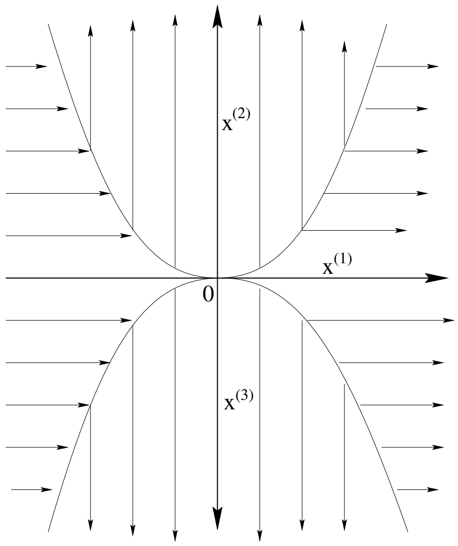

figure 5

EXAMPLE 5. Consider the patchy vector field on the plane (fig. 5) defined by (17), by taking

$$\Omega_1 \doteq \mathbb{R}^2, \qquad \Omega_2 \doteq \{x_2 > x_1^2\}, \qquad \Omega_3 \doteq \{x_2 < -x_1^2\},$$
$$g_1(x_1, x_2) \equiv (1, 0), \quad g_2(x_1, x_2) \equiv (0, 1), \quad g_3(x_1, x_2) \equiv (0, -1).$$

Then the Cauchy problem starting from the origin at time $t = 0$ has exactly three forward Carathéodory solutions, namely

$$x^{(1)}(t) = (t, 0), \qquad x^{(2)}(t) = (0, t), \qquad x^{(3)}(t) = (0, -t) \qquad t \geq 0.$$

The only backward Carathéodory solution is

$$x^{(1)}(t) = (t, 0) \qquad t \leq 0.$$

On the other hand there exist infinitely many Filippov solutions. In particular, for every $\tau < 0 < \tau'$, the function

$$x(t) = \begin{cases} (t - \tau, 0) & \text{if } t < \tau, \\ (0, 0) & \text{if } t \in [\tau, \tau'], \\ (t - \tau', 0) & \text{if } t > \tau' \end{cases}$$

provides a Filippov solution, and hence a Krasovskii solution as well.

5. Directionally continuous vector fields

Following [3], we say that a vector field g on $\mathbb{R}^n$ is *directionally continuous* if, at every point x where $g(x) \neq 0$ there holds

$$\lim_{n\to\infty} g(x_n) = g(x) \tag{22}$$

for every sequence $x_n \to x$ such that

$$\left| \frac{x_n - x}{|x_n - x|} - \frac{g(x)}{|g(x)|} \right| < \delta \text{ for all } n \geq 1\,. \tag{23}$$

Here $\delta = \delta(x) > 0$ is a function uniformly positive on compact sets. In other words (fig. 6), one requires $g(x_n) \to g(x)$ only for the sequences converging to x contained inside a cone with vertex at x and opening δ around an axis having the direction of $g(x)$.

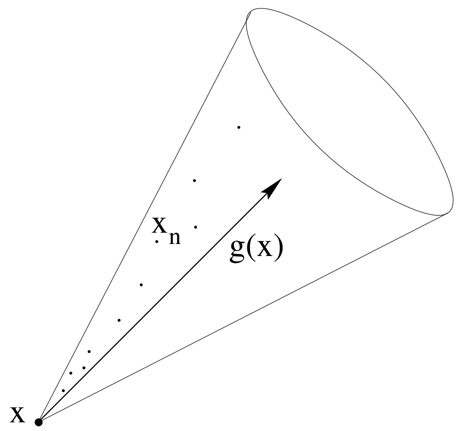

figure 6

For these vector fields, the local existence of Carathéodory trajectories is known [16]. It seems natural to ask whether the stabilization problems (AS) or (SOC) can be solved in terms of feedback controls generating a directionally continuous vector field. The following lemma reduces the problem to the construction of a patchy vector field.

LEMMA 2. *Let* $\big(\Omega, g, (\Omega_\alpha, g_\alpha)_{\alpha\in\mathcal{A}}\big)$ *be a patchy vector field. Then the map* $\tilde g$ *defined by*

$$\tilde g(x) = g_\alpha(x) \text{ if } x \in \overline{\Omega}_\alpha \setminus \bigcup_{\beta>\alpha} \overline{\Omega}_\beta \tag{24}$$

is directionally continuous. Every Carathéodory solution of

$$\dot x = \tilde g(x) \tag{25}$$

is also a solution of $\dot x = g(x)$. *The set of solutions of (25) may not be closed.*

Since directionally continuous vector fields form a much broader class of maps than patchy vector fields, solving a stabilization problem in terms of patchy fields thus provides a much better result. To see that the solution set of (25) may not be closed, consider

EXAMPLE 6. Consider the patchy vector field on $\mathbb{R}^2$ defined as follows.

$$\Omega_1 \doteq \mathbb{R}^2, \quad \Omega_2 \doteq \{x_2 < 0\}, \qquad g_1(x_1, x_2) = (1, 0), \quad g_2(x_1, x_2) = (0, -1) .$$

$$g(x_1, x_2) = \begin{cases} (1, 0) & \text{if } x_2 \geq 0 , \\ (0, -1) & \text{if } x_2 < 0 . \end{cases} \tag{26}$$

The corresponding directionally continuous field is (fig. 7)

$$\tilde{g}(x_1, x_2) = \begin{cases} (1, 0) & \text{if } x_2 > 0 , \\ (0, -1) & \text{if } x_2 \leq 0 . \end{cases} \tag{27}$$

The functions $t \mapsto x_\varepsilon(t) = (t, \varepsilon)$ are trajectories of both (26) and (27). However, as $\varepsilon \to 0$, the limit function $t \mapsto x(t) = (t, 0)$ is a trajectory of (26) but not of (27).

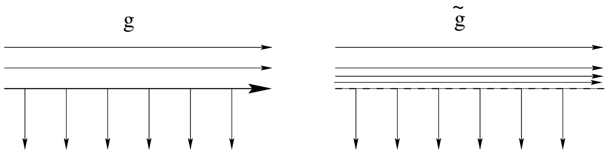

figure 7

6. Stabilizing feedback controls

In this section we discuss the applicability of the previous theory of discontinuous O.D.E's toward the construction of a stabilizing feedback. We first recall a basic definition [7, 18].

DEFINITION 3. *The system (1) is said to be globally* asymptotically controllable *to the origin if the following holds.*

1 - Attractivity. *For each $\bar{x} \in \mathbb{R}^n$ there exists some admissible control $u = u^{\bar{x}}(t)$ such that the corresponding solution of*

$$\dot{x}(t) = f\left(x(t), u^{\bar{x}}(t)\right), \qquad x(0) = \bar{x} \tag{28}$$

either tends to the origin as $t \to \infty$ or reaches the origin in finite time.

2 - Lyapunov stability. *For each $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds. For every $\bar{x} \in \mathbb{R}^n$ with $|\bar{x}| < \delta$ there is an admissible control $u^{\bar{x}}$ as in* **1.** *steering the system from $\bar{x}$ to the origin, such that the corresponding trajectory of (28) satisfies $|x(t)| < \varepsilon$ for all $t \geq 0$.*

The next definition singles out a particular class of piecewise constant feedback controls, generating a "patchy" dynamics.

DEFINITION 4. *Let $\left(\Omega, g, (\Omega_\alpha, g_\alpha)_{\alpha \in \mathcal{A}}\right)$ be a patchy vector field. Assume that there exist control values $k_\alpha \in K$ such that, for each $\alpha \in \mathcal{A}$*

$$g_\alpha(x) \doteq f(x, k_\alpha) \text{ for all } x \in \Omega_\alpha \setminus \bigcup_{\beta > \alpha} \Omega_\beta . \tag{29}$$

Then the piecewise constant map

$$U(x) \doteq k_\alpha \text{ if } x \in \Omega_\alpha \setminus \bigcup_{\beta>\alpha} \Omega_\beta \tag{30}$$

is called a patchy feedback *control on* Ω.

The main results concerning stabilization by discontinuous feedback controls can be stated as follows. For the proofs, see [7] and [1] respectively.

THEOREM 4. *If the system (1) is asymptotically controllable, then there exists a feedback control* $U : \mathbb{R}^n \setminus \{0\} \mapsto K$ *such that every uniform limit of sampling solutions either tends asymptotically to the origin, or reaches the origin in finite time.*

THEOREM 5. *If the system (1) is asymptotically controllable, then there exists a patchy feedback control U such that every Carathéodory solution of (2) either tends asymptotically to the origin, or reaches the origin in finite time.*

Proof. In view of part (iv) of Theorem 3, the result stated in Theorem 4 can be obtained as a consequence of Theorem 5. The main part of the proof of Theorem 5 consists in showing that, given two closed balls $B' \subset B$ centered at the origin, there exists a patchy feedback that steers every point $\bar{x} \in B$ inside B' within finite time. The basic steps of this construction are sketched below. Further details can be found in [1].

1. By assumption, for each point $\bar{x} \in B$, there exists an open loop control $t \mapsto u^{\bar{x}}(t)$ that steers the system from $\bar{x}$ into a point x' in the interior of B' at some time $\tau > 0$. By a density and continuity argument, we can replace $u^{\bar{x}}$ with a piecewise constant open loop control $\bar{u}$ (fig. 8), say

$$\bar{u}(t) = k_\alpha \in K \text{ if } t \in]t_\alpha, t_{\alpha+1}],$$

for some finite partition $0 = t_0 < t_1 < \cdots < t_m = \tau$. Moreover, it is not restrictive to assume that the corresponding trajectory $t \mapsto \gamma(t) \doteq x(t;\ \bar{x}, \bar{u})$ has no self-intersections.

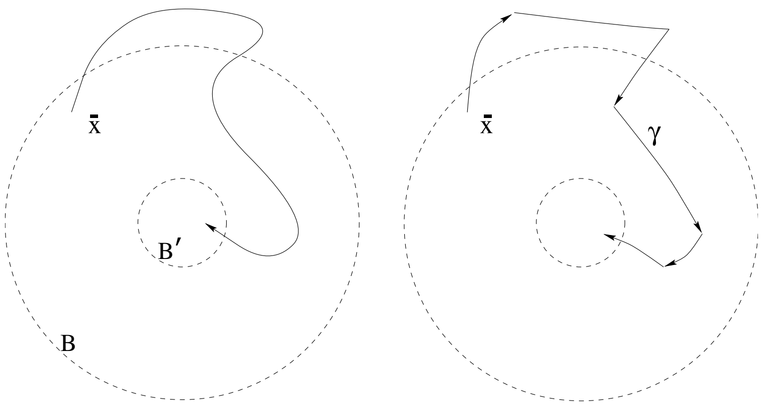

figure 8

2. We can now define a piecewise constant feedback control $u = U(x)$, taking the constant values $k_{\alpha_1}, \ldots, k_{\alpha_m}$ on a narrow tube Γ around γ, so that all trajectories starting inside Γ eventually reach the interior of B' (fig. 9).

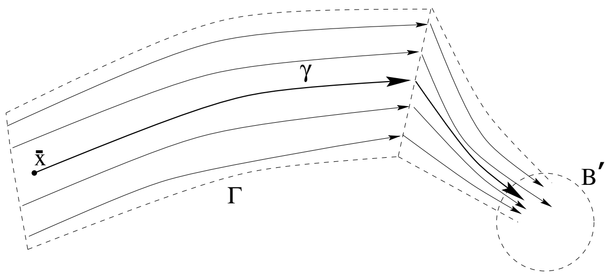

figure 9

3. By slightly bending the outer surface of each section of the tube Γ, we can arrange so that the vector fields $g_\alpha(x) \doteq f(x, k_\alpha)$ point strictly inward along the portion $\partial\Omega_\alpha \setminus \Omega_{\alpha+1}$. Recalling Lemma 1, we thus obtain a patchy vector field (fig. 10) defined on a small neighborhood of the tube Γ, which steers all points of a neighborhood of $\bar{x}$ into the interior of B'.

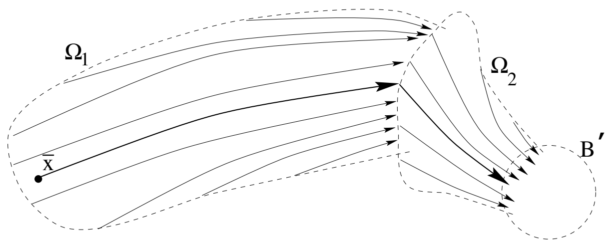

figure 10

4. The above construction can be repeated for every point $\bar{x}$ in the compact set B. We now select finitely many points $x_1, \ldots, x_N$ and patchy vector fields, $\Big(\Omega_i, g_i, (\Omega_{i,\alpha}, g_{i,\alpha})_{\alpha\in\mathcal{A}_i}\Big)$ with the properties that the domains Ω_i cover B, and that all trajectories of each field g_i eventually reach the interior of B'. We now define the patchy feedback obtained by the superposition of the g_i, in lexicographic order:

$$g(x) = g_{i,\alpha}(x) \text{ if } x \in \Omega_{i,\alpha} \setminus \bigcup_{(j,\beta)\succ(i,\alpha)} \Omega_{j,\beta}\,.$$

This achieves a patchy feedback control (fig. 11) defined on a neighborhood of $B \setminus B'$ which steers each point of B into the interior of B'.

5. For every integer ν, call B^ν be the closed ball centered at the origin with radius $2^{-\nu}$. By the previous steps, for every ν there exists a patchy feedback control U_ν steering each point in B_ν inside $B_{\nu+1}$, say

$$U_\nu(x) = k_{\nu,\alpha} \text{ if } x \in \Omega_{\nu,\alpha} \setminus \bigcup_{\beta>\alpha} \Omega_{\nu,\beta}\,. \tag{31}$$

The property of Lyapunov stability guarantees that the family of all open sets $\{\Omega_{\nu,\alpha} : \nu \in \mathbb{Z},\ \alpha = 1, \ldots, N_\nu\}$ forms a locally finite covering of $\mathbb{R}^n \setminus \{0\}$. We now define the patchy feedback control

$$U_\nu(x) = k_{\nu,\alpha} \text{ if } x \in \Omega_{\nu,\alpha} \setminus \bigcup_{(\mu,\beta)\succ(\nu,\alpha)} \Omega_{\mu,\beta}\,, \tag{32}$$

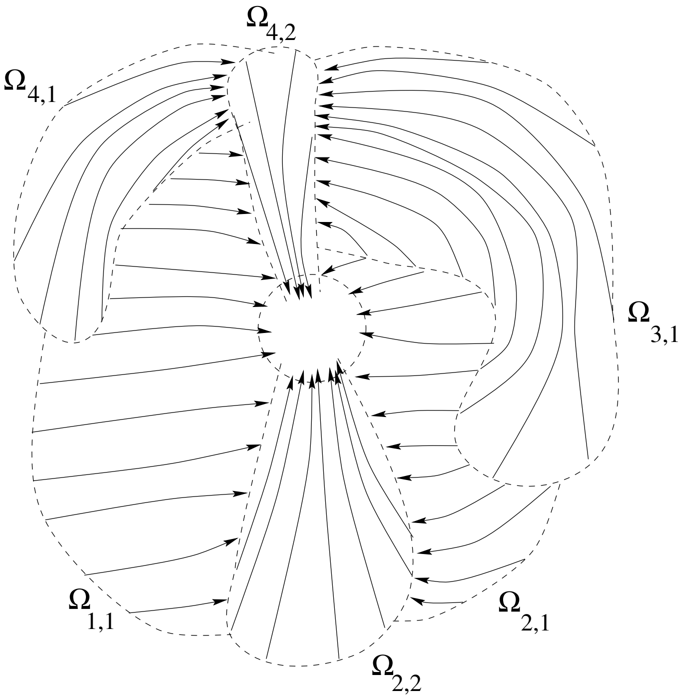

figure 11

where the set of indices (ν, α) is again ordered lexicographically. By construction, the patchy feedback (32) steers each point $x \in B^\nu$ into the interior of the smaller ball $B^{\nu+1}$ within finite time. Hence, every trajectory either tends to the origin as $t \to \infty$ or reaches the origin in finite time.

□

7. Some open problems

By Theorem 5, the asymptotic stabilization problem can be solved within the class of patchy feedback controls. We conjecture that the same is true for the problem of suboptimal controllability to zero.

Conjecture 1. Consider the smooth control system (1). For a fixed $\tau > 0$, call $R(\tau)$ the set of points that can be steered to the origin within time τ. Then, for every $\varepsilon > 0$, there exists a patchy feedback $u = U(x)$, defined on a neighborhood V of $R(\tau)$, with the following property. For every $\bar{x} \in V$, every trajectory of (2) starting at $\bar{x}$ reaches a point inside B_ε within time $T(\bar{x}) + \varepsilon$.

Although the family of patchy vector fields forms a very particular subclass of all discontinuous maps, the dynamics generated by such fields may still be very complicated and structurally unstable. In this connection, one should observe that the boundaries of the sets Ω_α may be taken in generic position. More precisely, one can slightly modify these boundaries so that the following property holds. If $x \in \partial\Omega_{\alpha_1} \cap \cdots \cap \partial\Omega_{\alpha_m}$, then the unit normals $\mathbf{n}_{\alpha_1}, \ldots, \mathbf{n}_{\alpha_m}$ are linearly independent. However, since no assumption is placed on the behavior of a vector field g_α at boundary points of a different domain Ω_β with $\beta \neq \alpha$, even the local behavior of the set of trajectories may be quite difficult to classify. More detailed results may be achieved for the special case of planar systems with control entering linearly:

$$\dot{x} = \sum_{i=1}^{m} f_i(x)\, u_i, \qquad u = (u_1, \ldots, u_m) \in K\,, \tag{33}$$

where $K \subset \mathbb{R}^m$ is a compact convex set. In this case, it is natural to conjecture the existence of stabilizing feedbacks whose dynamics has a very limited set of singular points. More precisely, consider the following four types of singularities illustrated in fig. 12. By a *cut* we mean a smooth curve γ along which the field g has a jump, pointing outward from both sides. At points at the of a cut, the field g is always tangent to γ. We call the endpoint an *incoming edge* or an *outgoing edge* depending on the orientation of g. A point where three distinct cuts join is called a *triple point*. Notice that the Cauchy problem with initial data along a cut, or an incoming edge of a cut, has two forward local solutions. Starting from a triple point there are three forward solutions.

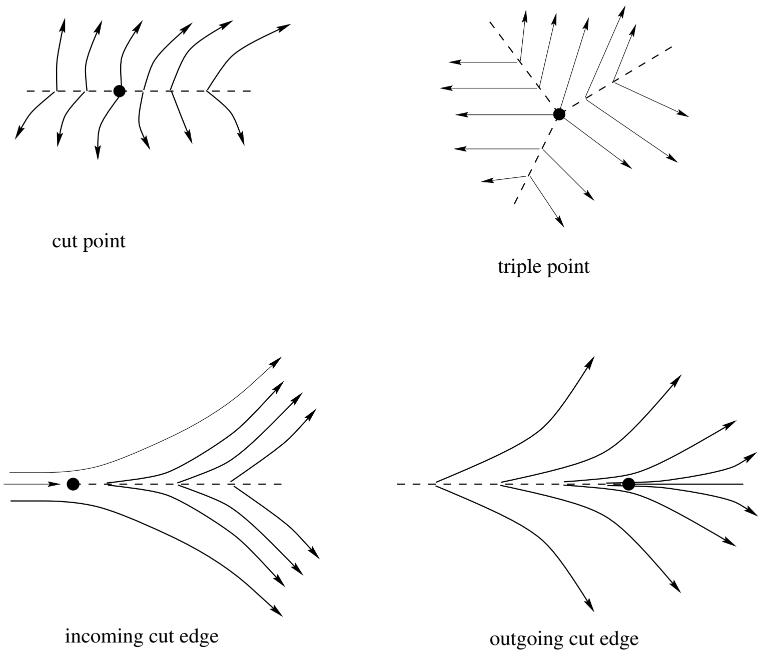

figure 12

Conjecture 2. Let the planar control system (33) be asymptotically controllable, with smooth coefficients. Then both the asymptotic stabilization problem (AS) and the suboptimal zero controllability problem (SOC) admit a solution in terms of a feedback $u = U(x) = (U_1(x), \ldots, U_n(x)) \in K$, such that the corresponding vector field

$$g(x) \doteq \sum_{i=1}^{m} f_i(x) U_i(x)$$

has singularities only of the four types described in fig. 12.

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