Rend. Sem. Mat. Univ. Pol. Torino
Vol. 56, 4 (1998)

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EXTERNAL STABILIZATION OF DISCONTINUOS SYSTEMS AND NONSMOOTH CONTROL LYAPUNOV-LIKE FUNCTIONS

Abstract.
The main result of this note is an external stabilizability theorem for discontinuous systems affine in the control (with solutions intended in the Filippov's sense). In order to get it we first prove a sufficient condition for external stability which makes use of nonsmooth Lyapunov-like functions.

1. Introduction

In this note we deal with discontinuous time-dependent systems affine in the control:

$$\dot{x} = f(t,x) + G(t,x)u = f(t,x) + \sum_{i=1}^{m} u_i g_i(t,x) \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f \in L^\infty_{loc}(\mathbb{R}^{n+1};\mathbb{R}^n)$, for all $i \in \{1,\ldots,m\}$, $g_i \in C(\mathbb{R}^{n+1};\mathbb{R}^n)$ and G is the matrix whose columns are $g_1,\ldots,g_m$.

Admissible inputs are $u \in L^\infty_{loc}(\mathbb{R};\mathbb{R}^m)$.

Solutions of system (1) (as well as solutions of all the systems considered in the following) are intended in the Filippov's sense. In other words, for each admissible input $u(t)$, (1) is replaced by the differential inclusion

$$\dot{x} \in K(f+Gu)(t,x) = \bigcap_{\delta>0}\bigcap_{\mu(N)=0} \overline{\text{co}}\,\{(f+Gu)(t, B(x,\delta)\backslash N)\}\,,$$

where $B(x,\delta)$ is the ball of center x and radius δ, $\overline{\text{co}}$ denotes the convex closure and μ is the usual Lebesgue measure in $\mathbb{R}^n$.

For the general theory of Filippov's solutions we refer to [6]. We denote by $S_{t_0,x_0,u}$ the set of solutions $\varphi(\cdot)$ of system (1) with the initial condition $\varphi(t_0) = x_0$ and the function $u : \mathbb{R} \to \mathbb{R}^m$ as input.

We are interested in the external behaviour of system (1), in particular in its uniform bounded input bounded state (UBIBS) stability.

Roughly speaking a system is said to be UBIBS stable if its trajectories are bounded whenever the input is bounded. More precisely we have the following definition.

DEFINITION 1. *System (1) is said to be* UBIBS stable *if for each $R > 0$ there exists $S > 0$ such that for each $(t_0, x_0) \in \mathbb{R}^{n+1}$, $t_0 \geq 0$, and each input $u \in L^\infty_{loc}(\mathbb{R};\mathbb{R}^m)$ one has*

$$\|x_0\| < R, \quad \|u\|_\infty < R \Rightarrow \forall\varphi(\cdot) \in S_{t_0,x_0,u} \quad \|\varphi(t)\| < S \quad \forall t \geq t_0\,.$$

We associate to system (1) the unforced system

$$\dot{x} = f(t, x) \tag{2}$$

obtained from (1) by setting $u = 0$. We denote $S_{t_0,x_0,0} = S_{t_0,x_0}$.

DEFINITION 2. *System (2) is said to be* uniformly Lagrange stable *if for each $R > 0$ there exists $S > 0$ such that for each $(t_0, x_0) \in \mathbb{R}^{n+1}$, $t_0 \geq 0$, one has*

$$\|x_0\| < R \Rightarrow \forall \varphi(\cdot) \in S_{t_0,x_0} \quad \|\varphi(t)\| < S \quad \forall t \geq t_0 .$$

If system (1) is UBIBS stable, then system (2) is uniformly Lagrange stable, but the converse is not true in general. In Section 3 we prove that, if not only system (2) is uniformly Lagrange stable, but some additional conditions on f and G are satisfied, then there exists an externally stabilizing feedback law for system (1), in the sense of the following definition.

DEFINITION 3. *System (1) is said to be* UBIBS stabilizable *if there exists a function $k \in L^{\infty}_{loc}(\mathbb{R}^{n+1}; \mathbb{R}^m)$ such that the closed loop system*

$$\dot{x} = f(t, x) + G(t, x)k(t, x) + G(t, x)v \tag{3}$$

(with v as input) is UBIBS stable.

The same problem has been previously treated in [1, 2, 4, 9]. We give our result (Theorem 2) and discuss the differences with the results obtained in the mentioned papers in Section 3.

In order to achieve Theorem 2 we need a preliminary theorem (Theorem 1 in Section 2). It is a different version of Theorem 1 in [13] and Theorem 6.2 in [4]. It provides a sufficient condition for UBIBS stability of system (1) by means of a nonsmooth control Lyapunov-like function. Finally the proof of the main result is given in Section 4.

2. UBIBS Stability

In this section we give a sufficient condition for UBIBS stability of system (1) by means of a nonsmooth control Lyapunov-like function. (See [11, 12] for control Lyapunov functions).

The following Theorem 1 (and also its proof) is analogous to Theorem 1 in [13] and Theorem 6.2 in [4]. It differs from both for the fact that it involves a control Lyapunov-like function which is not of class C^1, but just locally Lipschitz continuous and regular in the sense of Clarke (see [5], page 39).

DEFINITION 4. *We say that a function $V : \mathbb{R}^{n+1} \to \mathbb{R}$ is* regular *at $(t, x) \in \mathbb{R}^{n+1}$ if*

(i) *for all $v \in \mathbb{R}^n$ there exists the usual right directional derivative $V'_+((t, x), (1, v))$,*

(ii) *for all $v \in \mathbb{R}^n$, $V'_+((t, x), (1, v)) = \limsup_{(s,y)\to(t,x)\, h\downarrow 0} \frac{V(s+h, y+hv)-V(s,y)}{h}$.*

The fact that the control Lyapunov-like function for system (1) is regular allows us to characterize it by means of its set-valued derivative with respect to the system instead of by means of Dini derivatives.

Let us recall the definition of set-valued derivative of a function with respect to a system introduced in [10] and then used (with some modifications) in [3]. Let us denote by $\partial V(t, x)$ Clarke generalized gradient of V at (t, x) (see [5], page 27).

DEFINITION 5. *Let $t > 0$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ be fixed, $V : \mathbb{R}^{n+1} \to \mathbb{R}$. We call* set-valued derivative of V with respect to system (1) *the set*

$$\dot{\overline{V}}^{(1)}(t, x, u) = \{a \in \mathbb{R} : \exists v \in K(f(t, x) + G(t, x)u) \text{ such that } \forall p \in \partial V(t, x)\ p \cdot (1, v) = a\}.$$

Analagously, if $t > 0$, $x \in \mathbb{R}^n$, $u \in L^{\infty}_{loc}(\mathbb{R}, \mathbb{R}^m)$ are fixed, we set

$$\dot{\overline{V}}^{(1)}_{u(\cdot)}(t, x) = \{a \in \mathbb{R} : \exists v \in K(f(t, x) + G(t, x)u(t)) \text{ such that } \forall p \in \partial V(t, x)\ p \cdot (1, v) = a\}$$

and, if $t > 0$ and $x \in \mathbb{R}^n$ are fixed, we define

$$\dot{\overline{V}}^{(2)}(t, x) = \{a \in \mathbb{R} : \exists v \in Kf(t, x) \text{ such that } \forall p \in \partial V(t, x)\ p \cdot (1, v) = a\}.$$

Let us remark that $\dot{\overline{V}}^{(1)}(t, x, u)$ is a closed and bounded interval, possibly empty and

$$\max \dot{\overline{V}}^{(1)}(t, x, u) \leq \max_{v \in K(f(t,x)+G(t,x)u)} \overline{D^+} V((t, x), (1, v)),$$

where $\overline{D^+} V((t, x), (1, v))$ is the Dini derivative of V at (t, x) in the direction of $(1, v)$.

LEMMA 1. *Let $\varphi(\cdot)$ be a solution of the differential inclusion (1) corresponding to the input $u(\cdot)$ and let $V : \mathbb{R}^{n+1} \to \mathbb{R}$ be a locally Lipschitz continuous and regular function. Then $\frac{d}{dt} V(t, \varphi(t))$ exists almost everywhere and $\frac{d}{dt} V(t, \varphi(t)) \in \dot{\overline{V}}^{(1)}_{u(\cdot)}(t, \varphi(t))$ almost everywhere.*

We omit the proof of the previous lemma since it is completely analogous to the proofs of Theorem 2.2 in [10] (which involves a slightly different kind of set-valued derivative with respect to the system) and of Lemma 1 in [3] (which is given for autonomous differential inclusions and V not depending on time).

We can now state the main theorem of this section.

THEOREM 1. *Let $V : \mathbb{R}^{n+1} \to \mathbb{R}$ be such that there exists $L > 0$ such that*

(V0) *there exist two continuous, strictly increasing, positive functions $a, b : \mathbb{R} \to \mathbb{R}$ such that $\lim_{r \to +\infty} a(r) = +\infty$ and for all $t > 0$ and for all x*

$$\|x\| > L \Rightarrow a(\|x\|) \leq V(t, x) \leq b(\|x\|)$$

(V1) *V is locally Lipschitz continuous and regular in $\mathbb{R}^+ \times \{x \in \mathbb{R}^n : \|x\| > L\}$.*

If

(fG) *for all $R > 0$ there exists $\rho > L$ such that for all $x \in \mathbb{R}^n$ and for all $u \in \mathbb{R}^m$ the following holds:*

$$\|x\| > \rho,\ \|u\| < R \Rightarrow \max \dot{\overline{V}}^{(1)}(t, x, u) \leq 0 \text{ for a.e. } t \geq 0$$

then system (1) is UBIBS stable.

Proof. We prove the statement by contradiction, by assuming that there exists $\overline{R}$ such that for all $S > 0$ there exist $\overline{x_0}$ and $\overline{u} : [0, +\infty) \to \mathbb{R}^m$ such that $\|\overline{x_0}\| < \overline{R}$, $\|\overline{u}\|_\infty < \overline{R}$ and there exist $\overline{\varphi}(\cdot) \in S_{t_0, x_0, \overline{u}}$, and $\overline{t} > 0$ such that $\|\overline{\varphi}(\overline{t})\| \geq S$.

Let us choose $\overline{\rho}$ corresponding to $\overline{R}$ as in (fG). Without loss of generality we can suppose that $\overline{\rho} > \overline{R}$. Because of (V0), there exists $S_M > 0$ such that if $\|x\| > S_M$, then $V(t,x) > M = b(\overline{\rho}) \geq \max\{V(t,x),\ \|x\| = \overline{\rho},\ t \geq 0\}$ for all t.

Let us consider $S > \overline{\rho}, S_M$. There exist $t_1, t_2 > 0$ such that $\overline{t} \in [t_1, t_2]$, $\|\overline{\varphi}(t_1)\| = \overline{\rho}$, $\|\overline{\varphi}(t)\| \geq \overline{\rho}$ in $[t_1, t_2]$, $\|\overline{\varphi}(t_2)\| \geq S$. Then

$$V(t_2, \overline{\varphi}(t_2)) > M \geq V(t_1, \overline{\varphi}(t_1)) . \tag{4}$$

On the other hand, by Lemma 1, $\frac{d}{dt}V(t,\overline{\varphi}(t)) \in \dot{\overline{V}}^{(1)}_{\overline{u}(\cdot)}(t,\overline{\varphi}(t))$ a.e. It is clear that $\dot{\overline{V}}^{(1)}_{\overline{u}(\cdot)}(t,\overline{\varphi}(t)) \subseteq \dot{\overline{V}}^{(1)}(t,\overline{\varphi}(t),\overline{u}(t))$. Since $|\overline{u}(t)| < \overline{R}$ a.e. and $\|\varphi(t)\| > \overline{\rho}$ for all $t \in [t_1, t_2]$, by virtue of (fG) we have $\frac{d}{dt}V(t,\overline{\varphi}(t)) \leq 0$ for a.e. $t \in [t_1, t_2]$. By [7] (page 207) we get that $V \circ \overline{\varphi}$ is decreasing in $[t_1, t_2]$, then

$$V(t_2, \overline{\varphi}(t_2)) \leq V(t_1, \overline{\varphi}(t_1))$$

that contradicts (4).

□

REMARK 1. In order to get a sufficient condition for system (2) to be uniformly Lagrange stable, one can state Theorem 1 in the case $u = 0$. In this case the control Lyapunov-like function simply becomes a Lyapunov-like function.

REMARK 2. For sake of simplicity we have given the definition of UBIBS stability and stated Theorem 1 for systems affine in the control. Let us remark that exactly analogous definition, theorem and proof hold for more general systems of the form

$$\dot{x} = f(t,x,u)$$

where $f : \mathbb{R}^{m+n+1} \to \mathbb{R}^n$ is locally bounded and measurable with respect to the variables t and x and continuous with respect to u.

REMARK 3. If system (1) is autonomous it is possible to state a theorem analogous to Theorem 1 for a control Lyapunov-like function V not depending on time.

3. The Main Result

The main result of this note is the following Theorem 2. It essentially recalls Theorem 6.2 in [4] and Theorem 5 in [9], with the difference that the control Lyapunov-like function involved is not smooth.

We don't give a unique condition for system (1) to be externally stabilizable, but some alternative conditions which, combined together, give the external stabilizability of the system. Before stating the theorem we list these conditions. Note that the variable x is not yet quantified. Since its role depend on different situations, it is convenient to specify it later.

- (f1) $\max \dot{\overline{V}}^{(2)}(t,x) \leq 0$;
- (f2) for all $z \in Kf(t,x)$ there exists $\overline{p} \in \partial V(t,x)$ such that $\overline{p} \cdot (1,z) \leq 0$;
- (f3) for all $z \in Kf(t,x)$ and for all $p \in \partial V(t,x)$, $p \cdot (1,z) \leq 0$;

- (G1) for each $i \in \{1,\ldots,m\}$ there exists $c^i_{t,x} \in \mathbb{R}$ such that for all $p \in \partial V(t,x)$, $p \cdot (1, g_i(t,x)) = c^i_{t,x}$;

(G2) for each $i \in \{1, \ldots, m\}$ only one of the following mutually exclusive conditions holds:

- for all $p \in \partial V(t, x)$ $p \cdot (1, g_i(t, x)) > 0$,
- for all $p \in \partial V(t, x)$ $p \cdot (1, g_i(t, x)) < 0$,
- for all $p \in \partial V(t, x)$ $p \cdot (1, g_i(t, x)) = 0$;

(G3) there exists $\bar{\imath} \in \{1, \ldots, m\}$ such that for each $i \in \{1, \ldots, m\} \backslash \{\bar{\imath}\}$ only one of the following mutually exclusive conditions holds:

- for all $p \in \partial V(t, x)$ $p \cdot (1, g_i(t, x)) > 0$,
- for all $p \in \partial V(t, x)$ $p \cdot (1, g_i(t, x)) < 0$,
- for all $p \in \partial V(t, x)$ $p \cdot (1, g_i(t, x)) = 0$;

Let us remark that (f3) $\Rightarrow$ (f2) $\Rightarrow$ (f1) and (G1) $\Rightarrow$ (G2) $\Rightarrow$ (G3).

THEOREM 2. *Let $V : \mathbb{R}^{n+1} \to \mathbb{R}$ be such that there exists $L > 0$ such that* (V0) *and* (V1) *hold.*

If for all $x \in \mathbb{R}^n$ with $\|x\| > L$ one of the following couples of conditions holds for a.e. $t \geq 0$:

(i) (f1) *and* (G1), (ii) (f2) *and* (G2), (iii) (f3) *and* (G3),

then system (1) is UBIBS stabilizable.

Let us make some remarks.

If for all $x \in \mathbb{R}^n$ with $\|x\| > L$ assumption (f1) (or (f2) or (f3)) holds for a.e. $t \geq 0$, then, by Theorem 1 in Section 2, system (2) is uniformly Lagrange stable. Actually in [4] the authors introduce the concept of robust uniform Lagrange stability and prove that it is equivalent to the existence of a locally Lipschitz continuous Lyapunov-like function. Then assumption (f1) (or (f2) or (f3)) implies more than uniform Lagrange stability of system (2). In [9], the author has also proved that, under mild additional assumptions on f, robust Lagrange stability implies the existence of a C^∞ Lyapunov-like function, but the proof of this result is not actually constructive. Then we could still have to deal with nonsmooth Lyapunov-like functions even if we know that there exist smooth ones.

Moreover Theorem 2 can be restated for autonomous systems with the function V not depending on time. In this case the feedback law is autonomous and it is possible to deal with a situation in which the results in [9] don't help.

Finally let us remark that if f is locally Lipschitz continuous, then, by [14] (page 105), the Lagrange stability of system (2) implies the existence of a time-dependent Lyapunov-like function of class C^∞. In this case, in order to get UBIBS stabilizability of system (1), the regularity assumption on G can be weakened to $G \in L^\infty_{loc}(\mathbb{R}^{n+1}; \mathbb{R}^m)$ (as in [2]).

4. Proof of Theorem 2

We first state and prove a lemma.

LEMMA 2. *Let $V : \mathbb{R}^{n+1} \to \mathbb{R}$ be such that there exists $L > 0$ such that* (V0) *and* (V1) *hold. If $(\bar{t}, \bar{x})$, with $\|\bar{x}\| > L$, is such that, for all $p \in \partial V(\bar{t}, \bar{x})$ $p \cdot (1, g_i(\bar{t}, \bar{x})) > 0$, then there exists $\delta_{\bar{x}} > 0$ such that, for all $x \in B(\bar{x}, \delta_{\bar{x}})$, for all $p \in \partial V(\bar{t}, x)$, $p \cdot (1, g_i(\bar{t}, x)) > 0$.*

Analagously if $(\bar{t}, \bar{x})$, with $\|\bar{x}\| > L$, is such that for all $p \in \partial V(\bar{t}, \bar{x})$, $p \cdot (1, g_i(\bar{t}, \bar{x})) < 0$, then there exists $\delta_{\bar{x}} > 0$ such that, for all $x \in B(\bar{x}, \delta_{\bar{x}})$, for all $p \in \partial V(\bar{t}, x)$, $p \cdot (1, g_i(\bar{t}, x)) < 0$.

Proof. Let $\gamma > 0$ be such that $\|\bar{x}\| > L + \gamma$, and let $L_{\bar{x}} > 0$ be the Lipschitz constant of V in the set $\{\bar{t}\} \times B(\bar{x}, \gamma)$. For all $(\bar{t}, x) \in \{\bar{t}\} \times B(\bar{x}, \gamma)$ and for all $p \in \partial V(\bar{t}, x)$ $\|p\| \le L_{\bar{x}}$ (see [5], page 27).

Since g_i is continuous there exist η and M such that $\|(1, g_i(\bar{t}, x))\| \le M$ in $\{\bar{t}\} \times B(\bar{x}, \eta)$.

Let $d = \min\{p \cdot (1, g_i(\bar{t}, \bar{x})),\ p \in \partial V(\bar{t}, \bar{x})\}$. By assumption $d > 0$.

Let us consider $\epsilon < \frac{d}{2(L_{\bar{x}}+M)}$.

By the continuity of g_i, there exists δ_i such that, if $\|x - \bar{x}\| < \delta_i$, then $\|(1, g_i(\bar{t}, x)) - (1, g_i(\bar{t}, \bar{x}))\| < \epsilon$.

By the upper semi-continuity of ∂V (see [5], page 29), there exists $\delta_V > 0$ such that, if $\|x - \bar{x}\| < \delta_V$, then $\partial V(\bar{t}, x) \subseteq \partial V(\bar{t}, \bar{x}) + \epsilon B(0, 1)$, i.e. for all $p \in \partial V(\bar{t}, x)$ there exists $\bar{p} \in \partial V(\bar{t}, \bar{x})$ such that $\|p - \bar{p}\| < \epsilon$.

Let $\delta_{\bar{x}} = \min\{\gamma, \eta, \delta_i, \delta_V\}$, x be such that $\|x - \bar{x}\| < \delta_{\bar{x}}$ and $p \in \partial V(\bar{t}, x)$, $\bar{p} \in \partial V(\bar{t}, \bar{x})$ be such that $\|p - \bar{p}\| < \epsilon$.

It is easy to see that $|p \cdot (1, g_i(\bar{t}, x)) - \bar{p} \cdot (1, g_i(\bar{t}, \bar{x}))| < \frac{d}{2}$, hence $p \cdot (1, g_i(\bar{t}, x)) > \bar{p} \cdot (1, g_i(\bar{t}, \bar{x})) - \frac{d}{2} = \frac{d}{2} > 0$.

The second part of the lemma can be proved in a perfectly analogous way.

□

Proof of Theorem 2. For each $x \in \mathbb{R}^n$, let N_x be the zero-measure subset of $\mathbb{R}^+$ in which no one of the couples of conditions (i), (ii) and (iii) holds. Let $k : \mathbb{R}^{n+1} \to \mathbb{R}^m$, $k(t, x) = (k_1(t, x), \ldots, k_m(t, x))$, be defined by

$$k_i(t, x) = \begin{cases} -\|x\| & \text{if } \forall p \in \partial V(t, x)\ p \cdot (1, g_i(t, x)) > 0 \\ 0 & \text{if } \forall p \in \partial V(t, x)\ p \cdot g_i(t, x) = 0\,, \\ & \text{or (f3) and (G3) hold and } i = \bar{\imath},\ \text{ or } t \in N_x \\ \|x\| & \text{if } \forall p \in \partial V(t, x)\ p \cdot (1, g_i(t, x)) < 0\,. \end{cases}$$

It is clear that $k \in L^{\infty}_{loc}(\mathbb{R}^{n+1}, \mathbb{R}^m)$.

By Theorem 1 it is sufficient to prove that for all $R > 0$ there exists $\rho > L, R$ such that for all $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ the following holds:

$$\|x\| > \rho, \quad \|v\| < R \Rightarrow \max \dot{\bar{V}}^{(3)}(t, x) \le 0 \text{ for all } t \in \mathbb{R}^+ \backslash N_x$$

where $\dot{\bar{V}}^{(3)}(t, x) = \{a \in \mathbb{R} : \exists w \in K(f(t, x) + G(t, x)k(t, x) + G(t, x)v)$ such that $\forall p \in \partial V(t, x)\ p \cdot (1, w) = a\}$.

Let x be fixed and $t \in \mathbb{R}^+ \backslash N_x$. Let $a \in \dot{\bar{V}}^{(3)}(t, x)$, $w \in K(f(t, x) + G(t, x)k(t, x) + G(t, x)v)$ be such that for all $p \in \partial V(t, x)$ $p \cdot w = a$.

By Theorem 1 in [8] we have that
$K(f(t, x) + G(t, x))(k(t, x) + v)(x) \subseteq Kf(t, x) + \sum_{i=1}^m g_i(t, x) K(k_i(t, x) + v_i)$, then there exists $z \in Kf(t, x)$, $z_i \in K(k_i(t, x) + v_i)$, $i \in \{1, \ldots, m\}$, such that $w = z + \sum_{i=1}^m g_i(t, x) z_i$.

Let us show that $a \le 0$. We distinguish the three cases (i), (ii), (iii).

(i) $b = p \cdot (1, z) = a - \sum_{i=1}^m c^i_{t,x} z_i$ does not depend on p, then $b \in \dot{\bar{V}}^{(2)}(t, x)$ and, by (f1), $b \le 0$.

Let us now show that for each $i \in \{1, \dots, m\}$ $c^i_{t,x} z_i \le 0$. If i is such that $c^i_{t,x} = 0$, obviously $c^i_{t,x} z_i \le 0$. If i is such that $c^i_{t,x} > 0$ then, by Lemma 1, there exists δ_x such that $k_i(t, y) = -\|y\|$ in $\{\bar{t}\} \times B(x, \delta_x)$, then k_i is continuous at x with respect to y. This implies that $K(k_i(t, x) + v_i) = -\|x\| + v_i$, i.e. $z_i = -\|x\| + v_i$ and $c^i_{t,x} z_i \le 0$, provided that $\|v\| > \rho \ge \max\{L, R\}$.

The case in which i is such that $c^i_{t,x} < 0$ can be treated analogously. We finally get that $a = b + \sum_{i=1}^m c^i_{t,x} z_i \le 0$.

(ii) By (f2) there exists $\overline{p} \in \partial V(t, x)$ such that $\overline{p} \cdot (1, z) \le 0$. $a = \overline{p} \cdot (1, z) + \sum_{i=1}^m \overline{p} \cdot (1, g_i(t, x)) z_i$. The fact that for each $i \in \{1, \dots, m\}$ we have $\overline{p} \cdot (1, g_i(t, x)) z_i \le 0$ can be proved as in (i) we have proved that for each $i \in \{1, \dots, m\}$ $c^i_{t,x} z_i \le 0$. We finally get that $a \le 0$.

(iii) Let us remark that if (G2) is not verified, i.e. we are not in the case (ii), there exists $\overline{p} \in \partial V(t, x)$ corresponding to $\bar{\imath}$ such that $\overline{p} \cdot (1, g_{\bar{\imath}}(t, x)) = 0$. Indeed, because of the convexity of $\partial V(t, x)$, for all $v \in \mathbb{R}^n$, if there exist $p_1, p_2 \in \partial V(t, x)$ such that $p_1 \cdot v > 0$ and $p_2 \cdot v < 0$, then there also exists $p_3 \in \partial V(t, x)$ such that $p_3 \cdot v = 0$.

Let $\overline{p} \in \partial V(t, x)$ be such that $\overline{p} \cdot (1, g_{\bar{\imath}}(t, x)) = 0$. For all $p \in \partial V(t, x)$ $a = p \cdot (1, w)$. In particular we have $a = \overline{p} \cdot (1, w) = \overline{p} \cdot (1, z) + \sum_{i \ne \bar{\imath}} \overline{p} \cdot (1, g_i(t, x)) z_i + \overline{p} \cdot (1, g_{\bar{\imath}}(t, x)) z_{\bar{\imath}}$. By (f3), $\overline{p} \cdot (1, z) \le 0$. If $i \ne \bar{\imath}$ the proof that $\overline{p} \cdot (1, g_i(t, x)) z_i \le 0$ is the same as in (ii). If $i = \bar{\imath}$, because of the choice of $\overline{p}$, $\overline{p} \cdot (1, g_{\bar{\imath}}(t, x)) = 0$. Also in this case we can then conclude that $a \le 0$.

□

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AMS Subject Classification: ???.

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