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ON THE DIRICHLET PROBLEM FOR NONLINEAR DEGENERATE ELLIPTIC EQUATIONS AND APPLICATIONS TO OPTIMAL CONTROL

Abstract.

We construct a generalized viscosity solution of the Dirichlet problem for fully nonlinear degenerate elliptic equations in general domains by the Perron-Wiener-Brelot method. The result is designed for the Hamilton-Jacobi-Bellman-Isaacs equations of time-optimal stochastic control and differential games with discontinuous value function. We study several properties of the generalized solution, in particular its approximation via vanishing viscosity and regularization of the domain. The connection with optimal control is proved for a deterministic minimum-time problem and for the problem of maximizing the expected escape time of a degenerate diffusion process from an open set.

Introduction

The theory of viscosity solutions provides a general framework for studying the partial differential equations arising in the Dynamic Programming approach to deterministic and stochastic optimal control problems and differential games. This theory is designed for scalar fully nonlinear PDEs

$$F(x, u(x), Du(x), D^2u(x)) = 0 \text{ in } \Omega\,, \tag{1}$$

where Ω is a general open subset of $\mathbb{R}^N$, with the monotonicity property

$$\begin{gathered} F(x, r, p, X) \leq F(x, s, p, Y) \\ \text{if } r \leq s \text{ and } X - Y \text{ is positive semidefinite,} \end{gathered} \tag{2}$$

so it includes $1st$ order Hamilton-Jacobi equations and $2nd$ order PDEs that are degenerate elliptic or parabolic in a very general sense [18, 5].

The Hamilton-Jacobi-Bellman (briefly, HJB) equations in the theory of optimal control of diffusion processes are of the form

$$\sup_{\alpha \in A} \mathcal{L}^{\alpha} u = 0\,, \tag{3}$$

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where α is the control variable and, for each α, $\mathcal{L}^\alpha$ is a linear nondivergence form operator

$$\mathcal{L}^\alpha u := -a_{ij}^\alpha \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i^\alpha \frac{\partial u}{\partial x_i} + c^\alpha u - f^\alpha, \tag{4}$$

where f and c are the running cost and the discount rate in the cost functional, b is the drift of the system, $a = \frac{1}{2}\sigma\sigma^T$ and σ is the variance of the noise affecting the system (see Section 3.2). These equations satisfy (2) if and only if

$$a_{ij}^\alpha(x)\xi_i\xi_j \geq 0 \text{ and } c^\alpha(x) \geq 0, \text{ for all } x \in \Omega,\ \alpha \in A,\ \xi \in \mathbb{R}^N, \tag{5}$$

and these conditions are automatically satisfied by operators coming from control theory. In the case of deterministic systems we have $a_{ij}^\alpha \equiv 0$ and the PDE is of $1st$ order. In the theory of two-person zero-sum deterministic and stochastic differential games the Isaacs' equation has the form

$$\sup_{\alpha \in A} \inf_{\beta \in B} \mathcal{L}^{\alpha,\beta} u = 0, \tag{6}$$

where β is the control of the second player and $\mathcal{L}^{\alpha,\beta}$ are linear operators of the form (4) and satisfying assumptions such as (5).

For many different problems it was proved that the value function is the unique continuous viscosity solution satisfying appropriate boundary conditions, see the books [22, 8, 4, 5] and the references therein. This has a number of useful consequences, because we have PDE methods available to tackle several problems, such as the numerical calculation of the value function, the synthesis of approximate optimal feedback controls, asymptotic problems (vanishing noise, penalization, risk-sensitive control, ergodic problems, singular perturbations ...). However, the theory is considerably less general for problems with *discontinuous* value function, because it is restricted to deterministic systems with a single controller, where the HJB equation is of first order with convex Hamiltonian in the p variables. The pioneering papers on this issue are due to Barles and Perthame [10] and Barron and Jensen [11], who use different definitions of non-continuous viscosity solutions, see also [27, 28, 7, 39, 14], the surveys and comparisons of the different approaches in the books [8, 4, 5], and the references therein.

For cost functionals involving the exit time of the state from the set Ω, the value function is discontinuous if the noise vanishes near some part of the boundary and there is not enough controllability of the drift; other possible sources of discontinuities are the lack of smoothness of $\partial\Omega$, even for nondegenerate noise, and the discontinuity or incompatibility of the boundary data, even if the drift is controllable (see [8, 4, 5] for examples). For these functionals the value should be the solution of the Dirichlet problem

$$\begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where $g(x)$ is the cost of exiting Ω at x and we assume $g \in C(\partial\Omega)$. For $2nd$ order equations, or $1st$ order equations with nonconvex Hamiltonian, there are no local definitions of weak solution and weak boundary conditions that ensure existence and uniqueness of a possibly discontinuous solution. However a global definition of generalized solution of (7) can be given by the following variant of the classical Perron-Wiener-Brelot method in potential theory. We define

$$\begin{aligned} \mathcal{S} &:= \{w \in BUSC(\overline{\Omega}) \text{ subsolution of (1)},\ w \leq g \text{ on } \partial\Omega\} \\ \mathcal{Z} &:= \{W \in BLSC(\overline{\Omega}) \text{ supersolution of (1)},\ W \geq g \text{ on } \partial\Omega\}, \end{aligned}$$

where $BUSC(\overline{\Omega})$ (respectively, $BLSC(\overline{\Omega})$) denote the sets of bounded upper (respectively, lower) semicontinuous functions on $\overline{\Omega}$, and we say that $u : \overline{\Omega} \to \mathbb{R}$ is a generalized solution of (7) if

$$u(x) = \sup_{w \in \mathcal{S}} w(x) = \inf_{W \in \mathcal{Z}} W(x)\,. \tag{8}$$

With respect to the classical Wiener's definition of generalized solution of the Dirichlet problem for the Lapalce equation in general nonsmooth domains [45] (see also [16, 26]), we only replace sub- and superharmonic functions with viscosity sub- and supersolutions. In the classical theory the inequality $\sup_{w\in\mathcal{S}} w \leq \inf_{W\in\mathcal{Z}} W$ comes from the maximum principle, here it comes from the *Comparison Principle* for viscosity sub- and supersolutions; this important result holds under some additional assumptions that are very reasonable for the HJB equations of control theory, see Section 1.1; for this topic we refer to Jensen [29] and Crandall, Ishii and Lions [18]. The main difference with the classical theory is that the PWB solution for the Laplace equation is harmonic in Ω and can be discontinuous only at boundary points where $\partial\Omega$ is very irregular, whereas here u can be discontinuous also in the interior and even if the boundary is smooth: this is because the very degenerate ellipticity (2) neither implies regularizing effects, nor it guarantees that the boundary data are attained continuously. Note that if a continuous viscosity solution of (7) exists it coincides with u, and both the sup and the inf in (8) are attained.

Perron's method was extended to viscosity solutions by Ishii [27] (see Theorem 1), who used it to prove general existence results of continuous solutions. The PWB generalized solution of (7) of the form (8) was studied indipendently by the authors and Capuzzo-Dolcetta [4, 1] and by M. Ramaswamy and S. Ramaswamy [38] for some special cases of equations of the form (1), (2). In [4] this notion is called *envelope solution* and several properties are studied, in particular the equivalence with the generalized minimax solution of Subbotin [41, 42] and the connection with deterministic optimal control. The connection with pursuit-evasion games can be found in [41, 42] within the Krasovskii-Subbotin theory, and in our paper with Falcone [3] for the Fleming value; in [3] we also study the convergence of a numerical scheme.

The purposes of this paper are to extend the existence and basic properties of the PWB solution in [4, 1, 38] to more general operators, to prove some new continuity properties with respect to the data, in particular for the vanishing viscosity method and for approximations of the domain, and finally to show a connection with stochastic optimal control. For the sake of completeness we give all the proofs even if some of them follow the same argument as in the quoted references.

Let us now describe the contents of the paper in some detail. In Subsection 1.1 we recall some known definitions and results. In Subsection 1.2 we prove the existence theorem under an assumption on the boundary data g that is reminiscent of the compatibility conditions in the theory of $1st$ order Hamilton-Jacobi equations [34, 4]; this condition implies that the PWB solution is either the minimal supersolution or the maximal subsolution (i.e., either the inf or the sup in (8) is attained), and it is verified in time-optimal control problems. We recall that the classical Wiener Theorem asserts that for the Laplace equation any continuous boundary function g is *resolutive* (i.e., the PWB solution of the corresponding Dirichlet problem exists), and this was extended to some quasilinear nonuniformly elliptic equations, see the book of Heinonen, Kilpeläinen and Martio [25]. We do not know at the moment if this result can be extended to some class of fully nonlinear degenerate equations; however we prove in Subsection 2.1 that the set of resolutive boundary functions in our context is closed under uniform convergence as in the classical case (cfr. [26, 38]).

In Subsection 1.3 we show that the PWB solution is consistent with the notions of generalized solution by Subbotin [41, 42] and Ishii [27], and it satisfies the Dirichlet boundary condition in the weak viscosity sense [10, 28, 18, 8, 4]. Subsection 2.1 is devoted to the stability of the PWB solution with respect to the uniform convergence of the boundary data and the operator F. In Subsection 2.2 we consider merely local uniform perturbations of F, such as the vanishing viscosity, and prove a kind of stability provided the set Ω is simultaneously approximated from the interior.

In Subsection 2.3 we prove that for a nested sequence of open subsets Ω_n of Ω such that $\bigcup_n \Omega_n = \Omega$, if u_n is the PWB solution of the Dirichlet problem in Ω_n, the solution u of (7) satisfies

$$u(x) = \lim_n u_n(x)\,, \quad x \in \Omega\,. \tag{9}$$

This allows to approximate u with more regular solutions u_n when $\partial\Omega$ is not smooth and Ω_n are chosen with smooth boundary. This approximation procedure goes back to Wiener [44] again, and it is standard in elliptic theory for nonsmooth domains where (9) is often used to *define* a generalized solution of (7), see e.g. [30, 23, 12, 33]. In Subsection 2.3 we characterize the boundary points where the data are attained continuously in terms of the existence of suitable local barriers.

The last section is devoted to two applications of the previous theory to optimal control. The first (Subsection 3.1) is the classical minimum time problem for deterministic nonlinear systems with a closed target. In this case the lower semicontinuous envelope of the value function is the PWB solution of the homogeneous Dirichlet problem for the Bellman equation. The proof we give here is different from the one in [7, 4] and simpler. The second application (Subsection 3.2) is about the problem of maximizing the expected discounted time that a controlled degenerate diffusion process spends in Ω. Here we prove that the value function itself is the PWB solution of the appropriate problem. In both cases $g \equiv 0$ is a subsolution of the Dirichlet problem, which implies that the PWB solution is also the minimal supersolution.

It is worth to mention some recent papers using related methods. The thesis of Bettini [13] studies upper and lower semicontinuous solutions of the Cauchy problem for degenerate parabolic and $1st$ order equations with applications to finite horizon differential games. Our paper [2] extends some results of the present one to boundary value problems where the data are prescribed only on a suitable part of $\partial\Omega$. The first author, Goatin and Ishii [6] study the boundary value problem for (1) with Dirichlet conditions in the viscosity sense; they construct a PWB-type generalized solution that is also the limit of approximations of Ω from the outside, instead of the inside. This solution is in general different from ours and it is related to control problems involving the exit time from $\overline{\Omega}$, instead of Ω.

1. Generalized solutions of the Dirichlet problem

1.1. Preliminaries

Let F be a continuous function

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times S(N) \to \mathbb{R}\,,$$

where Ω is an open subset of $\mathbb{R}^N$, $S(N)$ is the set of symmetric $N \times N$ matrices equipped with its usual order, and assume that F satisfies (2). Consider the partial differential equation

$$F(x, u(x), Du(x), D^2u(x)) = 0 \ \text{ in } \Omega\,, \tag{10}$$

where $u : \Omega \to \mathbb{R}$, Du denotes the gradient of u and D^2u denotes the Hessian matrix of second derivatives of u. From now on subsolutions, supersolutions and solutions of this equation will be understood in the viscosity sense; we refer to [18, 5] for the definitions. For a general subset E of $\mathbb{R}^N$ we indicate with $USC(E)$, respectively $LSC(E)$, the set of all functions $E \to \mathbb{R}$ upper, respectively lower, semicontinuous, and with $BUSC(E)$, $BLSC(E)$ the subsets of functions that are also bounded.

DEFINITION 1. *We will say that equation (10) satisfies the* Comparison Principle *if for all subsolutions $w \in BUSC(\overline{\Omega})$ and supersolutions $W \in BLSC(\overline{\Omega})$ of (10) such that $w \leq W$ on $\partial\Omega$, the inequality $w \leq W$ holds in $\overline{\Omega}$.*

We refer to [29, 18] for the strategy of proof of some comparison principles, examples and references. Many results of this type for first order equations can be found in [8, 4].

The main examples we are interested in are the Isaacs equations:

$$\sup_{\alpha} \inf_{\beta} \mathcal{L}^{\alpha,\beta} u(x) = 0 \tag{11}$$

and

$$\inf_{\beta} \sup_{\alpha} \mathcal{L}^{\alpha,\beta} u(x) = 0\,, \tag{12}$$

where

$$\mathcal{L}^{\alpha,\beta} u(x) = -a_{ij}^{\alpha,\beta}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i^{\alpha,\beta}(x) \frac{\partial u}{\partial x_i} + c^{\alpha,\beta}(x) u - f^{\alpha,\beta}(x)\,.$$

Here F is

$$F(x, r, p, X) = \sup_{\alpha} \inf_{\beta} \{-\mathrm{trace}\,(a^{\alpha,\beta}(x) X) + b^{\alpha,\beta}(x) \cdot p + c^{\alpha,\beta}(x) r - f^{\alpha,\beta}(x)\}\,.$$

If, for all $x \in \overline{\Omega}$, $a^{\alpha,\beta}(x) = \frac{1}{2}\sigma^{\alpha,\beta}(x)(\sigma^{\alpha,\beta}(x))^T$, where $\sigma^{\alpha,\beta}(x)$ is a matrix of order $N \times M$, ^{T} denotes the transpose matrix, $\sigma^{\alpha,\beta},\ b^{\alpha,\beta},\ c^{\alpha,\beta},\ f^{\alpha,\beta}$ are bounded and uniformly continuous in $\overline{\Omega}$, uniformly with respect to α, β, then F is continuous, and it is proper if in addition $c^{\alpha,\beta} \geq 0$ for all α, β.

Isaacs equations satisfy the Comparison Principle if Ω is bounded and there are positive constants K_1, K_2, and C such that

$$F(x, t, p, X) - F(x, s, q, Y) \leq \max\{K_1 \mathrm{trace}\,(Y - X),\ K_1(t - s)\} + K_2|p - q|\,, \tag{13}$$

for all $Y \leq X$ and $t \leq s$,

$$\|\sigma^{\alpha,\beta}(x) - \sigma^{\alpha,\beta}(y)\| \leq C|x - y|, \text{ for all } x, y \in \overline{\Omega} \text{ and all } \alpha, \beta \tag{14}$$

$$|b^{\alpha,\beta}(x) - b^{\alpha,\beta}(y)| \leq C|x - y|, \text{ for all } x, y \in \overline{\Omega} \text{ and all } \alpha, \beta\,, \tag{15}$$

see Corollary 5.11 in [29]. In particular condition (13) is satisfied if and only if

$$\max\{\lambda^{\alpha,\beta}(x), c^{\alpha,\beta}(x)\} \geq K > 0 \text{ for all } x \in \overline{\Omega},\ \alpha \in A,\ \beta \in B\,,$$

where $\lambda^{\alpha,\beta}(x)$ is the smallest eigenvalue of $A^{\alpha,\beta}(x)$. Note that this class of equations contains as special cases the Hamilton-Jacobi-Bellman equations of optimal stochastic control (3) and linear degenerate elliptic equations with Lipschitz coefficients.

Given a function $u : \Omega \to [-\infty, +\infty]$, we indicate with u^* and u_*, respectively, the upper and the lower semicontinuous envelope of u, that is,

$$\begin{aligned} u^*(x) &:= \lim_{r \searrow 0} \sup\{u(y) : y \in \Omega,\ |y - x| \leq r\}, \\ u_*(x) &:= \lim_{r \searrow 0} \inf\{u(y) : y \in \Omega,\ |y - x| \leq r\}. \end{aligned}$$

PROPOSITION 1. *Let S (respectively Z) be a set of functions such that for all $w \in S$ (respectively $W \in Z$) w^* is a subsolution (respectively W_* is a supersolution) of (10). Define the function*

$$u(x) := \sup_{w \in S} w(x),\ x \in \Omega,\ (\textit{respectively } u(x) := \inf_{W \in Z} W(x)).$$

If u is locally bounded, then u^* is a subsolution (respectively u_* is a supersolution) of (10).

The proof of Proposition 1 is an easy variant of Lemma 4.2 in [18].

PROPOSITION 2. *Let $w_n \in BUSC(\Omega)$ be a sequence of subsolutions (respectively $W_n \in BLSC(\Omega)$ a sequence of supersolutions) of (10), such that $w_n(x) \searrow u(x)$ for all $x \in \Omega$ (respectively $W_n(x) \nearrow u(x)$) and u is a locally bounded function. Then u is a subsolution (respectively supersolution) of (10).*

For the proof see, for instance, [4]. We recall that, for a generale subset E of $\mathbb{R}^N$ and $\hat{x} \in E$, the second order superdifferential of u at $\hat{x}$ is the subset $J_E^{2,+}u(\hat{x})$ of $\mathbb{R}^N \times S(N)$ given by the pairs (p, X) such that

$$u(x) \leq u(\hat{x}) + p \cdot (x - \hat{x}) + \frac{1}{2} X(x - \hat{x}) \cdot (x - \hat{x}) + o(|x - \hat{x}|^2)$$

for $E \ni x \to \hat{x}$. The opposite inequality defines the second order subdifferential of u at $\hat{x}$, $J_E^{2,-}u(\hat{x})$.

LEMMA 1. *Let u^* be a subsolution of (10). If u_* fails to be a supersolution at some point $\hat{x} \in \Omega$, i.e. there exist $(p, X) \in J_\Omega^{2,-}u_*(\hat{x})$ such that*

$$F(\hat{x}, u_*(\hat{x}), p, X) < 0,$$

then for all $k > 0$ small enough, there exists $U_k : \Omega \to \mathbb{R}$ such that U_k^* is subsolution of (10) and

$$\begin{cases} U_k(x) \geq u(x),\ \ \sup_\Omega(U_k - u) > 0, \\ U_k(x) = u(x) \textit{ for all } x \in \Omega \textit{ such that } |x - \hat{x}| \geq k. \end{cases}$$

The proof is an easy variant of Lemma 4.4 in [18]. The last result of this subsection is Ishii's extension of Perron's method to viscosity solutions [27].

THEOREM 1. *Assume there exists a subsolution u_1 and a supersolution u_2 of (10) such that $u_1 \leq u_2$, and consider the functions*

$$\begin{aligned} U(x) &:= \sup\{w(x) : u_1 \leq w \leq u_2,\ w^* \textit{ subsolution of } (10)\}, \\ W(x) &:= \inf\{w(x) : u_1 \leq w \leq u_2,\ w_* \textit{ supersolution of } (10)\}. \end{aligned}$$

Then U^*, W^* are subsolutions of (10) and U_*, W_* are supersolutions of (10).

1.2. Existence of solutions by the PWB method

In this section we present a notion of weak solution for the boundary value problem

$$(16)\qquad \begin{cases} F(x,u,Du,D^2u)=0 & \text{in } \Omega\,,\\ u=g & \text{on } \partial\Omega\,,\end{cases}$$

where F satisfies the assumptions of Subsection 1.1 and $g:\partial\Omega\to\mathbb{R}$ is continuous. We recall that $\mathcal{S}$, $\mathcal{Z}$ are the sets of all subsolutions and all supersolutions of (16) defined in the Introduction.

DEFINITION 2. *The function defined by*

$$\underline{H}_g(x) := \sup_{w\in\mathcal{S}} w(x)\,,$$

is the lower envelope viscosity solution*, or* Perron-Wiener-Brelot lower solution*, of (16). We will refer to it as the* lower e-solution*. The function defined by*

$$\overline{H}_g(x) := \inf_{W\in\mathcal{Z}} W(x)\,,$$

is the upper envelope viscosity solution*, or* PWB upper solution*, of (16), briefly* upper e-solution*. If $\underline{H}_g=\overline{H}_g$, then*

$$H_g := \underline{H}_g = \overline{H}_g$$

is the envelope viscosity solution *or* PWB solution *of (16), briefly* e-solution*. In this case the data g are called* resolutive.

Observe that $\underline{H}_g\le\overline{H}_g$ by the Comparison Principle, so the e-solution exists if the inequality $\ge$ holds as well. Next we prove the existence theorem for e-solutions, which is the main result of this section. We will need the following notion of global barrier, that is much weaker than the classical one.

DEFINITION 3. *We say that w is a* lower *(respectively,* upper*)* barrier *at a point $x\in\partial\Omega$ if $w\in\mathcal{S}$ (respectively, $w\in\mathcal{Z}$) and*

$$\lim_{y\to x} w(y) = g(x)\,.$$

THEOREM 2. *Assume that the Comparison Principle holds, and that $\mathcal{S}$, $\mathcal{Z}$ are nonempty.*

i) If there exists a lower barrier at all points $x\in\partial\Omega$, then $H_g=\min_{W\in\mathcal{Z}} W$ is the e-solution of (16).

ii) If there exists an upper barrier at all points $x\in\partial\Omega$, then $H_g=\max_{w\in\mathcal{S}} w$ is the e-solution of (16).

Proof. Let w be the lower barrier at $x\in\partial\Omega$, then by definition $w\le\underline{H}_g$. Thus

$$(\underline{H}_g)_*(x) = \liminf_{y\to x}\underline{H}_g(y)\ge\liminf_{y\to x} w(y) = g(x)\,.$$

By Theorem 1 $(\underline{H}_g)_*$ is a supersolution of (10), so we can conclude that $(\underline{H}_g)_*\in\mathcal{Z}$. Then $(\underline{H}_g)_*\ge\overline{H}_g\ge\underline{H}_g$, so $\underline{H}_g=\overline{H}_g$ and $\underline{H}_g\in\mathcal{Z}$.

□

EXAMPLE 1. Consider the problem

$$
(17) \qquad \begin{cases} -a_{ij}(x)u_{x_ix_j}(x) + b_i(x)u_{x_i}(x) + c(x)u(x) = 0 & \text{in } \Omega\,, \\ u(x) = g(x) & \text{on } \partial\Omega\,, \end{cases}
$$

with the matrix $a_{ij}(x)$ such that $a_{11}(x) \geq \mu > 0$ for all $x \in \Omega$. In this case we can show that all continuous functions on $\partial\Omega$ are resolutive. The proof follows the classical one for the Laplace equation, the only hard point is checking the superposition principle for viscosity sub- and supersolutions. This can be done by the same methods and under the same assumptions as the Comparison Principle.

1.3. Consistency properties and examples

Next results give a characterization of the e-solution as pointwise limit of sequences of sub and supersolutions of (16). If the equation (10) is of first order, this property is essentially Subbotin's definition of (generalized) minimax solution of (16) [41, 42].

THEOREM 3. *Assume that the Comparison Principle holds, and that $\mathcal{S}$, $\mathcal{Z}$ are nonempty.*

i) If there exists $\underline{u} \in \mathcal{S}$ continuous at each point of $\partial\Omega$ and such that $\underline{u} = g$ on $\partial\Omega$, then there exists a sequence $w_n \in \mathcal{S}$ such that $w_n \nearrow H_g$.

ii) If there exists $\overline{u} \in \mathcal{Z}$ continuous at each point of $\partial\Omega$ and such that $\overline{u} = g$ on $\partial\Omega$, then there exists a sequence $W_n \in \mathcal{Z}$ such that $W_n \searrow H_g$.

Proof. We give the proof only for i), the same proof works for ii). By Theorem 2 $H_g = \min_{W\in\mathcal{Z}} W$. Given $\epsilon > 0$ the function

$$
(18) \qquad u_\epsilon(x) := \sup\{w(x) : w \in \mathcal{S},\ w(x) = \underline{u}(x) \text{ if } \operatorname{dist}(x, \partial\Omega) < \epsilon\}\,,
$$

is bounded, and $u_\delta \leq u_\epsilon$ for $\epsilon < \delta$. We define

$$
V(x) := \lim_{n\to\infty} (u_{1/n})_*(x)\,,
$$

and note that, by definition, $H_g \geq u_\epsilon \geq (u_\epsilon)_*$, and then $H_g \geq V$. We claim that $(u_\epsilon)_*$ is supersolution of (10) in the set

$$
\Omega_\epsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\}\,.
$$

To prove this claim we assume by contradiction that $(u_\epsilon)_*$ fails to be a supersolution at $y \in \Omega_\epsilon$. Note that, by Proposition 1, $(u_\epsilon)^*$ is a subsolution of (10). Then by Lemma 1, for all $k > 0$ small enough, there exists U_k such that U_k^* is subsolution of (10) and

$$
(19) \qquad \sup_\Omega (U_k - u_\epsilon) > 0, \quad U_k(x) = u_\epsilon(x) \text{ if } |x - y| \geq k\,.
$$

We fix $k \leq \operatorname{dist}(y, \partial\Omega) - \epsilon$, so that $U_k(x) = u_\epsilon(x) = \underline{u}(x)$ for all x such that $\operatorname{dist}(x, \partial\Omega) < \epsilon$. Then $U_k^*(x) = \underline{u}(x)$, so $U_k^* \in \mathcal{S}$ and by the definition of u_ϵ we obtain $U_k^* \leq u_\epsilon$. This gives a contradiction with (19) and proves the claim.

By Proposition 2 V is a supersolution of (10) in Ω. Moreover if $x \in \partial\Omega$, for all $\epsilon > 0$, $(u_\epsilon)_*(x) = g(x)$, because $u_\epsilon(x) = \underline{u}(x)$ if $\operatorname{dist}(x, \partial\Omega) < \epsilon$ by definition, $\underline{u}$ is continuous and $\underline{u} = g$ on $\partial\Omega$. Then $V \geq g$ on $\partial\Omega$, and so $V \in \mathcal{Z}$.

To complete the proof we define $w_n := (u_{1/n})^*$, and observe that this is a nondecreasing sequence in $\mathcal{S}$ whose pointwise limit is $\geq V$ by definition of V. On the other hand $w_n \leq H_g$ by definition of H_g, and we have shown that $H_g = V$, so $w_n \nearrow H_g$.

□

COROLLARY 1. *Assume the hypotheses of Theorem 3. Then H_g is the e-solution of (16 if and only if there exist two sequences of functions $w_n \in \mathcal{S}$, $W_n \in \mathcal{Z}$, such that $w_n = W_n = g$ on $\partial\Omega$ and for all $x \in \overline{\Omega}$*

$$w_n(x) \to H_g(x), \quad W_n(x) \to H_g(x) \text{ as } n \to \infty .$$

REMARK 1. It is easy to see from the proof of Theorem 3, that in case $i)$, the e-solution H_g satisfies

$$H_g(x) = \sup_{\epsilon} u_\epsilon(x) \quad x \in \overline{\Omega} ,$$

where

$$u_\epsilon(x) := \sup\{w(x) : w \in \mathcal{S},\ w(x) = \underline{u}(x) \text{ for } x \in \overline{\Omega} \setminus \overline{\Theta}_\epsilon\} , \tag{20}$$

and Θ_ϵ, $\epsilon \in]0, 1]$, is any family of open sets such that $\overline{\Theta}_\epsilon \subseteq \Omega$, $\Theta_\epsilon \supseteq \Theta_\delta$ for $\epsilon < \delta$ and $\bigcup_\epsilon \Theta_\epsilon = \Omega$.

EXAMPLE 2. Consider the Isaacs equation (11) and assume the sufficient conditions for the Comparison Principle.

- If

$$g \equiv 0 \text{ and } f^{\alpha,\beta}(x) \geq 0 \text{ for all } x \in \overline{\Omega},\ \alpha \in A,\ \beta \in B ,$$

  then $\underline{u} \equiv 0$ is subsolution of the PDE, so the assumption $i)$ of Theorem 3 is satisfied.
- If the domain Ω is bounded with smooth boundary and there exist $\overline{\alpha} \in A$ and $\mu > 0$ such that

$$a_{ij}^{\overline{\alpha},\beta}(x)\xi_i\xi_j \geq \mu|\xi|^2 \text{ for all } \beta \in B,\ x \in \overline{\Omega},\ \xi \in \mathbb{R}^N ,$$

  then there exists a classical solution $\underline{u}$ of

$$\begin{cases} \inf_{\beta \in B} \mathcal{L}^{\overline{\alpha},\beta}\underline{u} = 0 & \text{in } \Omega , \\ \underline{u} = g & \text{on } \partial\Omega , \end{cases}$$

  see e.g. Chapt. 17 of [24]. Then $\underline{u}$ is a supersolution of (11), so the hypothesis $ii)$ of Theorem 3 is satisfied.

Next we compare e-solutions with Ishii's definitions of non-continuous viscosity solution and of boundary conditions in viscosity sense. We recall that a function $u \in BUSC(\overline{\Omega})$ (respectively $u \in BLSC(\overline{\Omega})$) is a *viscosity subsolution* (respectively a *viscosity supersolution*) *of the boundary condition*

$$u = g \text{ or } F(x, u, Du, D^2u) = 0 \text{ on } \partial\Omega , \tag{21}$$

if for all $x \in \partial\Omega$ and $\phi \in C^2(\overline{\Omega})$ such that $u-\phi$ attains a local maximum (respectively minimum) at x, we have

$$(u-g)(x) \leq 0 \text{ (resp. } \geq 0) \text{ or } F(x, u(x), D\phi(x), D^2\phi(x)) \leq 0 \text{ (resp. } \geq 0)\,.$$

An equivalent definition can be given by means of the semijets $J^{2,+}_{\overline{\Omega}} u(x)$, $J^{2,-}_{\overline{\Omega}} u(x)$ instead of the test functions, see [18].

PROPOSITION 3. *If $\underline{H}_g : \overline{\Omega} \to \mathbb{R}$ is the lower e-solution (respectively, $\overline{H}_g$ is the upper e-solution) of (16), then $\underline{H}_g^*$ is a subsolution (respectively, $\overline{H}_{g*}$ is a supersolution) of (10) and of the boundary condition (21).*

Proof. If $\underline{H}_g$ is the lower e-solution, then by Proposition 1, $\underline{H}_g^*$ is a subsolution of (10). It remains to check the boundary condition.

Fix an $y \in \partial\Omega$ such that $\underline{H}_g^*(y) > g(y)$, and $\phi \in C^2(\overline{\Omega})$ such that $\underline{H}_g^* - \phi$ attains a local maximum at y. We can assume, without loss of generality, that

$$\underline{H}_g^*(y) = \phi(y), \quad (\underline{H}_g^* - \phi)(x) \leq -|x-y|^3 \text{ for all } x \in \overline{\Omega} \cap B(y, r)\,.$$

By definition of $\underline{H}_g^*$, there exists a sequence of points $x_n \to y$ such that

$$(\underline{H}_g - \phi)(x_n) \geq -\frac{1}{n} \text{ for all } n\,.$$

Moreover, since $\underline{H}_g$ is the lower e-solution, there exists a sequence of functions $w_n \in S$ such that

$$\underline{H}_g(x_n) - \frac{1}{n} < w_n(x_n) \text{ for all } n\,.$$

Since the function $w_n - \phi$ is upper semicontinuous, it attains a maximum at $y_n \in \overline{\Omega} \cap \overline{B}(y, r)$, such that, for n big enough,

$$-\frac{2}{n} < (w_n - \phi)(y_n) \leq -|y_n - y|^3.$$

So as $n \to \infty$

$$y_n \to y, \quad w_n(y_n) \to \phi(y) = \underline{H}_g^*(y) > g(y)\,.$$

Note that $y_n \notin \partial\Omega$, because $y_n \in \partial\Omega$ would imply $w_n(y_n) \leq g(y_n)$, which gives a contradiction to the continuity of g at y. Therefore, since w_n is a subsolution of (10), we have

$$F(y_n, w_n(y_n), D\phi(y_n), D^2\phi(y_n)) \leq 0\,,$$

and letting $n \to \infty$ we get

$$F(y, \underline{H}_g^*(y), D\phi(y), D^2\phi(y)) \leq 0\,,$$

by the continuity of F.

□

REMARK 2. By Proposition 3, if the e-solution H_g of (16) exists, it is a non-continuous viscosity solution of (10) (21) in the sense of Ishii [27]. These solutions, however, are not unique in general. An e-solution satisfies also the Dirichlet problem in the sense that it is a non-continuous solution of (10) in Ishii's sense and $H_g(x) = g(x)$ for all $x \in \partial\Omega$, but neither this property characterizes it. We refer to [4] for explicit examples and more details.

REMARK 3. Note that, by Proposition 3, if the e-solution H_g is continuous at all points of $\partial\Omega_1$ with $\Omega_1 \subset \Omega$, we can apply the Comparison Principle to the upper and lower semicontinuous envelopes of H_g and obtain that it is continuous in Ω_1. If the equation is uniformly elliptic in Ω_1 we can also apply in Ω_1 the local regularity theory for continuous viscosity solutions developed by Caffarelli [17] and Trudinger [43].

2. Properties of the generalized solutions

2.1. Continuous dependence under uniform convergence of the data

We begin this section by proving a result about continuous dependence of the e-solution on the boundary data of the Dirichlet Problem. It states that the set of resolutive data is closed with respect to uniform convergence. Throughout the paper we denote with $\rightrightarrows$ the uniform convergence.

THEOREM 4. *Let $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times S(N) \to \mathbb{R}$ be continuous and proper, and let $g_n : \partial\Omega \to \mathbb{R}$ be continuous. Assume that $\{g_n\}_n$ is a sequence of resolutive data such that $g_n \rightrightarrows g$ on $\partial\Omega$. Then g is resolutive and $H_{g_n} \rightrightarrows H_g$ on Ω.*

The proof of this theorem is very similar to the classical one for the Laplace equation [26]. We need the following result:

LEMMA 2. *For all $c > 0$, $\underline{H}_{(g+c)} \leq \underline{H}_g + c$ and $\overline{H}_{(g+c)} \leq \overline{H}_g + c$.*

Proof. Let

$$\mathcal{S}_c := \{w \in BUSC(\overline{\Omega}) : w \text{ is subsolution of (10)},\ w \leq g + c \text{ on } \partial\Omega\}.$$

Fix $u \in \mathcal{S}_c$, and consider the function $v(x) = u(x) - c$. Since F is proper it is easy to see that $v \in \mathcal{S}$. Then

$$\underline{H}_{(g+c)} := \sup_{u \in \mathcal{S}_c} u \leq \sup_{v \in \mathcal{S}} v + c := \underline{H}_g + c\,.$$

□

of Theorem 4. Fix $\epsilon > 0$, the uniform convergence implies $\exists m : \forall n \geq m$: $g_n - \epsilon \leq g \leq g_n + \epsilon$. Since g_n is resolutive by Lemma 2, we get

$$H_{g_n} - \epsilon \leq \underline{H}_{(g_n - \epsilon)} \leq \underline{H}_g \leq \underline{H}_{(g_n + \epsilon)} \leq H_{g_n} + \epsilon\,.$$

Therefore $H_{g_n} \rightrightarrows \underline{H}_g$. The proof that $H_{g_n} \rightrightarrows \overline{H}_g$, is similar.

□

Next result proves the continuous dependence of e-solutions with respect to the data of the Dirichlet Problem, assuming that the equations F_n are strictly decreasing in r, uniformly in n.

THEOREM 5. *Let $F_n : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times S(N) \to \mathbb{R}$ is continuous and proper, $g : \partial\Omega \to \mathbb{R}$ is continuous. Suppose that $\forall n,\ \forall\delta > 0\ \exists\epsilon$ such that*

$$F_n(x, r - \delta, p, X) + \epsilon \le F_n(x, r, p, X)$$

for all $(x, r, p, X) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times S(N)$, and $F_n \rightrightarrows F$ on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times S(N)$. Suppose g is resolutive for the problems

$$\begin{cases} F_n(x, u, Du, D^2u) = 0 & \text{in } \Omega\,, \\ u = g & \text{on } \partial\Omega\,. \end{cases} \tag{22}$$

Suppose $g_n : \partial\Omega \to \mathbb{R}$ is continuous, $g_n \rightrightarrows g$ on $\partial\Omega$ and g_n is resolutive for the problem

$$\begin{cases} F_n(x, u, Du, D^2u) = 0 & \text{in } \Omega\,, \\ u = g_n & \text{on } \partial\Omega\,. \end{cases} \tag{23}$$

Then g is resolutive for (16) and $H^n_{g_n} \rightrightarrows H_g$, where $H^n_{g_n}$ is the e-solution of (23).

Proof. Step 1. For fixed $\delta > 0$ we want to show that there exists m such that for all $n \ge m$: $|\underline{H}^n_g - \underline{H}_g| \le \delta$, where H^n_g is the e-solution of (22).

We claim that there exists m such that $\underline{H}^n_g - \delta \le \underline{H}_g$ and $\overline{H}_g \le \overline{H}^n_g + \delta$ for all $n \ge m$. Then

$$\underline{H}^n_g - \delta \le \underline{H}_g \le \overline{H}_g \le \overline{H}^n_g + \delta = \underline{H}^n_g + \delta\,.$$

This proves in particular $H^n_g \rightrightarrows \underline{H}_g$ and $H^n_g \rightrightarrows \overline{H}_g$, and then $\overline{H}_g = \underline{H}_g$, so g is resolutive for (16).

It remains to prove the claim. Let

$$\mathcal{S}^n_g := \{v \text{ subsolution of } F_n = 0 \text{ in } \Omega,\ v \le g \text{ on } \partial\Omega\}\,.$$

Fix $v \in \mathcal{S}^n_g$, and consider the function $u = v - \delta$. By hypothesis there exists an ϵ such that $F_n(x, u(x), p, X) + \epsilon \le F_n(x, v(x), p, X)$, for all $(p, X) \in J^{2,+}_\Omega u(x)$. Then using uniform convergence of F_n at F we get

$$F(x, u(x), p, X) \le F_n(x, u(x), p, X) + \epsilon \le F_n(x, v(x), p, X) \le 0\,,$$

so v is a subsolution of the equation $F_n = 0$ because $J^{2,+}_\Omega v(x) = J^{2,+}_\Omega u(x)$.

We have shown that for all $v \in \mathcal{S}^n_g$ there exists $u \in \mathcal{S}$ such that $v = u + \delta$, and this proves the claim.

Step 2. Using the argument of proof of Theorem 4 with the problem

$$\begin{cases} F_m(x, u, Du, D^2u) = 0 & \text{in } \Omega\,, \\ u = g_n & \text{on } \partial\Omega\,, \end{cases} \tag{24}$$

we see that fixing $\delta > 0$, there exists p such that for all $n \ge p$: $|\underline{H}^m_{g_n} - \underline{H}^m_g| \le \delta$ for all m.

Step 3. Using again arguments of proof of Theorem 4, we see that fixing $\delta > 0$ there exists q such that for all $n, m \ge q$: $|\underline{H}^m_{g_n} - \underline{H}^m_{g_m}| \le \delta$.

Step 4. Now take $\delta > 0$, then there exists p such that for all $n, m \geq p$:

$$|\underline{H}^m_{g_m} - \underline{H}_g| \leq |\underline{H}^m_{g_m} - \underline{H}^m_{g_n}| + |\underline{H}^m_{g_n} - \underline{H}^m_g| + |\underline{H}^m_g - \underline{H}_g| \leq 3\delta\,.$$

Similarly $|\overline{H}^m_{g_m} - \overline{H}_g| \leq 3\delta$. But $\underline{H}^m_{g_m} = \overline{H}^m_{g_m}$, and this complete the proof.

□

2.2. Continuous dependence under local uniform convergence of the operator

In this subsection we study the continuous dependence of e-solutions with respect to perturbations of the operator, depending on a parameter h, that are not uniform over all $\Omega \times \mathbb{R} \times \mathbb{R}^N \times S(N)$ as they were in Theorem 5, but only on compact subsets of $\Omega \times \mathbb{R} \times \mathbb{R}^N \times S(N)$. A typical example we have in mind is the vanishing viscosity approximation, but similar arguments work for discrete approximation schemes, see [3]. We are able to pass to the limit under merely local perturbations of the operator by approximating Ω with a nested family of open sets Θ_ϵ, solving the problem in each Θ_ϵ, and then letting $\epsilon,\ h$ go to 0 "with h linked to ϵ" in the following sense.

DEFINITION 4. *Let v^ϵ_h, $u : Y \to \mathbb{R}$, for $\epsilon > 0$, $h > 0$, $Y \subseteq \mathbb{R}^N$. We say that v^ϵ_h* converges to u as $(\epsilon, h) \searrow (0,0)$ with h linked to ϵ *at the point x, and write*

$$\lim_{\substack{(\epsilon,h)\searrow(0,0)\\ h\leq h(\epsilon)}} v^\epsilon_h(x) = u(x) \tag{25}$$

if for all $\gamma > 0$, there exist a function $\tilde{h} :]0, +\infty[\to]0, +\infty[$ and $\overline{\epsilon} > 0$ such that

$$|v^\epsilon_h(y) - u(x)| \leq \gamma,\ \textit{for all } y \in Y : |x - y| \leq \tilde{h}(\epsilon)$$

for all $\epsilon \leq \overline{\epsilon}$, $h \leq \tilde{h}(\epsilon)$.

To justify this definition we note that:

i) it implies that for any x and $\epsilon_n \searrow 0$ there is a sequence $h_n \searrow 0$ such that $v^{\epsilon_n}_{h_n}(x_n) \to u(x)$ for any sequence x_n such that $|x - x_n| \leq h_n$, e.g. $x_n = x$ for all n, and the same holds for any sequence $h'_n \geq h_n$;

ii) if $\lim_{h\searrow 0} v^\epsilon_h(x)$ exists for all small ϵ and its limit as $\epsilon \searrow 0$ exists, then it coincides with the limit of Definition 4, that is,

$$\lim_{\substack{(\epsilon,h)\searrow(0,0)\\ h\leq h(\epsilon)}} v^\epsilon_h(x) = \lim_{\epsilon\searrow 0}\lim_{h\searrow 0} v^\epsilon_h(x)\,.$$

REMARK 4. If the convergence of Definition 4 occurs on a compact set K where the limit u is continuous, then (25) can be replaced, for all $x \in K$ and redefining $\tilde{h}$ if necessary, with

$$|v^\epsilon_h(y) - u(y)| \leq 2\gamma,\ \text{for all } y \in K : |x - y| \leq \tilde{h}(\epsilon)\,,$$

and by a standard compactness argument we obtain the uniform convergence in the following sense:

DEFINITION 5. *Let K be a subset of $\mathbb{R}^N$ and v^ϵ_h, $u : K \to \mathbb{R}$ for all $\epsilon, h > 0$. We say that v^ϵ_h* converge uniformly *on K to u as $(\epsilon, h) \searrow (0,0)$ with h* linked to ϵ *if for any $\gamma > 0$ there are $\overline{\epsilon} > 0$ and $\tilde{h} :]0, +\infty[\to]0, +\infty[$ such that*

$$\sup_K |v^\epsilon_h - u| \leq \gamma$$

for all $\epsilon \leq \overline{\epsilon}$, $h \leq \tilde{h}(\epsilon)$.

The main result of this subsection is the following. Recall that a family of functions $v_h^\epsilon : \Omega \to \mathbb{R}$ is locally uniformly bounded if for each compact set $K \subseteq \Omega$ there exists a constant C_K such that $\sup_K |v_h^\epsilon| \leq C_K$ for all $h, \epsilon > 0$. In the proof we use the weak limits in the viscosity sense and the stability of viscosity solutions and of the Dirichlet boundary condition in viscosity sense (21) with respect to such limits.

THEOREM 6. *Assume the Comparison Principle holds,* $\mathcal{Z} \neq \emptyset$ *and let* $\underline{u}$ *be a continuous subsolution of (16) such that* $\underline{u} = g$ *on* $\partial\Omega$. *For any* $\epsilon \in]0,1]$, *let* Θ_ϵ *be an open set such that* $\overline{\Theta_\epsilon} \subseteq \Omega$, *and for* $h \in]0,1]$ *let* v_h^ϵ *be a non-continuous viscosity solution of the problem*

$$\begin{cases} F_h(x,u,Du,D^2u) = 0 & \text{in } \Theta_\epsilon\,, \\ u(x) = \underline{u}(x) \text{ or } F_h(x,u,Du,D^2u) = 0 & \text{on } \partial\Theta_\epsilon\,, \end{cases} \tag{26}$$

where $F_h : \overline{\Theta_\epsilon} \times \mathbb{R} \times \mathbb{R}^N \times S(N) \to \mathbb{R}$ *is continuous and proper. Suppose* $\{v_h^\epsilon\}$ *is locally uniformly bounded,* $v_h^\epsilon \geq \underline{u}$ *in* $\overline{\Omega}$, *and extend* $v_h^\epsilon := \underline{u}$ *in* $\overline{\Omega} \setminus \Theta_\epsilon$. *Finally assume that* F_h *converges uniformly to* F *on any compact subset of* $\Omega \times \mathbb{R} \times \mathbb{R}^N \times S(N)$ *as* $h \searrow 0$, *and* $\Theta_\epsilon \supseteq \Theta_\delta$ *if* $\epsilon < \delta$, $\bigcup_{0<\epsilon\leq 1} \Theta_\epsilon = \Omega$.

Then v_h^ϵ *converges to the e-solution* H_g *of (16) with* h *linked to* ϵ, *that is, (25) holds for all* $x \in \overline{\Omega}$; *moreover the convergence is uniform (as in Def. 5) on any compact subset of* $\overline{\Omega}$ *where* H_g *is continuous.*

Proof. Note that the hypotheses of Theorem 3 are satisfied, so the e-solution H_g exists. Consider the weak limits

$$\begin{aligned} \underline{v}_\epsilon(x) &:= \liminf_{h\searrow 0}{}_* v_h^\epsilon(x) := \sup_{\delta>0} \inf\{v_h^\epsilon(y) : |x-y| < \delta,\ 0 < h < \delta\}\,, \\ \overline{v}_\epsilon(x) &:= \limsup_{h\searrow 0}{}^* v_h^\epsilon(x) := \inf_{\delta>0} \sup\{v_h^\epsilon(y) : |x-y| < \delta,\ 0 < h < \delta\}\,. \end{aligned}$$

By a standard result in the theory of viscosity solutions, see [10, 18, 8, 4], $\underline{v}_\epsilon$ and $\overline{v}_\epsilon$ are respectively supersolution and subsolution of

$$\begin{cases} F(x,u,Du,D^2u) = 0 & \text{in } \Theta_\epsilon\,, \\ u(x) = \underline{u}(x) \text{ or } F(x,u,Du,D^2u) = 0 & \text{on } \partial\Theta_\epsilon\,. \end{cases} \tag{27}$$

We claim that $\overline{v}_\epsilon$ is also a subsolution of (16). Indeed $v_h^\epsilon \equiv \underline{u}$ in $\Omega \setminus \Theta_\epsilon$, so $\overline{v}_\epsilon \equiv \underline{u}$ in the interior of $\Omega \setminus \Theta_\epsilon$ and then in this set it is a subsolution. In Θ_ϵ we have already seen that $\overline{v}_\epsilon = (\overline{v}_\epsilon)^*$ is a subsolution. It remains to check what happens on $\partial\Theta_\epsilon$. Given $\hat{x} \in \partial\Theta_\epsilon$, we must prove that for all $(p,X) \in J_\Omega^{2,+}\overline{v}_\epsilon(\hat{x})$ we have

$$F_h(\hat{x}, \overline{v}_\epsilon(\hat{x}), p, X) \leq 0\,. \tag{28}$$

1st Case: $\overline{v}_\epsilon(\hat{x}) > \underline{u}(\hat{x})$. Since $\overline{v}_\epsilon$ satisfies the boundary condition on $\partial\Theta_\epsilon$ of problem (27), then for all $(p,X) \in J_{\overline{\Theta_\epsilon}}^{2,+}\overline{v}_\epsilon(\hat{x})$ (28) holds. Then the same inequality holds for all $(p,X) \in J_\Omega^{2,+}\overline{v}_\epsilon(\hat{x})$ as well, because $J_\Omega^{2,+}\overline{v}_\epsilon(\hat{x}) \subseteq J_{\overline{\Theta_\epsilon}}^{2,+}\overline{v}_\epsilon(\hat{x})$.

2nd Case: $\overline{v}_\epsilon(\hat{x}) = \underline{u}(\hat{x})$. Fix $(p,X) \in J_\Omega^{2,+}\overline{v}_\epsilon(\hat{x})$, by definition

$$\overline{v}_\epsilon(x) \leq \overline{v}_\epsilon(\hat{x}) + p\cdot(x-\hat{x}) + \frac{1}{2}X(x-\hat{x})\cdot(x-\hat{x}) + o(|x-\hat{x}|^2)$$

for all $x \to \hat{x}$. Since $\overline{v}_\epsilon \geq \underline{u}$ and $\overline{v}_\epsilon(\hat{x}) = \underline{u}(\hat{x})$, we get

$$\underline{u}(x) \leq \underline{u}(\hat{x}) + p \cdot (x - \hat{x}) + \frac{1}{2} X(x - \hat{x}) \cdot (x - \hat{x}) + o(|x - \hat{x}|^2)\,,$$

that is $(p, X) \in J^{2,+}_{\Omega}\underline{u}(\hat{x})$. Now, since $\underline{u}$ is a subsolution, we conclude

$$F(\hat{x}, \overline{v}_\epsilon(\hat{x}), p, X) = F(\hat{x}, \underline{u}(\hat{x}), p, X) \leq 0\,.$$

We now claim that

$$u_\epsilon \leq \underline{v}_\epsilon \leq \overline{v}_\epsilon \leq H_g \text{ in } \overline{\Omega}\,, \tag{29}$$

where u_ϵ is defined by (20). Indeed, since $\underline{v}_\epsilon$ is a supersolution in Θ_ϵ and $\underline{v}_\epsilon \geq \underline{u}$, by the Comparison Principle $\underline{v}_\epsilon \geq w$ in Θ_ϵ for any $w \in \mathcal{S}$ such that $w = \underline{u}$ on $\partial\Theta_\epsilon$. Moreover $\underline{v}_\epsilon \equiv \underline{u}$ on $\Omega \backslash \Theta_\epsilon$, so we get $\underline{v}_\epsilon \geq u_\epsilon$ in $\overline{\Omega}$. To prove the last inequality we note that H_g is a supersolution of (16) by Theorem 3, which implies $\overline{v}_\epsilon \leq H_g$ by Comparison Principle.

Now fix $x \in \overline{\Omega}$, $\epsilon > 0$, $\gamma > 0$ and note that, by definition of lower weak limit, there exists $\overline{h} = \overline{h}(x, \epsilon, \gamma) > 0$ such that

$$\underline{v}_\epsilon(x) - \gamma \leq v_h^\epsilon(y)$$

for all $h \leq \overline{h}$ and $y \in \overline{\Omega} \cap B(x, \overline{h})$. Similarly there exists $\overline{k} = \overline{k}(x, \epsilon, \gamma) > 0$ such that

$$v_h^\epsilon(y) \leq \overline{v}_\epsilon(x) + \gamma$$

for all $h \leq \overline{k}$ and $y \in \overline{\Omega} \cap B(x, \overline{k})$. From Remark 1, we know that $H_g = \sup_\epsilon u_\epsilon$, so there exists $\overline{\epsilon}$ such that

$$H_g(x) - \gamma \leq u_\epsilon(x), \text{ for all } \epsilon \leq \overline{\epsilon}\,.$$

Then, using (29), we get

$$H_g(x) - 2\gamma \leq v_h^\epsilon(y) \leq H_g(x) + \gamma$$

for all $\epsilon \leq \overline{\epsilon}$, $h \leq \tilde{h} := \min\{\overline{h}, \overline{k}\}$ and $y \in \overline{\Omega} \cap B(x, \tilde{h})$, and this completes the proof.

□

REMARK 5. Theorem 6 applies in particular if v_h^ϵ are the solutions of the following vanishing viscosity approximation of (10)

$$\begin{cases} -h\Delta v + F(x, v, Dv, D^2 v) = 0 & \text{in } \Theta_\epsilon\,, \\ v = \underline{u} & \text{on } \partial\Theta_\epsilon\,. \end{cases} \tag{30}$$

Since F is degenerate elliptic, the PDE in (30) is uniformly elliptic for all $h > 0$. Therefore we can choose a family of nested Θ_ϵ with smooth boundary and obtain that the approximating v_h^ϵ are much smoother than the e-solution of (16). Indeed (30) has a classical solution if, for instance, either F is smooth and $F(x, \cdot, \cdot, \cdot)$ is convex, or the PDE (10 is a Hamilton-Jacobi-Bellman equation (3 where the linear operators $\mathcal{L}^\alpha$ have smooth coefficients, see [21, 24, 31]. In the nonconvex case, under some structural assumptions, the continuity of the solution of (30) follows from a barrier argument (see, e.g., [5]), and then it is twice differentiable almost everywhere by a result in [43], see also [17].

2.3. Continuous dependence under increasing approximation of the domain

In this subsection we prove the continuity of the e-solution of (16) with respect to approximations of the domain Ω from the interior. Note that, if $v_h^\epsilon = v^\epsilon$ for all h in Theorem 6, then $v^\epsilon(x) \to H_g(x)$ for all $x \in \Omega$ as $\epsilon \searrow 0$. This is the case, for instance, if v^ϵ is the unique e-solution of

$$\begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \Theta_\epsilon\,, \\ u = \underline{u} & \text{on } \partial\Theta_\epsilon\,, \end{cases}$$

by Proposition 3. The main result of this subsection extends this remark to more general approximations of Ω from the interior, where the condition $\overline{\Theta}_\epsilon \subseteq \Omega$ is dropped. We need first a monotonicity property of e-solutions with respect to the increasing of the domain.

LEMMA 3. *Assume the Comparison Principle holds and let $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{R}^N$, H_g^1, respectively H_g^2, be the e-solution in Ω_1, respectively Ω_2, of the problem*

$$\begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \Omega_i\,, \\ u = g & \text{on } \partial\Omega_i\,, \end{cases} \tag{31}$$

with $g : \overline{\Omega}_2 \to \mathbb{R}$ continuous and subsolution of (31) with $i = 2$. If we define

$$\tilde{H}_g^1(x) = \begin{cases} H_g^1(x) & \text{if } x \in \overline{\Omega}_1 \\ g(x) & \text{if } x \in \Omega_2 \setminus \overline{\Omega}_1\,, \end{cases}$$

then $H_g^2 \geq \tilde{H}_g^1$ in Ω_2.

Proof. By definition of e-solution $H_g^2 \geq g$ in Ω_2, so H_g^2 is also supersolution of (31) in Ω_1. Therefore $H_g^2 \geq H_g^1$ in Ω_1 because H_g^1 is the smallest supersolution in Ω_1, and this completes the proof.

□

THEOREM 7. *Assume that the hypotheses of Theorem 3 $i)$ hold with $\underline{u}$ continuous and Ω bounded. Let $\{\Omega_n\}$ be a sequence of open subsets of Ω, such that $\Omega_n \subseteq \Omega_{n+1}$ and $\bigcup_n \Omega_n = \Omega$. Let u_n be the e-solution of the problem*

$$\begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \Omega_n\,, \\ u = \underline{u} & \text{on } \partial\Omega_n\,. \end{cases} \tag{32}$$

If we extend $u_n := \underline{u}$ in $\Omega \setminus \Omega_n$, then $u_n(x) \nearrow H_g(x)$ for all $x \in \Omega$, where H_g is the e-solution of (16).

Proof. Note that for all n there exists an $\epsilon_n > 0$ such that $\Omega_{\epsilon_n} = \{x \in \Omega : \text{dist}\,(x, \partial\Omega) \geq \epsilon_n\} \subseteq \Omega_n$. Consider the e-solution u_{ϵ_n} of problem

$$\begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \Omega_{\epsilon_n}\,, \\ u = \underline{u} & \text{on } \partial\Omega_{\epsilon_n}\,. \end{cases}$$

If we set $u_{\epsilon_n} \equiv \underline{u}$ in $\Omega \setminus \Omega_{\epsilon_n}$, by Theorem 6 we get $u_{\epsilon_n} \to H_g$ in Ω, as remarked at the beginning of this subsection. Finally by Lemma 3 we have $H_g \geq u_n \geq u_{\epsilon_n}$ in Ω, and so $u_n \to H_g$ in Ω.

□

REMARK 6. If $\partial\Omega$ is not smooth and F is uniformly elliptic Theorem 7 can be used as an approximation result by choosing Ω_n with smooth boundary. In fact, under some structural assumptions, the solution u_n of (32) turns out to be continuous by a barrier argument (see, e.g., [5]), and then it is twice differentiable almost everywhere by a result in [43], see also [17]. If, in addition, F is smooth and $F(x,\cdot,\cdot,\cdot)$ is convex, or the PDE (10) is a HJB equation (3) where the linear operators $\mathcal{L}^\alpha$ have smooth coefficients, then u_n is of class C^2, see [21, 24, 31, 17] and the references therein. The Lipschitz continuity of u_n holds also if F is not uniformly elliptic but it is coercive in the p variables.

2.4. Continuity at the boundary

In this section we study the behavior of the e-solution at boundary points and characterize the points where the boundary data are attained continuously by means of barriers.

PROPOSITION 4. *Assume that hypothesis $i)$ (respectively $ii)$) of Theorem 2 holds. Then the e-solution H_g of (16) takes up the boundary data g continuously at $x_0 \in \partial\Omega$, i.e.* $\lim_{x\to x_0} H_g(x) = g(x_0)$*, if and only if there is an upper (respectively lower) barrier at x_0 (see Definition 3).*

Proof. The necessity is obvious because Theorem 2 $i)$ implies that $H_g \in \mathcal{Z}$, so H_g is an upper barrier at x if it attains continuously the data at x.

Now we assume W is an upper barrier at x. Then $W \geq H_g$, because $W \in \mathcal{Z}$ and H_g is the minimal element of $\mathcal{Z}$. Therefore

$$g(x) \leq H_g(x) \leq \liminf_{y\to x} H_g(y) \leq \limsup_{y\to x} H_g(y) \leq \lim_{y\to x} W(y) = g(x)\,,$$

so $\lim_{y\to x} H_g(y) = g(x) = H_g(x)$.

□

In the classical theory of linear elliptic equations, local barriers suffice to characterize boundary continuity of weak solutions. Similar results can be proved in our fully nonlinear context. Here we limit ourselves to a simple result on the Dirichlet problem with homogeneous boundary data for the Isaacs equation

$$\begin{cases} \sup_\alpha \inf_\beta \{-a_{ij}^{\alpha,\beta} u_{x_i x_j} + b_i^{\alpha,\beta} u_{x_i} + c^{\alpha,\beta} u - f^{\alpha,\beta}\} = 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{33}$$

DEFINITION 6. *We say that $W \in BLSC(\overline{B(x_0,r)\cap\Omega})$ with $r > 0$ is an* upper local barrier *for problem (33) at $x_0 \in \partial\Omega$ if*

$i)$ *$W \geq 0$ is a supersolution of the PDE in (33) in $B(x_0,r)\cap\Omega$,*

$ii)$ *$W(x_0) = 0$, $W(x) \geq \mu > 0$ for all $|x - x_0| = r$,*

$iii)$ *W is continuous at x_0.*

PROPOSITION 5. *Assume the Comparison Principle holds for (33), $f^{\alpha,\beta} \geq 0$ for all α, β, and let H_g be the e-solution of problem (33). Then H_g takes up the boundary data continuously at $x_0 \in \partial\Omega$ if and only if there exists an upper local barrier W at x_0.*

Proof. We recall that H_g exists because the function $u \equiv 0$ is a lower barrier for all points $x \in \partial\Omega$ by the fact that $f^{\alpha,\beta} \geq 0$, and so we can apply Theorem 2. Consider a supersolution w of (33). We claim that the function V defined by

$$V(x) = \begin{cases} \rho W(x) \wedge w(x) & \text{if } x \in \overline{B(x_0, r) \cap \Omega}\,, \\ w(x) & \text{if } x \in \Omega \setminus \overline{B}(x_0, r)\,, \end{cases}$$

is an upper barrier at x_0 for $\rho > 0$ large enough. It is easy to check that ρW is a supersolution of (33) in $B(x_0, r) \cap \Omega$, so V is a supersolution in $B(x_0, r) \cap \Omega$ (by Proposition 1) and in $\Omega \setminus \overline{B}(x_0, r)$. Since w is bounded, by property $ii)$ in Definition 6, we can fix ρ and $\epsilon > 0$ such that $V(x) = w(x)$ for all $x \in \Omega$ satisfying $r - \epsilon < |x - x_0| \leq r$. Then V is supersolution even on $\partial B(x_0, r) \cap \Omega$. Moreover it is obvious that $V \geq 0$ on $\partial\Omega$ and $V(x_0) = 0$. We have proved that V is supersolution of (33) in Ω.

It remains to prove that $\lim_{x \to x_0} V(x) = 0$. Since the constant 0 is a subsolution of (33) and w is a supersolution, we have $w \geq 0$. Then we reach the conclusion by $ii)$ and $iii)$ of Definition 6.

□

EXAMPLE 3. We construct an upper local barrier for (33) under the assumptions of Proposition 5 and supposing in addition

$$\partial\Omega \text{ is } C^2 \text{ in a neighbourhood of } x_0 \in \partial\Omega\,,$$

there exists an α^* such that for all β either

$$a_{ij}^{\alpha^*,\beta}(x_0) n_i(x_0) n_j(x_0) \geq c > 0 \tag{34}$$

or

$$-a_{ij}^{\alpha^*,\beta}(x_0) d_{x_i x_j}(x_0) + b_i^{\alpha^*,\beta}(x_0) n_i(x_0) \geq c > 0 \tag{35}$$

where n denotes the exterior normal to Ω and d is the *signed distance* from $\partial\Omega$

$$d(x) = \begin{cases} \operatorname{dist}(x, \partial\Omega) & \text{if } x \in \Omega\,, \\ -\operatorname{dist}(x, \partial\Omega) & \text{if } x \in \mathbb{R}^N \setminus \Omega\,. \end{cases}$$

Assumptions (34) and (35) are the natural counterpart for Isaacs equation in (33) of the conditions for boundary regularity of solutions to linear equations in Chapt. 1 of [37]. We claim that

$$W(x) = 1 - e^{-\delta(d(x) + \lambda|x - x_0|^2)}$$

is an upper local barrier at x_0 for a suitable choice of $\delta, \lambda > 0$. Indeed it is easy to compute

$$\begin{aligned} &-a_{ij}^{\alpha,\beta}(x_0) W_{x_i x_j}(x_0) + b_i^{\alpha,\beta}(x_0) W_{x_i}(x_0) + c^{\alpha,\beta}(x_0) W(x_0) - f^{\alpha,\beta}(x_0) = \\ &\quad -\delta a_{ij}^{\alpha,\beta}(x_0) d_{x_i x_j}(x_0) + \delta^2 a_{ij}^{\alpha,\beta}(x_0) d_{x_i}(x_0) d_{x_j}(x_0) + \delta b_i^{\alpha,\beta}(x_0) d_{x_i}(x_0) \\ &\quad -2\delta\lambda \operatorname{Tr}[a^{\alpha,\beta}(x_0)] - f^{\alpha,\beta}(x_0)\,. \end{aligned}$$

Next we choose α^* as above and assume first (34). In this case, since the coefficients are bounded and continuous and d is C^2, we can make W a supersolution of the PDE in (33) in a neighborhood of x_0 by taking δ large enough. If, instead, (35) holds, we choose first λ small and then δ large to get the same conclusion.

3. Applications to Optimal Control

3.1. A deterministic minimum-time problem

Our first example of application of the previous theory is the time-optimal control of nonlinear deterministic systems with a closed and nonempty target $\Gamma \subset \mathbb{R}^N$. For this minimum-time problem we prove that the lower semicontinuous envelope of the value function is the e-solution of the associated Dirichlet problem for the Bellman equation. This result can be also found in [7] and [4], but we give here a different and simpler proof. Consider the system

$$\left\{ \begin{array}{l} y\prime(t) = f(y(t), a(t)) \quad t > 0\,, \\ y(0) = x\,, \end{array} \right. \tag{36}$$

where $a \in \mathcal{A} := \{a : [0, \infty) \to A \text{ measurable}\}$ is the set of admissible controls, with

$$\begin{array}{l} A \text{ a compact space, } f : \mathbb{R}^N \times A \to \mathbb{R}^N \text{ continuous} \\ \exists L > 0 \text{ such that } (f(x, a) - f(y, a)) \cdot (x - y) \leq L|x - y|^2\,, \end{array} \tag{37}$$

for all $x, y \in \mathbb{R}^N$, $a \in A$. Under these assumptions, for any $a \in \mathcal{A}$ there exists a unique trajectory of the system (36) defined for all t, that we denote $y_x(t, a)$ or $y_x(t)$. We also define the minimum time for the system to reach the target using the control $a \in \mathcal{A}$:

$$t_x(a) := \left\{ \begin{array}{ll} \inf\{t \geq 0 : y_x(t, a) \in \Gamma\}\,, & \text{if } \{t \geq 0 : y_x(t, a) \in \Gamma\} \neq \emptyset\,, \\ +\infty & \text{otherwise}\,. \end{array} \right.$$

The *value function* for this problem, named *minimum time* function, is

$$T(x) = \inf_{a \in \mathcal{A}} t_x(a), \quad x \in \mathbb{R}^N\,.$$

Consider now the Kruzkov transformation of the minimum time

$$v(x) := \left\{ \begin{array}{ll} 1 - e^{-T(x)}\,, & \text{if } T(x) < \infty\,, \\ 1\,, & \text{otherwise}\,. \end{array} \right.$$

The new unknown v is itself the value function of a time-optimal control problem with a discount factor, and from its knowledge one recovers immediately the minimum time function T. We remark that in general v has no continuity properties without further assumptions; however, it is lower semicontinuous if $f(x, A)$ is a convex set for all x, so in such a case $v = v_*$ (see, e.g., [7, 4]).

The Dirichlet problem associated to v by the Dynamic Programming method is

$$\left\{ \begin{array}{ll} v + H(x, Dv) = 0\,, & \text{in } \mathbb{R}^N \setminus \Gamma\,, \\ v = 0\,, & \text{in } \partial\Gamma\,, \end{array} \right. \tag{38}$$

where

$$H(x, p) := \max_{a \in A}\{-f(x, a) \cdot p\} - 1\,.$$

A Comparison Principle for this problem can be found, for instance, in [4].

THEOREM 8. *Assume (37). Then v_* is the e-solution and the minimal supersolution of (38).*

Proof. Note that by (37) and the fact that $w \equiv 0$ is a subsolution of (38), the hypotheses of Theorem 3 are satisfied, so the e-solution exists and it is a supersolution. It is well known that v_* is a supersolution of $v + H(x, Dv) = 0$ in $\mathbb{R}^N \setminus \Gamma$, see, e.g., [28, 8, 4]; moreover $v_* \geq 0$ on $\partial\Gamma$, so v_* is a supersolution of (38). In order to prove that v_* is the lower e-solution we construct a sequence of subsolutions of (38) converging to v_*.

Fix $\epsilon > 0$, and consider the set

$$\Gamma_\epsilon := \{x \in \mathbb{R}^N : \operatorname{dist}(x, \partial\Gamma) \leq \epsilon\},$$

let T_ϵ be the minimum time function for the problem with target Γ_ϵ, and v_ϵ its Kruzkov transformation. By standard results [28, 8, 4] v_ϵ is a non-continuous viscosity solution of

$$\begin{cases} v + H(x, Dv) = 0\,, & \text{in } \mathbb{R}^N \setminus \Gamma_\epsilon\,, \\ v = 0 \text{ or } v + H(x, Dv) = 0\,, & \text{in } \partial\Gamma_\epsilon\,. \end{cases}$$

With the same argument we used in Theorem 6, we can see that v_ϵ^* is a subsolution of (38). We define

$$u(x) := \sup_\epsilon v_\epsilon^*(x)$$

and will prove that $u = v_*$.

By the Comparison Principle $v_\epsilon^* \leq v_*$ for all $\epsilon > 0$, then $u(x) \leq v_*(x)$. To prove the opposite inequality we observe it is obvious in Γ and assume by contradiction there exists a point $\hat{x} \notin \Gamma$ such that:

$$\sup_\epsilon v_\epsilon(\hat{x}) \leq \sup_\epsilon v_\epsilon^*(\hat{x}) < v_*(\hat{x})\,. \tag{39}$$

Consider first the case $v_*(\hat{x}) < 1$, that is, $T_*(\hat{x}) < +\infty$. Then there exists $\delta > 0$ such that

$$T_\epsilon^*(\hat{x}) < T_*(\hat{x}) - \delta < +\infty, \text{ for all } \epsilon > 0\,. \tag{40}$$

By definition of minimum time, for all ϵ there is a control a_ϵ such that

$$t_{\hat{x}}^\epsilon(a_\epsilon) \leq T_\epsilon(\hat{x}) + \frac{\delta}{2} < +\infty\,. \tag{41}$$

Let $z_\epsilon \in \Gamma_\epsilon$ be the point reached at time $t_{\hat{x}}^\epsilon(a_\epsilon)$ by the trajectory starting from $\hat{x}$, using control a_ϵ. By standard estimates on the trajectories, we have for all ϵ

$$|z_\epsilon| = |y_{\hat{x}}(t_{\hat{x}}^\epsilon(a_\epsilon))| \leq \left(|\hat{x}| + \sqrt{2MT(\hat{x})}\right) e^{MT(\hat{x})}\,,$$

where $M := L + \sup\{|f(0, a)| : a \in A\}$. So, for some $R > 0$, $z_\epsilon \in \overline{B}(0, R)$ for all ϵ. Then we can find subsequences such that

$$z_{\epsilon_n} \to z \in \partial\Gamma, \quad t_n := t_{\hat{x}}^{\epsilon_n}(a_{\epsilon_n}) \to \bar{t}, \text{ as } n \to \infty\,. \tag{42}$$

From this, (40) and (41) we get

$$\bar{t} < T_*(\hat{x}) - \frac{\delta}{2}\,. \tag{43}$$

Let $\overline{y}_{\epsilon_n}$ be the solution of the system

$$\begin{cases} y\prime = f(y, a_{\epsilon_n}) & t < t_n \,, \\ y(t_{\hat{x}}(a_{\epsilon_n})) = z \,, \end{cases}$$

that is, the trajectory moving backward from z using control a_{ϵ_n}, and set $x_n := \overline{y}_{\epsilon_n}(0)$. In order to prove that $x_n \to \hat{x}$ we consider the solution y_{ϵ_n} of

$$\begin{cases} y\prime = f(y, a_{\epsilon_n}) & t < t_n \,, \\ y(t_n) = z_{\epsilon_n} \,, \end{cases}$$

that is, the trajectory moving backward from z_{ϵ_n} and using control a_{ϵ_n}. Note that $y_{\epsilon_n}(0) = \hat{x}$. By differentiating $|y_{\epsilon_n} - \overline{y}_{\epsilon_n}|^2$, using (37) and then integrating we get, for all $t < t_n$,

$$|y_{\epsilon_n}(t) - \overline{y}_{\epsilon_n}(t)|^2 \leq |z_{\epsilon_n} - z|^2 + \int_t^{t_n} 2L|y_{\epsilon_n}(s) - \overline{y}_{\epsilon_n}(s)|^2 ds \,.$$

Then by Gronwall's lemma, for all $t < t_n$,

$$|y_{\epsilon_n}(t) - \overline{y}_{\epsilon_n}(t)| \leq |z_{\epsilon_n} - z| e^{L(t_n - t)} \,,$$

which gives, for $t = 0$,

$$|\hat{x} - x_n| \leq |z_{\epsilon_n} - z| e^{L t_n} \,.$$

By letting $n \to \infty$, we get that $x_n \to \hat{x}$.

By definition of minimum time $T(x_n) \leq t_n$, so letting $n \to \infty$ we obtain $T_*(\hat{x}) \leq \overline{t}$, which gives the desired contradiction with (43).

The remaining case is $v_*(\hat{x}) = 1$. By (39) $T_\epsilon^*(\hat{x}) \leq K < +\infty$ for all ϵ. By using the previous argument we get (42) with $\overline{t} < +\infty$ and $T_*(\hat{x}) \leq \overline{t}$. This is a contradiction with $T_*(\hat{x}) = +\infty$ and completes the proof.

□

3.2. Maximizing the mean escape time of a degenerate diffusion process

In this subsection we study a stochastic control problem having as a special case the problem of maximizing the expected discounted time spent by a controlled diffusion process in a given open set $\Omega \subseteq \mathbb{R}^N$. A number of engineering applications of this problem are listed in [19], where, however, a different cost criterion is proposed and a nondegeneracy assumption is made on the diffusion matrix. We consider a probability space $(\Omega', \mathcal{F}, P)$ with a right-continuous increasing filtration of complete sub-σ fields $\{\mathcal{F}_t\}$, a Brownian motion B_t in $\mathbb{R}^M$ $\mathcal{F}_t$-adapted, a compact set A, and call $\mathcal{A}$ the set of progressively measurable processes α_t taking values in A. We are given bounded and continuous maps σ from $\mathbb{R}^N \times A$ into the set of $N \times M$ matrices and $b : \mathbb{R}^N \times A \to \mathbb{R}^N$ satisfying (14), (15) and consider the controlled stochastic differential equation

$$(SDE) \begin{cases} dX_t = \sigma^{\alpha_t}(X_t) dB_t - b^{\alpha_t}(X_t) dt \,, & t > 0 \,, \\ X_0 = x \,. \end{cases}$$

For any $\alpha_. \in \mathcal{A}$ (SDE) has a pathwise unique solution X_t which is $\mathcal{F}_t$-progressively measurable and has continuous sample paths. We are given also two bounded and uniformly continuous maps $f, c : \mathbb{R}^N \times A \to \mathbb{R}$, $c^\alpha(x) \geq c_0 > 0$ for all x, α, and consider the payoff functional

$$J(x, \alpha_.) := E\left(\int_0^{t_x(\alpha_.)} f^{\alpha_t}(X_t) e^{-\int_0^t c^{\alpha_s}(X_s)\,ds} dt\right),$$

where E denotes the expectation and

$$t_x(\alpha_.) := \inf\{t \geq 0 : X_t \notin \Omega\},$$

where, as usual, $t_x(\alpha_.) = +\infty$ if $X_t \in \Omega$ for all $t \geq 0$. We want to maximize this payoff, so we consider the value function

$$v(x) := \sup_{\alpha_. \in \mathcal{A}} J(x, \alpha_.).$$

Note that for $f = c \equiv 1$ the problem becomes the maximization of the mean discounted time $E(1 - e^{-t_x(\alpha_.)})$ spent by the trajectories of (SDE) in Ω.

The Hamilton-Jacobi-Bellman operator and the Dirichlet problem associated to v by the Dynamic Programming method are

$$F(x, u, Du, D^2u) := \min_{\alpha \in A}\{-a^\alpha_{ij}(x) u_{x_i x_j} + b^\alpha(x) \cdot Du + c^\alpha(x) u - f^\alpha(x)\},$$

where the matrix (a_{ij}) is $\frac{1}{2}\sigma\sigma^T$, and

$$\text{(44)} \qquad \begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

see, for instance, [40, 35, 36, 22, 32] and the references therein. The proof that the value function satisfies the Hamilton-Jacobi-Bellman PDE is based on the Dynamic Programming Principle

$$\text{(45)} \qquad v(x) = \sup_{\alpha_. \in \mathcal{A}} E\left(\int_0^{\theta \wedge t_x} f^{\alpha_t}(X_t) e^{-\int_0^t c^{\alpha_s}(X_s)\,ds} dt + v(X_{\theta \wedge t_x}) e^{-\int_0^{\theta \wedge t_x} c^{\alpha_s}(X_s)\,ds}\right),$$

where $t_x = t_x(\alpha_.)$, for all $x \in \overline{\Omega}$ and all $\mathcal{F}_t$-measurable stopping times θ. Although the DPP (45) is generally believed to be true under the current assumptions (see, e.g., [35]), we were able to find its proof in the literature only under some additional conditions, such as the convexity of the set

$$\{(a^\alpha(x), b^\alpha(x), f^\alpha(x), c^\alpha(x)) : \alpha \in A\}$$

for all $x \in \overline{\Omega}$, see [20] (this is true, in particular, when relaxed controls are used), or the independence of the variance of the noise from the control [15], i.e., $\sigma^\alpha(x) = \sigma(x)$ for all x, or the continuity of v [35]. As recalled in Subsection 1.1 a Comparison Principle for (44) can be found in [29], see also [18] and the references therein.

In order to prove that v is the e-solution of (44), we approximate Ω with a nested family of open sets with the properties

$$\text{(46)} \qquad \Theta_\epsilon \subseteq \Omega,\ \epsilon \in]0, 1],\ \Theta_\epsilon \supseteq \overline{\Theta}_\delta \text{ for } \epsilon < \delta,\ \bigcup_\epsilon \Theta_\epsilon = \Omega.$$

For each $\epsilon > 0$ we call v_ϵ the value function of the same control problem with t_x replaced with

$$t^\epsilon_x(\alpha_.) := \inf\{t \geq 0 : X_t \notin \Theta_\epsilon\}$$

in the definition of the payoff J. In the next theorem we assume that each v_ϵ satisfies the DPP (45) with t_x replaced with t_x^ϵ.

Finally, we make the additional assumption

$$f^\alpha(x) \geq 0 \text{ for all } x \in \Omega,\ \alpha \in A\,. \tag{47}$$

which ensures that $\underline{u} \equiv 0$ is a subsolution of (44). The main result of this subsection is the following.

THEOREM 9. *Under the previous assumptions the value function v is the e-solution and the minimal supersolution of (44), and*

$$v = \sup_{0<\epsilon\leq 1} v_\epsilon = \lim_{\epsilon\searrow 0} v_\epsilon\,.$$

Proof. Note that v_ϵ is nondecreasing as $\epsilon \searrow 0$, so $\lim_{\epsilon\searrow 0} v_\epsilon$ exists and equals the sup. By Theorem 3 with $g \equiv 0$, $\underline{u} \equiv 0$, there exists the e-solution H_0 of (44). We consider the functions u_ϵ defined by (20) and claim that

$$u_{2\epsilon} \leq (v_\epsilon)_* \leq v_\epsilon^* \leq H_0\,.$$

Then

$$H_0 = \sup_{0<\epsilon\leq 1} v_\epsilon\,, \tag{48}$$

because $H_0 = \sup_\epsilon u_{2\epsilon}$ by Remark 1. We prove the claim in three steps.

Step 1. By standard methods [35, 9], the Dynamic Programming Principle for v_ϵ implies that v_ϵ is a non-continuous viscosity solution of the Hamilton-Jacobi-Bellman equation $F = 0$ in Θ_ϵ and v_ϵ^* is a viscosity subsolution of the boundary condition

$$u = 0 \text{ or } F(x, u, Du, D^2u) = 0 \text{ on } \partial\Theta_\epsilon\,, \tag{49}$$

as defined in Subsection 1.3.

Step 2. Since $(v_\epsilon)_*$ is a supersolution of the PDE $F = 0$ in Θ_ϵ and $(v_\epsilon)_* \geq 0$ on $\partial\Theta_\epsilon$, the Comparison Principle implies $(v_\epsilon)_* \geq w$ for any subsolution w of (44) such that $w = 0$ on $\partial\Theta_\epsilon$. Since $\partial\Theta_\epsilon \subseteq \Omega \setminus \overline{\Theta}_{2\epsilon}$ by (46), we obtain $u_{2\epsilon} \leq v_{\epsilon *}$ by the definition (20) of $u_{2\epsilon}$.

Step 3. We claim that v_ϵ^* is a subsolution of (44). In fact we noted before that it is a subsolution of the PDE in Θ_ϵ, and this is true also in $\Omega \setminus \overline{\Theta}_\epsilon$ where $v_\epsilon^* \equiv 0$ by (47), whereas the boundary condition is trivial. It remains to check the PDE at all points of $\partial\Theta_\epsilon$. Given $\hat{x} \in \partial\Theta_\epsilon$, we must prove that for all $\phi \in C^2(\Omega)$ such that $v_\epsilon^* - \phi$ attains a local maximum at $\hat{x}$, we have

$$F(\hat{x}, v_\epsilon^*(\hat{x}), D\phi(\hat{x}), D^2\phi(\hat{x})) \leq 0\,. \tag{50}$$

$1st$ Case: $v_\epsilon^*(\hat{x}) > 0$. Since v_ϵ^* satisfies (49), for all $\phi \in C^2(\overline{\Theta}_\epsilon)$ such that $v_\epsilon^* - \phi$ attains a local maximum at $\hat{x}$ (50) holds. Then the same inequality holds for all $\phi \in C^2(\Omega)$ as well.

$2nd$ Case: $v_\epsilon^*(\hat{x}) = 0$. Since $v_\epsilon^* - \phi$ attains a local maximum at $\hat{x}$, for all x near $\hat{x}$ we have

$$v_\epsilon^*(x) - v_\epsilon^*(\hat{x}) \leq \phi(x) - \phi(\hat{x})\,.$$

By Taylor's formula for ϕ at $\hat{x}$ and the fact that $v_\epsilon^*(x) \geq 0$, we get

$$D\phi(\hat{x}) \cdot (x - \hat{x}) \geq o(|x - \hat{x}|)\,,$$

and this implies $D\phi(\hat{x}) = 0$. Then Taylor's formula for ϕ gives also

$$(x - \hat{x}) \cdot D^2\phi(\hat{x})(x - \hat{x}) \geq o(|x - \hat{x}|^2)\,,$$

and this implies $D^2\phi(\hat{x}) \geq 0$, as it is easy to check. Then

$$F(\hat{x}, v_\epsilon^*(\hat{x}), D\phi(\hat{x}), D^2\phi(\hat{x})) = F(\hat{x}, 0, 0, D^2\phi(\hat{x})) \leq 0$$

because $a^\alpha \geq 0$ and $f^\alpha \geq 0$ for all x and α. This completes the proof that v_ϵ^* is a subsolution of (44). Now the Comparison Principle yields $v_\epsilon^* \leq H_0$, since H_0 is a supersolution of (44).

It remains to prove that $v = \sup_{0<\epsilon\leq 1} v_\epsilon$. To this purpose we take a sequence $\epsilon_n \searrow 0$ and define

$$J_n(x, \alpha_.) := E\left(\int_0^{t_x^{\epsilon_n}(\alpha_.)} f^{\alpha_t}(X_t) e^{-\int_0^t c^{\alpha_s}(X_s)\,ds} dt\right).$$

We claim that

$$\lim_n J_n(x, \alpha_.) = \sup_n J_n(x, \alpha_.) = J(x, \alpha_.) \text{ for all } \alpha_. \text{ and } x\,.$$

The monotonicity of $t_x^{\epsilon_n}$ follows from (46) and it implies the monotonicity of J_n by (47). Let

$$\tau := \sup_n t_x^{\epsilon_n}(\alpha_.) \leq t_x(\alpha_.)\,,$$

and note that $t_x(\alpha_.) = +\infty$ if $\tau = +\infty$. In the case $\tau < +\infty$, $X_{t_x^{\epsilon_n}} \in \partial\Theta_{\epsilon_n}$ implies $X_\tau \in \partial\Omega$, so $\tau = t_x(\alpha_.)$ again. This and (47) yield the claim by the Lebesgue monotone convergence theorem. Then

$$v(x) = \sup_{\alpha_.} \sup_n J_n(x, \alpha_.) = \sup_n \sup_{\alpha_.} J_n(x, \alpha_.) = \sup_n v_{\epsilon_n} = \sup_\epsilon v_\epsilon\,,$$

so (48) gives $v = H_0$ and completes the proof.

□

REMARK 7. From Theorem 9 it is easy to get a *Verification theorem* by taking the supersolutions of (44) as verification functions. We consider a presynthesis $\alpha^{(x)}$, that is, a map $\alpha^{(\cdot)} : \Omega \to \mathcal{A}$, and say it is optimal at x_o if $J(x_o, \alpha^{(x_o)}) = v(x_o)$. Then Theorem 9 gives immediately the following sufficient condition of optimality: *if there exists a verification function W such that $W(x_o) \leq J(x_o, \alpha^{(x_o)})$, then $\alpha^{(\cdot)}$ is optimal at x_o*; moreover, a characterization of global optimality is the following: *$\alpha^{(\cdot)}$ is optimal in Ω if and only if $J(\cdot, \alpha^{(\cdot)})$ is a verification function.*

REMARK 8. We can combine Theorem 9 with the results of Subsection 2.2 to approximate the value function v with smooth value functions. Consider a Brownian motion $\tilde{B}_t$ in $\mathbb{R}^N$ $\mathcal{F}_t$-adapted and replace the stochastic differential equation in (SDE) with

$$dX_t = \sigma^{\alpha_t}(X_t)dB_t - b^{\alpha_t}(X_t)dt + \sqrt{2h}\,d\tilde{B}_t, \quad t > 0\,,$$

for $h > 0$. For a family of nested open sets with the properties (46) consider the value function v_h^ϵ of the problem of maximizing the payoff functional J with t_x replaced with t_x^ϵ. Assume for simplicity that $a^\alpha, b^\alpha, c^\alpha, f^\alpha$ are smooth (otherwise we can approximate them by mollification). Then v_h^ϵ is the classical solution of (30), where F is the HJB operator of this subsection and $\underline{u} \equiv 0$, by the results in [21, 24, 36, 31], and it is possible to synthesize an optimal Markov control policy for the problem with $\epsilon, h > 0$ by standard methods (see, e.g., [22]). By Theorem 6 v_h^ϵ converges to v as $\epsilon, h \searrow 0$ with h linked to ϵ.

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