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THE PERIODIC UNFOLDING METHOD IN PERFORATED DOMAINS

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Abstract: The periodic unfolding method was introduced in [4] by D. Cioranescu, A. Damlamian and G. Griso for the study of classical periodic homogenization. The main tools are the unfolding operator and a macro-micro decomposition of functions which allows to separate the macroscopic and microscopic scales.

In this paper, we extend this method to the homogenization in domains with holes, introducing the unfolding operator for functions defined on periodically perforated domains as well as a boundary unfolding operator.

As an application, we study the homogenization of some elliptic problems with a Robin condition on the boundary of the holes, proving convergence and corrector results.

1 – Introduction

The homogenization theory is a branch of the mathematical analysis which treats the asymptotic behavior of differential operators with rapidly oscillating coefficients.

We have now different methods related to this theory:

- The multiple-scale method introduced by A. Bensoussan, J.-L. Lions and G. Papanicolaou in [2].
- The oscillating test functions method due to L. Tartar in [13].
- The two-scale convergence method introduced by G. Nguetseng in [12], and further developed by G. Allaire in [1].

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Recently, the periodic unfolding method was introduced in [4] by D. Cioranescu, A. Damlamian and G. Griso for the study of classical periodic homogenization in the case of fixed domains. This method is based on two ingredients: the unfolding operator and a macro-micro decomposition of functions which allows to separate the macroscopic and microscopic scales. The interest of the method comes from the fact that it only deals with functions and classical notions of convergence in L^p spaces. This renders the proof of homogenization results quite elementary. It also provides error estimates and corrector results (see [10] for the case of fixed domains).

The aim of this paper is to adapt the method to the homogenization in domains with holes. To do so, we define in the upcoming section the unfolding operator for functions defined on periodically perforated domains. We also define in Section 5 a boundary unfolding operator, in order to treat problems with nonhomogeneous boundary conditions on the holes (Neumann or Robin type). The main feature is that, when treating such problems, we do not need any extension operator. Consequently, we can consider a larger class of geometrical situations than in [2], [5], and [7] for instance. In particular, for the homogeneous Neumann problem, we can admit some fractal holes like the two dimensional snowflake (see [16]). For a general nonhomogeneous Robin condition, we only assume a Lipschitz boundary, in order to give a sense to traces in Sobolev spaces.

The paper is organized as follows:

- In Section 2, we define the unfolding operator and prove some linked properties.
- In Section 3, we give the macro-micro decomposition of functions defined in perforated domains.
- In Section 4, we introduce the averaging operator and state a corrector result.
- In Section 5, we define the boundary unfolding operator and prove some related properties.
- In Section 6, as an application, we treat an elliptic problem with Robin boundary condition.

2 – The periodic unfolding operator in a perforated domain

In this section, we introduce the periodic unfolding operator in the case of perforated domains.

In the following we denote:

- Ω an open bounded set in $\mathbb{R}^N$,
- $Y = \prod_{i=1}^{N} [0, l_i[$ the reference cell, with $l_i > 0$ for all $1 \leq i \leq N$, or more generally a set having the paving property with respect to a basis $(b_1, \cdots, b_N)$ defining the periods,
- T an open set included in Y such that ∂T does not contain the summits of Y. We can be, sometimes, transported to this situation by a simple change of period,
- $Y^\star = Y \backslash \overline{T}$ a connected open set.

We define

$$T^\varepsilon = \bigcup_{\xi \in \mathbb{Z}^N} \varepsilon(\xi + T) \qquad \text{and} \qquad \Omega^\varepsilon = \Omega \backslash T^\varepsilon \ .$$

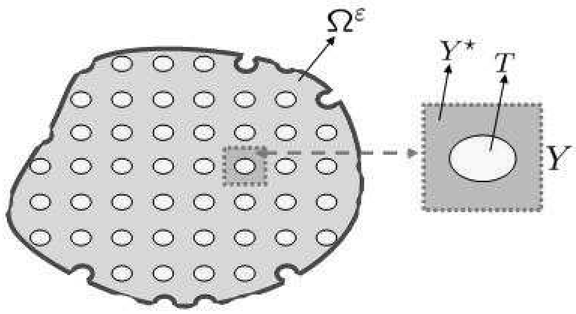

Figure 1 – The domain Ω^ε and the reference cell Y

We assume in the following that Ω^ε is a connected set. Unlike preceding papers treating perforated domains (see for example [5], [6], [7]) we can allow that the holes meet the boundary $\partial\Omega$. In the rest of this paper, we only take the regularity hypothesis

$$|\partial\Omega| = 0 \ . \tag{1}$$

Remark 2.1. The hypothesis aforementioned is equivalent to the fact that the number of cells intersecting the boundary of Ω is of order ε^{-N} (we refer to [11, Lemma 21]). ▫

Remark 2.2. An interesting example on the hypotheses aforementioned would be the lattice-type structures for which it is not possible, in some cases, to define extension operators. This situation happens if the holes intersect the exterior boundary $\partial\Omega$ (see [7], [8]). □

In the sequel, we will use the following notation:

- $\widetilde{\varphi}$ for the extension by 0 outside Ω^ε (resp. Ω) for any function φ in $L^p(\Omega^\varepsilon)$ (resp. $L^p(\Omega)$),
- χ^ε for the characteristic function of Ω^ε,
- θ for the proportion of the material in the elementary cell, i.e. $\theta = \dfrac{|Y^\star|}{|Y|}$,
- $\rho(Y)$ for the diameter of the cell Y. □

By analogy to the 1D notation, for $z \in \mathbb{R}^N$, $[z]_Y$ denotes the unique integer combination $\sum\limits_{j=1}^{j=N} k_j b_j$, such that $z - [z]_Y$ belongs to Y. Set $\{z\}_Y = z - [z]_Y$ (see Fig. 2). Then, for almost every $x \in \mathbb{R}^N$, there exists a unique element in $\mathbb{R}^N$, denoted by $\left[\dfrac{x}{\varepsilon}\right]_Y$, such that

$$x - \varepsilon\left[\frac{x}{\varepsilon}\right]_Y = \varepsilon\left\{\frac{x}{\varepsilon}\right\}_Y ,$$

where

$$\left\{\frac{x}{\varepsilon}\right\}_Y \in Y .$$

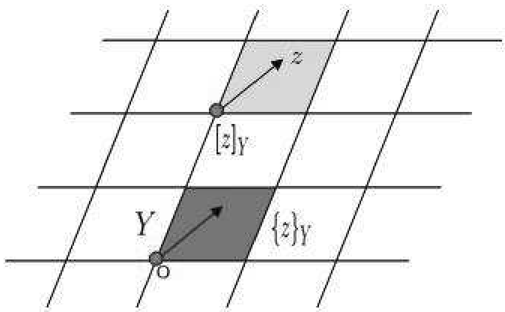

Figure 2 – The decomposition $z = [z]_Y + \{z\}_Y$

Definition 2.3. Let $\varphi \in L^p(\Omega^\varepsilon)$, $p \in [1, +\infty]$. We define the function $\mathcal{T}_\varepsilon(\varphi) \in L^p(\mathbb{R}^N \times Y^\star)$ by setting

$$\mathcal{T}_\varepsilon(\varphi)(x,y) \;=\; \widetilde{\varphi}\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right), \tag{2}$$

for every $x \in \mathbb{R}^N$ and $y \in Y^\star$. □

Remark 2.4. Notice that the oscillations due to perforations are shifted into the second variable y which belongs to the fixed domain $Y^\star$, while the first variable x belongs to $\mathbb{R}^N$.

One see immediately the interest of the unfolding operator. Indeed, when trying to pass to the limit in a sequence defined on Ω^ε, one needs first, while using standard methods, to extend it to a fixed domain. With $\mathcal{T}_\varepsilon$, such extensions are no more necessary. □

The main properties given in [4] for fixed domains can easily be adapted for the perforated ones without any major difficulty in the proofs. These properties are listed in the proposition below.

To do so, let us first define the following domain:

$$\widetilde{\Omega^\varepsilon} = \mathrm{Int}\left(\bigcup_{\xi \in \Lambda_\varepsilon} \varepsilon(\xi + Y)\right), \quad \text{where } \Lambda_\varepsilon = \left\{\xi \in \mathbb{Z}^N;\ \varepsilon(\xi + \overline{Y}) \cap \Omega \neq \phi\right\}.$$

The set $\widetilde{\Omega^\varepsilon}$ is the smallest finite union of εY cells containing Ω.

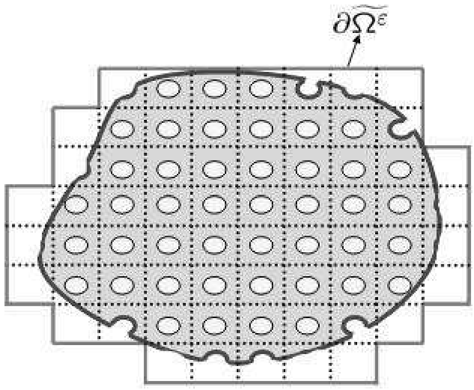

Figure 3 – The domain $\widetilde{\Omega^\varepsilon}$

Proposition 2.5. *The unfolding operator $\mathcal{T}_\varepsilon$ has the following properties:*

1. $\mathcal{T}_\varepsilon$ *is a linear operator.*

2. $\mathcal{T}_\varepsilon(\varphi)\left(x, \left\{\dfrac{x}{\varepsilon}\right\}_Y\right) = \varphi(x), \quad \forall \varphi \in L^p(\Omega^\varepsilon)$ *and* $x \in \mathbb{R}^N$.

3. $\mathcal{T}_\varepsilon(\varphi\psi) = \mathcal{T}_\varepsilon(\varphi)\, \mathcal{T}_\varepsilon(\psi), \quad \forall \varphi, \psi \in L^p(\Omega^\varepsilon)$.

4. *Let φ in $L^p(Y)$ or $L^p(Y^\star)$ be a Y-periodic function. Set* $\varphi^\varepsilon(x) = \varphi\left(\dfrac{x}{\varepsilon}\right)$. *Then,*

$$\mathcal{T}_\varepsilon(\varphi^\varepsilon)(x,y) = \varphi(y) \ .$$

5. *One has the integration formula*

$$\int_{\Omega^\varepsilon} \varphi \, dx = \frac{1}{|Y|} \int_{\mathbb{R}^N \times Y^\star} \mathcal{T}_\varepsilon(\varphi) \, dx \, dy = \frac{1}{|Y|} \int_{\widetilde{\Omega^\varepsilon} \times Y^\star} \mathcal{T}_\varepsilon(\varphi) \, dx \, dy \,, \quad \forall \varphi \in L^1(\Omega^\varepsilon) \,.$$

6. *For every $\varphi \in L^2(\Omega^\varepsilon)$, $\mathcal{T}_\varepsilon(\varphi)$ belongs to $L^2(\mathbb{R}^N \times Y^\star)$. It also belongs to $L^2(\widetilde{\Omega^\varepsilon} \times Y^\star)$.*

7. *For every $\varphi \in L^2(\Omega^\varepsilon)$, one has*

$$\|\mathcal{T}_\varepsilon(\varphi)\|_{L^2(\mathbb{R}^N \times Y^\star)} = \sqrt{|Y|}\, \|\varphi\|_{L^2(\Omega^\varepsilon)} \ .$$

8. $\nabla_y \mathcal{T}_\varepsilon(\varphi)(x,y) = \varepsilon \mathcal{T}_\varepsilon(\nabla_x \varphi)(x,y)$ *for every* $(x,y) \in \mathbb{R}^N \times Y^\star$.

9. *If $\varphi \in H^1(\Omega^\varepsilon)$, then $\mathcal{T}_\varepsilon(\varphi)$ is in $L^2(\mathbb{R}^N; H^1(Y^\star))$.*

10. *One has the estimate*

$$\|\nabla_y \mathcal{T}_\varepsilon(\varphi)\|_{(L^2(\mathbb{R}^N \times Y^\star))^N} = \varepsilon \sqrt{|Y|}\, \|\nabla_x \varphi\|_{(L^2(\Omega^\varepsilon))^N} \ .$$

Proof: The proof follows along the lines of the corresponding one in the case of fixed domains (see [4]). For the reader's convenience, we prove here the fifth assertion.

Let $\varphi \in L^1(\Omega^\varepsilon)$. One has

$$\begin{aligned}
\int_{\Omega^\varepsilon} \varphi(x) \, dx &= \int_{\widetilde{\Omega^\varepsilon}} \widetilde{\varphi}(x) \, dx = \sum_{\xi \in \Lambda_\varepsilon} \int_{\varepsilon(\xi+Y)} \widetilde{\varphi}(x) \, dx \\
&= \sum_{\xi \in \Lambda_\varepsilon} \int_Y \widetilde{\varphi}\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) \varepsilon^N dy \int_{\varepsilon(\xi+Y)} \frac{1}{|\varepsilon(\xi+Y)|} \, dx \\
&= \frac{1}{|Y|} \sum_{\xi \in \Lambda_\varepsilon} \int_{\varepsilon(\xi+Y) \times Y^\star} \widetilde{\varphi}\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) dx \, dy \ ,
\end{aligned}$$

since $\widetilde{\varphi}$ is null in the holes. The desired result is then straightforward. ∎

N.B. In the rest of this paper, when a function ψ is defined on a domain containing Ω^ε, and for simplicity, we may use the notation $\mathcal{T}_\varepsilon(\psi)$ instead of $\mathcal{T}_\varepsilon(\psi|_{\Omega^\varepsilon})$.

Proposition 2.6. *Let $\varphi \in L^2(\Omega)$. Then,*

1. $\mathcal{T}_\varepsilon(\varphi) \to \widetilde{\varphi}$ *strongly in* $L^2(\mathbb{R}^N{\times}Y^\star)$,

2. $\varphi\chi^\varepsilon \rightharpoonup \theta\varphi$ *weakly in* $L^2(\Omega)$,

3. *Let* (φ^ε) *be in* $L^2(\Omega)$ *such that*

$$\varphi^\varepsilon \to \varphi \quad \textit{strongly in} \;\; L^2(\Omega) \;.$$

Then,

$$\mathcal{T}_\varepsilon(\varphi^\varepsilon) \to \widetilde{\varphi} \quad \textit{strongly in} \;\; L^2(\mathbb{R}^N{\times}Y^\star) \;.$$

Proof: 1. The first assertion is obvious for every $\varphi \in D(\Omega)$.

If $\varphi \in L^2(\Omega)$, let $\varphi_k \in D(\Omega)$ such that $\varphi_k \to \varphi$ in $L^2(\Omega)$. Then

$$\begin{aligned}
&\|\mathcal{T}_\varepsilon(\varphi) - \widetilde{\varphi}\|_{L^2(\mathbb{R}^N\times Y^\star)} \le \\
&\quad \le \|\mathcal{T}_\varepsilon(\varphi) - \mathcal{T}_\varepsilon(\varphi_k)\|_{L^2(\mathbb{R}^N\times Y^\star)} + \|\mathcal{T}_\varepsilon(\varphi_k) - \varphi_k\|_{L^2(\mathbb{R}^N\times Y^\star)} + \|\varphi_k - \widetilde{\varphi}\|_{L^2(\mathbb{R}^N\times Y^\star)} \,,
\end{aligned}$$

from which the result is straightforward.

2. The sequence $\varphi\chi^\varepsilon$ is bounded in $L^2(\Omega)$. Let $\psi \in D(\Omega)$. From 3 and 5 of Proposition 2.5, one has

$$\begin{aligned}
\int_\Omega \varphi\chi^\varepsilon\psi\,dx = \int_{\Omega^\varepsilon} \varphi\psi\,dx &= \frac{1}{|Y|}\int_{\mathbb{R}^N\times Y^\star} \mathcal{T}_\varepsilon(\varphi\psi)\,dx\,dy \\
&= \frac{1}{|Y|}\int_{\mathbb{R}^N\times Y^\star} \mathcal{T}_\varepsilon(\varphi)\,\mathcal{T}_\varepsilon(\psi)\,dx\,dy \;.
\end{aligned}$$

Consequently,

$$\int_\Omega \varphi\chi^\varepsilon\psi\,dx \;\to\; \frac{1}{|Y|}\int_{\mathbb{R}^N\times Y^\star} \widetilde{\varphi}\psi\,dx\,dy = \frac{|Y^\star|}{|Y|}\int_\Omega \varphi\psi\,dx \;.$$

3. One has

$$\begin{aligned}
&\int_{\mathbb{R}^N\times Y^\star} \big(\mathcal{T}_\varepsilon(\varphi^\varepsilon) - \widetilde{\varphi}\big)^2\,dx\,dy \le \\
&\qquad \le 2\left(\int_{\mathbb{R}^N\times Y^\star} \big(\mathcal{T}_\varepsilon(\varphi^\varepsilon) - \mathcal{T}_\varepsilon(\varphi)\big)^2\,dx\,dy + \int_{\mathbb{R}^N\times Y^\star} \big(\mathcal{T}_\varepsilon(\varphi) - \widetilde{\varphi}\big)^2\,dx\,dy\right) .
\end{aligned}$$

On one hand, by using 1 and 7 of Proposition 2.5, we get as $\varepsilon \to 0$

$$\begin{aligned}\int_{\mathbb{R}^N\times Y^\star} \big(\mathcal{T}_\varepsilon(\varphi^\varepsilon) - \mathcal{T}_\varepsilon(\varphi)\big)^2\,dx\,dy \;&=\; \int_{\mathbb{R}^N\times Y^\star} \big(\mathcal{T}_\varepsilon(\varphi^\varepsilon - \varphi)\big)^2\,dx\,dy\\ &= |Y| \int_{\Omega^\varepsilon} (\varphi^\varepsilon - \varphi)^2\,dx\\ &\le |Y| \int_{\Omega} (\varphi^\varepsilon - \varphi)^2\,dx \;\to\; 0\;.\end{aligned}$$

On the other hand, by using 1, one has

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N\times Y^\star} \big(\mathcal{T}_\varepsilon(\varphi) - \widetilde{\varphi}\big)^2\,dx\,dy \;=\; 0\;.$$

Therefore, assertion 3 holds true. ∎

Proposition 2.7. *Let φ^ε be in $L^2(\Omega^\varepsilon)$ for every ε, such that*

$$\mathcal{T}_\varepsilon(\varphi^\varepsilon) \rightharpoonup \widehat{\varphi} \quad \text{weakly in } \; L^2(\mathbb{R}^N\times Y^\star)\;.$$

Then,

$$\widetilde{\varphi^\varepsilon} \rightharpoonup \frac{1}{|Y|} \int_{Y^\star} \widehat{\varphi}(\cdot, y)\,dy \quad \text{weakly in } \; L^2(\mathbb{R}^N)\;.$$

Proof: Let $\psi \in D(\Omega)$. Using 3 and 5 of Proposition 2.5, one has successively

$$\begin{aligned}\int_{\mathbb{R}^N} \widetilde{\varphi^\varepsilon}\psi\,dx \;=\; \int_{\Omega^\varepsilon} \varphi^\varepsilon\psi\,dx \;&=\; \frac{1}{|Y|} \int_{\mathbb{R}^N\times Y^\star} \mathcal{T}_\varepsilon(\varphi^\varepsilon\psi)\,dx\,dy\\ &= \frac{1}{|Y|} \int_{\mathbb{R}^N\times Y^\star} \mathcal{T}_\varepsilon(\varphi^\varepsilon)\,\mathcal{T}_\varepsilon(\psi)\,dx\,dy\;.\end{aligned}$$

This gives, using 1 of Proposition 2.6

$$\int_{\mathbb{R}^N} \widetilde{\varphi^\varepsilon}\psi\,dx \;\to\; \frac{1}{|Y|} \int_{\mathbb{R}^N\times Y^\star} \widehat{\varphi}(x,y)\,\psi(x)\,dx\,dy = \frac{1}{|Y|} \int_{\mathbb{R}^N} \left\{ \int_{Y^\star} \widehat{\varphi}(x,y)\,dy \right\} \psi(x)\,dx\,. \blacksquare$$

Proposition 2.8. *Let φ^ε be in $L^2(\Omega^\varepsilon)$ for every ε, with*

$$\|\varphi^\varepsilon\|_{L^2(\Omega^\varepsilon)} \le C\;,$$

$$\varepsilon\|\nabla_x\varphi^\varepsilon\|_{(L^2(\Omega^\varepsilon))^N} \le C\;.$$

Then, there exists $\widehat{\varphi}$ *in* $L^2(\mathbb{R}^N; H^1(Y^\star))$ *such that, up to subsequences*

1. $\mathcal{T}_\varepsilon(\varphi^\varepsilon) \rightharpoonup \widehat{\varphi}$ *weakly in* $L^2(\mathbb{R}^N; H^1(Y^\star))$,

2. $\varepsilon\mathcal{T}_\varepsilon(\nabla_x\varphi^\varepsilon) \rightharpoonup \nabla_y\widehat{\varphi}$ *weakly in* $L^2(\mathbb{R}^N \times Y^\star)$,
where

$$y \mapsto \widehat{\varphi}(.,y) \;\in\; L^2(\mathbb{R}^N; H^1_{per}(Y^\star)) \;.$$

Proof: Convergence 1 is immediate and 2 follows from 8 in Proposition 2.5. It remains to prove that $\widehat{\varphi}$ is periodic. To do so, let $\psi \in D(\Omega\times Y^\star)$. By using the definition of $\mathcal{T}_\varepsilon$ and a simple change of variables, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N\times Y^\star} \Big(\mathcal{T}_\varepsilon(\varphi^\varepsilon)\,(x, y + l_i\overrightarrow{e_i}) - \mathcal{T}_\varepsilon(\varphi^\varepsilon)\,(x,y)\Big)\,\psi(x,y)\;dx\,dy \;= \\
&\qquad = \int_{\mathbb{R}^N\times Y^\star} \bigg(\varphi^\varepsilon\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon l_i\overrightarrow{e_i} + \varepsilon y\Big) - \varphi^\varepsilon\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon y\Big)\bigg)\psi(x,y)\;dx\,dy \\
&\qquad = \int_{\mathbb{R}^N\times Y^\star} \varphi^\varepsilon\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon y\Big)\,\Big[\psi\,(x - \varepsilon l_i\overrightarrow{e_i}, y) - \psi\,(x,y)\Big]\;dx\,dy \;.
\end{aligned}$$

Passing to the limit, we obtain the result since $\psi(x - \varepsilon l_i\overrightarrow{e_i}, y) - \psi(x,y) \to 0$ when $\varepsilon \to 0$. ∎

3 – Macro-Micro decomposition

Following [4], we decompose any function φ in the form

$$\varphi = \mathcal{Q}_\varepsilon(\varphi) + \mathcal{R}_\varepsilon(\varphi) \;,$$

where $\mathcal{R}_\varepsilon$ is designed in order to capture the oscillations.

As in the case of fixed domains, we start by defining $\mathcal{Q}_\varepsilon(\varphi)$ on the nodes $\varepsilon\xi_k$ of the εY-lattice. Here, it is no longer possible to take the average on the entire cell Y as in [4], but it will be taken on a small ball B_ε centered on $\varepsilon\xi_k$ and not touching the holes. This is possible using the fact that ∂T does not contain the summits of Y. However, B_ε must be entirely contained in Ω^ε.

To guarantee that, we are let to define $\mathcal{Q}_\varepsilon(\varphi)$ on a subdomain of Ω^ε only. To do so, for every $\delta > 0$, let us set

$$\Omega^\varepsilon_\delta = \Big\{ x \in \Omega\,;\; d(x, \partial\Omega) > \delta \Big\} \qquad \text{and} \qquad \widehat{\Omega^\varepsilon_\delta} = \operatorname{Int}\bigg(\bigcup_{\xi \in \Pi^\delta_\varepsilon} \varepsilon(\xi + \overline{Y}) \bigg),$$

where

$$\Pi^\delta_\varepsilon = \Big\{ \xi \in \mathbb{Z}^N;\; \varepsilon(\xi + \overline{Y}) \subset \Omega^\varepsilon_\delta \Big\} \;.$$

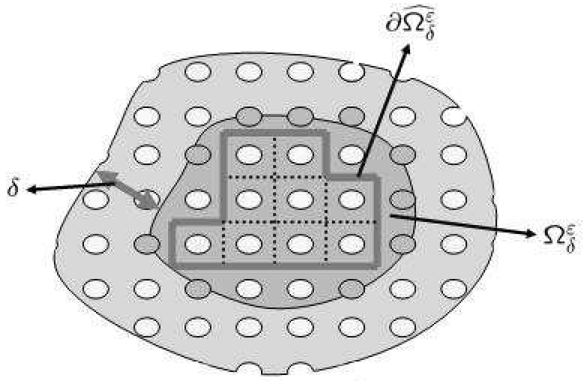

Figure 4 – The domains $\Omega^\varepsilon_\delta$ and $\widehat{\Omega^\varepsilon_\delta}$

The construction of the decomposition is as follows:

- For every node $\varepsilon\xi_k$ in $\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}$ we define

$$\mathcal{Q}_\varepsilon(\varphi)(\varepsilon\xi_k) \;=\; \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} \varphi(\varepsilon\xi_k + \varepsilon z)\, dz \;.$$

  Observe that by definition, any ball B_ε centered in a node of $\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}$ is entirely contained in Ω^ε, since actually they all belong to $\Omega^\varepsilon_{\varepsilon\rho(Y)}$.

- We define $\mathcal{Q}_\varepsilon(\varphi)$ on the whole $\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}$, by taking a Q_1-interpolate, as in the finite element method (FEM), of the discrete function $\mathcal{Q}_\varepsilon(\varphi)(\varepsilon\xi_k)$.

- On $\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}$, $\mathcal{R}_\varepsilon$ will be defined as the remainder: $\mathcal{R}_\varepsilon(\varphi) = \varphi - \mathcal{Q}_\varepsilon(\varphi)$.

Proposition 3.1. *For φ belonging to $H^1(\Omega^\varepsilon)$, one has the following properties:*

1. $\|\mathcal{Q}_\varepsilon(\varphi)\|_{H^1(\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)})} \leq C\,\|\varphi\|_{H^1(\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)})}$,
2. $\|\mathcal{R}_\varepsilon(\varphi)\|_{L^2(\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)})} \leq C\,\varepsilon\|\nabla_x\varphi\|_{(L^2(\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}))^N}$,
3. $\|\nabla_x\mathcal{R}_\varepsilon(\varphi)\|_{(L^2(\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}))^N} \leq C\,\|\nabla_x\varphi\|_{(L^2(\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}))^N}$.

Proof: These results are straightforward from the definition of $\mathcal{Q}_\varepsilon$. The proof, based on some FEM properties, is very similar to the corresponding one in the case of fixed domains (see [4]), with the simple replacement of Y by $Y^\star$. ∎

We can now state the main result of this section.

Theorem 3.2. *Let φ^ε be in $H^1(\Omega^\varepsilon)$ for every ε, with $\|\varphi^\varepsilon\|_{H^1(\Omega^\varepsilon)}$ bounded. There exists φ in $H^1(\Omega)$ and $\widehat{\varphi}$ in $L^2(\Omega; H^1_{per}(Y^\star))$ such that, up to subsequences*

1. $\mathcal{Q}_\varepsilon(\varphi^\varepsilon) \rightharpoonup \varphi$ *weakly in* $H^1_{loc}(\Omega)$,
2. $\mathcal{T}_\varepsilon(\varphi^\varepsilon) \rightharpoonup \varphi$ *weakly in* $L^2_{loc}(\Omega; H^1(Y^\star))$,
3. $\dfrac{1}{\varepsilon}\,\mathcal{T}_\varepsilon(\mathcal{R}_\varepsilon(\varphi^\varepsilon)) \rightharpoonup \widehat{\varphi}$ *weakly in* $L^2_{loc}(\Omega; H^1(Y^\star))$,
4. $\mathcal{T}_\varepsilon(\nabla_x(\varphi^\varepsilon)) \rightharpoonup \nabla_x\varphi + \nabla_y\widehat{\varphi}$ *weakly in* $L^2_{loc}(\Omega; L^2(Y^\star))$.

Remark 3.3. When comparing with the case of fixed domains, the main difference is that, since the decomposition was done on $\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}$, we have here local convergences only. □

Proof of Theorem 3.2: Assertions 2, 3 and 4 can be proved by using the same arguments as in the corresponding proofs for the case of fixed domains. We consider here just the first assertion.

Let K be a compact set in Ω. As $d(K, \partial\Omega) > 0$, there exists $\varepsilon_K > 0$ depending on K, such that

$$\forall\varepsilon \leq \varepsilon_K\ , \qquad K \subset \widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}\ .$$

Hence,

$$\|\mathcal{Q}_\varepsilon(\varphi^\varepsilon)\|_{H^1(K)} \leq \|\mathcal{Q}_\varepsilon(\varphi^\varepsilon)\|_{H^1(\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)})} \leq C\|\varphi^\varepsilon\|_{H^1(\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)})} \leq C\|\varphi^\varepsilon\|_{H^1(\Omega)} \leq C\,,$$

so that there exists some $\varphi \in H^1(\Omega)$ such that

$$\mathcal{Q}_\varepsilon(\varphi^\varepsilon) \rightharpoonup \varphi \quad \text{weakly in } \ H^1_{loc}(\Omega)\ .$$

What remains to be proved is that $\varphi \in H^1(\Omega)$. To do so, we make use of the Dominated Convergence theorem.

Let us consider the sequence $(\Omega^\varepsilon_{\frac{1}{N}})_N$. Observe that it is increasing. Indeed,

$$x \in \Omega^\varepsilon_{\frac{1}{N}} \ \Rightarrow\ d(x,\partial\Omega) > \frac{1}{N} > \frac{1}{N+1}, \qquad \text{hence } \ x \in \Omega^\varepsilon_{\frac{1}{N+1}} \ .$$

Moreover, for every N, there exists ε_N depending on $\Omega^\varepsilon_{\frac{1}{N}}$ such that

$$\forall \varepsilon \leq \varepsilon_N, \qquad \text{one has } \ \Omega^\varepsilon_{\frac{1}{N}} \subset \widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)} \ .$$

Let us define the sequence of functions $(\varphi_N)_N$ for every $N \in \mathbb{N}^\star$ as follows:

$$\varphi_N = |\varphi|^2 \, \chi_{\Omega^\varepsilon_{\frac{1}{N}}} \ .$$

Observe that

$$\text{the sequence } (\varphi_N)_N \text{ is increasing}\,. \tag{3}$$

Let us show that

$$\text{the sequence } (\varphi_N)_N \text{ belongs to } L^1(\Omega)\,. \tag{4}$$

One has successively

$$\int_\Omega |\varphi_N| \, dx \ = \int_\Omega |\varphi|^2 . \, \chi_{\Omega^\varepsilon_{\frac{1}{N}}} \, dx \ = \int_{\Omega^\varepsilon_{\frac{1}{N}}} |\varphi|^2 \, dx \ \leq \int_{\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}} |\varphi|^2 \, dx \ ,$$

for a suitable ε. Then, by Fatou's lemma, one has

$$\int_\Omega |\varphi_N| \, dx \ \leq \liminf \int_{\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}} |\mathcal{Q}_\varepsilon \varphi|^2 \, dx \ \leq\ \liminf \, \|\mathcal{Q}_\varepsilon(\varphi^\varepsilon)\|^2_{L^2(\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)})} \ .$$

Finally, Proposition 3.1(1) yields

$$\int_\Omega |\varphi_N| \ \leq\ C \, \|\varphi^\varepsilon\|^2_{H^1(\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)})} \ \leq\ C \ ,$$

whence (4).

The next step is to prove that

$$\text{the sequence } (\varphi_N)_N \text{ simply converges towards } \ |\varphi|^2 \,. \tag{5}$$

Let $x \in \Omega$, then $d(x, \partial\Omega) = \alpha > 0$ where $\alpha \in \mathbb{R}$. There exists $N_0 \in \mathbb{N}^\star$ such that $\alpha > \frac{1}{N_0}$, hence $d(x, \partial\Omega) > \frac{1}{N_0}$ and $x \in \Omega^\varepsilon_{\frac{1}{N_0}}$. As the sequence $(\Omega^\varepsilon_{\frac{1}{N}})_N$ is increasing, we deduce that $x \in \Omega^\varepsilon_{\frac{1}{N}}$ for all $N \geq N_0$. Hence,

$$\chi_{\Omega^\varepsilon_{\frac{1}{N}}}(x) = 1 \ , \qquad \forall N \geq N_0 \ ,$$

and this ends the proof of (5).

Thanks to (3),(4) and (5), we can apply the Dominated Convergence theorem to deduce that

$$|\varphi|^2 \in L^1(\Omega) \quad \text{ and } \quad \lim_{N\to\infty} \int_\Omega |\varphi_N| \, dx \, = \int_\Omega |\varphi|^2 \, dx \ .$$

Consequently $\varphi \in L^2(\Omega)$.

Similarly, we prove that $\nabla\varphi \in (L^2(\Omega))^N$. Thus, $\varphi \in H^1(\Omega)$. ∎

4 – The averaging operator $\mathcal{U}_\varepsilon$

Definition 4.1. For $\varphi \in L^2(\mathbb{R}^N \times Y^\star)$, we set

$$\mathcal{U}_\varepsilon(\varphi)(x) \ = \ \frac{1}{|Y^\star|} \int_{Y^\star} \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z\,,\, \left\{\frac{x}{\varepsilon}\right\}_Y\right) dz, \qquad \text{for every} \quad x \in \mathbb{R}^N \ . \ \square$$

Remark 4.2. For $V \in L^1(\mathbb{R}^N \times Y^\star)$, the function $x \mapsto V\left(x, \left\{\frac{x}{\varepsilon}\right\}_Y\right)$ is generally not measurable (for example, we refer to [5]-Chapter 9). Hence, it cannot be used as a test function. We replace it by the function $\mathcal{U}_\varepsilon(V)$. □

The next result extends the corresponding one given in [4].

Proposition 4.3. *One has the following properties:*

1. *The operator $\mathcal{U}_\varepsilon$ is linear and continuous from $L^2(\mathbb{R}^N \times Y^\star)$ into $L^2(\mathbb{R}^N)$, and one has for every $\varphi \in L^2(\mathbb{R}^N \times Y^\star)$*

$$\|\mathcal{U}_\varepsilon(\varphi)\|_{L^2(\mathbb{R}^N)} \leq \|\varphi\|_{L^2(\mathbb{R}^N \times Y^\star)} \ ,$$

2. *$\mathcal{U}_\varepsilon$ is the left inverse of $\mathcal{T}_\varepsilon$ on Ω^ε, which means that $\mathcal{U}_\varepsilon \circ \mathcal{T}_\varepsilon = Id$ on Ω^ε,*
3. $\mathcal{T}_\varepsilon\left(\chi^\varepsilon \mathcal{U}_\varepsilon(\varphi)\right)(x, y) = \dfrac{1}{|Y^\star|} \displaystyle\int_{Y^\star} \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z\,,\, y\right) dz, \quad \forall \varphi \in L^2(\mathbb{R}^N \times Y^\star)$,
4. *$\mathcal{U}_\varepsilon$ is the formal adjoint of $\mathcal{T}_\varepsilon$.*

Proof: 1. It is straightforward from Definition 4.1.

2. For every $\varphi \in L^2(\Omega^\varepsilon)$, one has

$$\begin{aligned}
\mathcal{U}_\varepsilon\left(\mathcal{T}_\varepsilon(\varphi)\right)(x) &= \frac{1}{|Y^\star|}\int_{Y^\star} \mathcal{T}_\varepsilon(\varphi)\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z\,,\,\left\{\frac{x}{\varepsilon}\right\}_Y\right) dz \\
&= \frac{1}{|Y^\star|}\int_{Y^\star} \varphi\left(\varepsilon\left[\left[\frac{x}{\varepsilon}\right]_Y + z\right]_Y + \varepsilon\left\{\frac{x}{\varepsilon}\right\}_Y\right) dz \\
&= \frac{1}{|Y^\star|}\int_{Y^\star} \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon\left\{\frac{x}{\varepsilon}\right\}_Y\right) dz \\
&= \frac{1}{|Y^\star|}\int_{Y^\star} \varphi(x)\, dz \;=\; \varphi(x)\;.
\end{aligned}$$

3. Let $\varphi \in L^2\left(\mathbb{R}^N\right)$, one has

$$\begin{aligned}
\mathcal{T}_\varepsilon\left(\chi^\varepsilon \mathcal{U}_\varepsilon(\varphi)\right)(x,y) &= \mathcal{U}_\varepsilon(\varphi)\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) \\
&= \frac{1}{|Y^\star|}\int_{Y^\star} \varphi\left(\varepsilon\left[\frac{\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y}{\varepsilon}\right]_Y + \varepsilon z\,,\,\left\{\frac{\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y}{\varepsilon}\right\}_Y\right) dz \\
&= \frac{1}{|Y^\star|}\int_{Y^\star} \varphi\left(\varepsilon\left[\left[\frac{x}{\varepsilon}\right]_Y + y\right]_Y + \varepsilon z\,,\,\left\{\left[\frac{x}{\varepsilon}\right]_Y + y\right\}_Y\right) dz \\
&= \frac{1}{|Y^\star|}\int_{Y^\star} \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z\,,\,y\right) dz\;.
\end{aligned}$$

4. For every $\varphi \in L^2\left(\mathbb{R}^N\right)$ and $\psi \in L^2\left(\mathbb{R}^N \times Y^\star\right)$, we have

$$\begin{aligned}
&\frac{1}{|Y^\star|}\int_{\mathbb{R}^N\times Y^\star} \mathcal{T}_\varepsilon(\varphi)\,(x,y)\,\psi(x,y)\;dx\,dy \;= \\
&\quad= \frac{1}{|Y^\star|}\sum_{\xi\in\mathbb{Z}^N}\int_{\varepsilon(\xi+Y)\times Y^\star} \varphi(\varepsilon\xi + \varepsilon y)\,\psi(x,y)\;dx\,dy \\
&\quad= \frac{1}{|Y^\star|}\sum_{\xi\in\mathbb{Z}^N}\int_{Y\times Y^\star} \varphi(\varepsilon\xi + \varepsilon y)\,\psi(\varepsilon\xi + \varepsilon z, y)\,\varepsilon^N\,dz\,dy \\
&\quad= \frac{1}{|Y^\star|}\sum_{\xi\in\mathbb{Z}^N}\int_{Y^\star\times\varepsilon(\xi+Y)} \varphi(t)\;\psi\left(\varepsilon\left[\frac{t}{\varepsilon}\right]_Y + \varepsilon z, \left\{\frac{t}{\varepsilon}\right\}_Y\right) dz\,dt \\
&\quad= \int_{\mathbb{R}^N} \varphi(t)\,\mathcal{U}_\varepsilon\psi(t)\;dt\;,
\end{aligned}$$

and the proof of Proposition 4.3 is complete. ∎

Proposition 4.4.

1. *Let $\varphi \in L^2(\mathbb{R}^N)$. One has*

$$\mathcal{U}_\varepsilon(\varphi) \to \varphi \quad \text{strongly in } \ L^2(\mathbb{R}^N)\ .$$

2. *Let $\varphi \in L^2(\mathbb{R}^N \times Y^\star)$. Then,*

$$\mathcal{T}_\varepsilon\left(\chi^\varepsilon \mathcal{U}_\varepsilon(\varphi)\right) \to \varphi \quad \text{strongly in } \ L^2(\mathbb{R}^N \times Y^\star)\ ,$$

and

$$\mathcal{U}_\varepsilon(\varphi) \rightharpoonup \frac{1}{|Y|} \int_{Y^\star} \varphi(.,y)\, dy \quad \text{weakly in } \ L^2(\mathbb{R}^N)\ .$$

Proof: 1. If $\varphi \in L^2(\mathbb{R}^N)$, one has by definition

$$\mathcal{U}_\varepsilon(\varphi)(x,y) = \frac{1}{|Y^\star|} \int_{Y^\star} \varphi\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z\right) dz\ , \quad \forall (x,y) \in \mathbb{R}^N \times Y^\star\ .$$

But $\varphi\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z\right) \to \varphi(x)$ when $\varepsilon \to 0$, and this explains the result.

2. It is a simple consequence of 1 in Proposition 2.6, and Proposition 2.7. ∎

As in the case of fixed domains, one has

Theorem 4.5. *Let φ^ε be in $L^2(\Omega^\varepsilon)$ for every ε, and let $\varphi \in L^2(\mathbb{R}^N \times Y^\star)$. Then,*

1. $\mathcal{T}_\varepsilon(\varphi^\varepsilon) \to \varphi$ *strongly in* $L^2(\mathbb{R}^N \times Y^\star)$
$\iff$ $\widetilde{\varphi^\varepsilon} - \mathcal{U}_\varepsilon(\varphi) \to 0$ *strongly in* $L^2(\mathbb{R}^N)$.

2. $\mathcal{T}_\varepsilon(\varphi^\varepsilon) \to \varphi$ *strongly in* $L^2_{loc}(\mathbb{R}^N; L^2(Y^\star))$
$\iff$ $\widetilde{\varphi^\varepsilon} - \mathcal{U}_\varepsilon(\varphi) \to 0$ *strongly in* $L^2_{loc}(\mathbb{R}^N)$.

Proof: 1. Observe that

$$\begin{aligned}
\|\widetilde{\varphi^\varepsilon} - \mathcal{U}_\varepsilon \varphi\|_{L^2(\mathbb{R}^N)} &\le C\, \big\| \mathcal{T}_\varepsilon(\varphi^\varepsilon) - \mathcal{T}_\varepsilon(\chi^\varepsilon \mathcal{U}_\varepsilon \varphi^\varepsilon) \big\|_{L^2(\mathbb{R}^N \times Y^\star)} \\
&\le C \Big(\big\| \mathcal{T}_\varepsilon(\varphi^\varepsilon) - \varphi \big\|_{L^2(\mathbb{R}^N \times Y^\star)} + \big\| \varphi - \mathcal{T}_\varepsilon(\chi^\varepsilon \mathcal{U}_\varepsilon \varphi^\varepsilon) \big\|_{L^2(\mathbb{R}^N \times Y^\star)} \Big) \\
&\to 0, \quad \text{when } \ \varepsilon \to 0\ .
\end{aligned}$$

The converse implication is immediate.

2. Let $w \subset\subset \Omega$, and $\psi \in D(\mathbb{R}^{\mathbb{N}})$ such that

$$\psi \geq 0 \quad \text{and} \quad \psi = 1 \ \text{ on } w \, .$$

Then, by using 1 of Proposition 2.6, one has

$$\begin{aligned}
\|\widetilde{\varphi^\varepsilon} - \mathcal{U}_\varepsilon \varphi\|_{L^2(w)} &\leq \|\psi(\widetilde{\varphi^\varepsilon} - \mathcal{U}_\varepsilon \varphi)\|_{L^2(\mathbb{R}^N)} \\
&\leq C \left\| \mathcal{T}_\varepsilon(\psi) \left(\mathcal{T}_\varepsilon(\varphi^\varepsilon) - \mathcal{T}_\varepsilon(\chi^\varepsilon \mathcal{U}_\varepsilon \varphi^\varepsilon)\right) \right\|_{L^2(\mathrm{supp}\psi \times Y^\star)} \\
&\leq C \Bigg(\left\| \mathcal{T}_\varepsilon(\psi) \left(\mathcal{T}_\varepsilon(\varphi^\varepsilon) - \varphi\right) \right\|_{L^2(\mathrm{supp}\psi \times Y^\star)} \\
&\quad + \left\| \mathcal{T}_\varepsilon(\psi) \left(\varphi - \mathcal{T}_\varepsilon(\chi^\varepsilon \mathcal{U}_\varepsilon \varphi^\varepsilon)\right) \right\|_{L^2(\mathrm{supp}\psi \times Y^\star)} \Bigg) \\
&\to 0, \quad \text{when } \ \varepsilon \to 0 \, . \ \blacksquare
\end{aligned}$$

Remark 4.6. This result is essential for proving corrector results when studying homogenization problems, as we show in Section 6. □

5 – The boundary unfolding operator

We define here the unfolding operator on the boundary of the holes ∂T^ε, which is specific to the case of perforated domains. To do that, we need to suppose that T has a Lipschitz boundary.

Definition 5.1. Suppose that T has a Lipschitz boundary, and let $\varphi \in L^p(\partial T^\varepsilon)$, $p \in [1, +\infty]$. We define the function $\mathcal{T}^b_\varepsilon(\varphi) \in L^p(\mathbb{R}^N \times \partial T)$ by setting

$$\mathcal{T}^b_\varepsilon(\varphi)(x,y) \,=\, \varphi\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) \,,$$

for every $x \in \mathbb{R}^N$ and $y \in \partial T$. □

The next assertions reformulate those presented in Proposition 2.5, when functions are defined on the boundary ∂T^ε.

Proposition 5.2. *The boundary unfolding operator has the following properties:*

1. $\mathcal{T}^b_\varepsilon$ *is a linear operator.*
2. $\mathcal{T}^b_\varepsilon(\varphi)\left(x, \left\{\dfrac{x}{\varepsilon}\right\}_Y\right) = \varphi(x)$, $\forall \varphi \in L^p(\partial T^\varepsilon)$ *and* $x \in \mathbb{R}^N$.

3. $\mathcal{T}^b_\varepsilon(\varphi\psi) = \mathcal{T}^b_\varepsilon(\varphi)\,\mathcal{T}^b_\varepsilon(\psi), \quad \forall \varphi,\psi \in L^p(\partial T^\varepsilon)$.

4. *Let* φ *in* $L^p(\partial T)$ *be a* Y*-periodic function. Set* $\varphi^\varepsilon(x) = \varphi\left(\dfrac{x}{\varepsilon}\right)$. *Then,*

$$\mathcal{T}^b_\varepsilon(\varphi^\varepsilon)(x,y) = \varphi(y)\ .$$

5. *For every* $\varphi \in L^1(\partial T^\varepsilon)$, *we have the integration formula*

$$\begin{aligned}\int_{\partial T^\varepsilon} \varphi(x)\,d\sigma(x) &= \frac{1}{\varepsilon|Y|}\int_{\mathbb{R}^N\times\partial T} \mathcal{T}^b_\varepsilon(\varphi)(x,y)\ dx\,d\sigma(y)\\ &= \frac{1}{\varepsilon|Y|}\int_{\widetilde{\Omega}^\varepsilon\times\partial T} \mathcal{T}^b_\varepsilon(\varphi)(x,y)\ dx\,d\sigma(y)\ .\end{aligned}$$

6. *For every* $\varphi \in L^2(\partial T^\varepsilon)$, $\mathcal{T}^b_\varepsilon(\varphi)$ *belongs to* $L^2(\mathbb{R}^N\times\partial T)$. *It also belongs to* $L^2(\widetilde{\Omega}^\varepsilon\times\partial T)$.

7. *For every* $\varphi \in L^2(\partial T^\varepsilon)$, *one has*

$$\|\mathcal{T}^b_\varepsilon(\varphi)\|_{L^2(\mathbb{R}^N\times\partial T)} = \sqrt{\varepsilon|Y|}\,\|\varphi\|_{L^2(\partial T^\varepsilon)}\ .$$

Proof: The proof follows by the same arguments that those used for Proposition 2.5. As an example, let us prove the integration formula.

Let $\varphi \in L^1(\partial T^\varepsilon)$. From the definition of T^ε, one has

$$\int_{\partial T^\varepsilon} \varphi(x)\,d\sigma(x) = \sum_{\xi\in\Lambda_\varepsilon}\int_{\varepsilon(\xi+\partial T)} u(x)\,d\sigma(x)\ .$$

By taking $x = \varepsilon(\xi+y)$, we have $d\sigma(x) = \varepsilon^{N-1}\,d\sigma(y)$. Hence,

$$\begin{aligned}\int_{\partial T^\varepsilon} \varphi(x)\,d\sigma(x) &= \sum_{\xi\in\Lambda_\varepsilon}\int_{\partial T} u\big(\varepsilon(\xi+y)\big)\,\varepsilon^{N-1}\,d\sigma(y)\\ &= \sum_{\xi\in\Lambda_\varepsilon}\int_{\varepsilon(\xi+Y)}\frac{1}{|\varepsilon(\xi+Y)|}\,dx\int_{\partial T} u\big(\varepsilon(\xi+y)\big)\,\varepsilon^{N-1}\,d\sigma(y)\\ &= \frac{1}{\varepsilon|Y|}\int_{\mathbb{R}^N\times\partial T} u\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon\left\{\frac{x}{\varepsilon}\right\}_Y\right)\,dx\,d\sigma(y)\\ &= \frac{1}{\varepsilon|Y|}\int_{\mathbb{R}^N\times\partial T} \mathcal{T}^b_\varepsilon(\varphi)(x,y)\ dx\,d\sigma(y)\\ &= \frac{1}{\varepsilon|Y|}\int_{\widetilde{\Omega}^\varepsilon\times\partial T} \mathcal{T}^b_\varepsilon(\varphi)(x,y)\ dx\,d\sigma(y)\ . \ \blacksquare\end{aligned}$$

Proposition 5.3. *Let $g \in L^2(\partial T)$ and $\varphi \in H^1(\Omega)$. One has the estimate*

$$\left| \int_{\mathbb{R}^N \times \partial T} g(y)\, \mathcal{T}_\varepsilon^b(\varphi)(x,y)\ dx\, d\sigma(y) \right| \leq C \left(|\mathcal{M}_{\partial T}(g)| + \varepsilon \right) \|\nabla \varphi\|_{(L^2(\Omega^\varepsilon))^N} ,$$

where $\mathcal{M}_{\partial T}(g) = \dfrac{1}{|\partial T|} \displaystyle\int_{\partial T} g(y)\, d\sigma(y)$.

Proof: Due to density properties, it is enough to prove this estimate for functions in $D(\mathbb{R}^N)$. Let $\varphi \in D(\mathbb{R}^N)$, one has

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N \times \partial T} g(y)\, \mathcal{T}_\varepsilon^b(\varphi)(x,y)\ dx\, d\sigma(y) \right| = \\
&\qquad = \left| \int_{\mathbb{R}^N \times \partial T} g(y)\, \varphi\left(\varepsilon \left[\frac{x}{\varepsilon} \right]_Y + \varepsilon y \right)\ dx\, d\sigma(y) \right| \\
&\qquad \leq \left| \int_{\mathbb{R}^N \times \partial T} g(y)\, \varphi\left(\varepsilon \left[\frac{x}{\varepsilon} \right]_Y \right)\ dx\, d\sigma(y) \right| \\
&\qquad\quad + \left| \int_{\mathbb{R}^N \times \partial T} g(y) \left(\varphi\left(\varepsilon \left[\frac{x}{\varepsilon} \right]_Y + \varepsilon y \right) - \varphi\left(\varepsilon \left[\frac{x}{\varepsilon} \right]_Y \right) \right)\ dx\, d\sigma(y) \right| \\
&\qquad \leq C \left(|\mathcal{M}_{\partial T}(g)|\, \|\varphi\|_{L^2(\Omega^\varepsilon)} + \varepsilon\, \|g\|_{L^2(\partial T)}\, \|\nabla \varphi\|_{(L^2(\Omega^\varepsilon))^N} \right) .
\end{aligned}$$

The desired result is then straightforward by using the Poincaré inequality. ∎

Corollary 5.4. *Let $g \in L^2(\partial T)$ be a Y-periodic function, and set $g^\varepsilon(x) = g\left(\dfrac{x}{\varepsilon}\right)$ for all $x \in \mathbb{R}^N \setminus \bigcup_{\xi \in \mathbb{Z}^N} \varepsilon(\xi + T)$. Then, for all $\varphi \in H^1(\Omega)$, one has*

$$\left| \int_{\partial T^\varepsilon} g^\varepsilon(x)\, \varphi(x)\, d\sigma(x) \right| \leq \frac{C}{\varepsilon} \left(|\mathcal{M}_{\partial T}(g)| + \varepsilon \right) \|\nabla \varphi\|_{(L^2(\Omega^\varepsilon))^N} .$$

Proof: The proof follows from 2 and 5 in Proposition 5.2 and Proposition 5.3. ∎

Remark 5.5. This result allows in particular to prove, in a much easier way than usual, accurate a priori estimates for several kinds of boundary conditions in perforated domains, as done for instance in Section 6 where we study an elliptic problem with Robin boundary condition. A priori estimates for this type of problems have been previously obtained in literature (see [6] for instance) by means of a suitable auxiliary problem due to Vanninathan [14], [15], allowing to transform surface integrals into volume integrals. □

Proposition 5.6. *Let $g\in L^2(\partial T)$ be a Y-periodic function, and set $g^\varepsilon(x)= g\Big(\dfrac{x}{\varepsilon}\Big)$. One has the following convergence results as $\varepsilon \to 0$:*

1. *If $\mathcal{M}_{\partial T}(g) \neq 0$, then*

$$\varepsilon \int_{\partial T^\varepsilon} g^\varepsilon(x)\,\varphi(x)\,d\sigma(x) \;\to\; \frac{|\partial T|}{|Y|}\,\mathcal{M}_{\partial T}(g) \int_\Omega \varphi(x)\,dx\;, \qquad \forall \varphi \in H^1(\Omega)\;.$$

2. *If $\mathcal{M}_{\partial T}(g) = 0$, then*

$$\int_{\partial T^\varepsilon} g^\varepsilon(x)\,\varphi(x)\,d\sigma(x) \;\to\; 0\;, \qquad \forall \varphi \in H^1(\Omega)\;.$$

Proof: We prove these two assertions for all $\varphi \in D(\mathbb{R}^N)$ and then we pass to the desired ones by density.

1. One has by unfolding

$$\begin{aligned}\varepsilon \int_{\partial T^\varepsilon} g^\varepsilon(x)\,\varphi(x)\,d\sigma(x) \;&=\; \varepsilon\,\frac{1}{\varepsilon|Y|} \int_{\widetilde{\Omega^\varepsilon}\times\partial T} \mathcal{T}^b_\varepsilon(g^\varepsilon)(x,y)\,\mathcal{T}^b_\varepsilon(\varphi)(x,y)\;dx\,d\sigma(y)\\ &=\; \frac{1}{|Y|} \int_{\widetilde{\Omega^\varepsilon}\times\partial T} g(y)\,\mathcal{T}^b_\varepsilon(\varphi)(x,y)\;dx\,d\sigma(y)\;.\end{aligned}$$

When $\varepsilon \to 0$, we obtain

$$\varepsilon\int_{\partial T^\varepsilon} g^\varepsilon(x)\,\varphi(x)\,d\sigma(x) \to \frac{1}{|Y|}\int_{\Omega\times\partial T} g(y)\,\varphi(x)\,dx\,d\sigma(y) = \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)\int_\Omega \varphi(x)\,dx\;.$$

2. As in the proof of Proposition 5.3, we have

$$\begin{aligned}\left|\int_{\partial T^\varepsilon} g^\varepsilon(x)\,\varphi(x)\,d\sigma(x)\right| \;\le\; &\frac{C}{\varepsilon}\left|\int_{\mathbb{R}^N\times\partial T} g(y)\,\varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y\right)\;dx\,d\sigma(y)\right|\\ &+\; C\left|\int_{\mathbb{R}^N\times\partial T} g(y)\frac{\varphi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big]_Y+\varepsilon y\Big)-\varphi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big]_Y\Big)}{\varepsilon}\;dx\,d\sigma(y)\right|.\end{aligned}$$

Observe first that

$$\int_{\mathbb{R}^N\times\partial T} g(y)\,\varphi\Big(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y\Big)\;dx\,d\sigma(y) \;=\; \int_{\partial T} g(y)\,d\sigma(y) \int_{\mathbb{R}^N} \varphi\Big(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y\Big)\;dx \;=\; 0\;,$$

since $\mathcal{M}_{\partial T}(g) = 0$. On the other hand

$$\begin{aligned}&\int_{\mathbb{R}^N\times\partial T} g(y)\,\frac{\varphi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y+\varepsilon y\right)-\varphi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y\right)}{\varepsilon}\;dx\,d\sigma(y) \;=\\ &\qquad=\int_{\mathbb{R}^N\times\partial T} y\,g(y)\,\frac{\varphi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y+\varepsilon y\right)-\varphi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y\right)}{\varepsilon y}\;dx\,d\sigma(y)\;.\end{aligned}$$

When passing to the limit as $\varepsilon \to 0$, and since $\varphi \in D(\mathbb{R}^N)$, this integral goes to

$$\int_{\mathbb{R}^N \times \partial T} y\, g(y)\, \nabla\varphi(x)\ dx\, d\sigma(y) \;=\; \int_{\mathbb{R}^N} \Big(\int_{\partial T} y\, g(y)\, d\sigma(y) \Big) \nabla\varphi(x)\, dx \;=\; 0\ . \ \blacksquare$$

The next result is the equivalent of Propositions 2.6(1) and 2.7, to the case of functions defined on the boundaries of the holes.

Proposition 5.7.

1. *Let $\varphi \in L^2(\Omega)$. Then, as $\varepsilon \to 0$, one has the convergence*

$$\int_{\mathbb{R}^N \times \partial T} \mathcal{T}^b_\varepsilon(\varphi)(x,y)\, dx\, d\sigma(y) \;\to\; \int_{\mathbb{R}^N \times \partial T} \widetilde{\varphi}\ dx\, d\sigma(y)\ .$$

2. *Let $\varphi \in L^2(\Omega)$. Then,*

$$\mathcal{T}^b_\varepsilon(\varphi) \to \widetilde{\varphi} \quad \textit{strongly in}\ \ L^2(\mathbb{R}^N \times \partial T)\ .$$

3. *Let φ^ε be in $L^2(\partial T^\varepsilon)$ for every ε, such that*

$$\mathcal{T}^b_\varepsilon(\varphi^\varepsilon) \rightharpoonup \widehat{\varphi} \quad \textit{weakly in}\ \ L^2(\mathbb{R}^N \times \partial T)\ .$$

Then,

$$\varepsilon \int_{\partial T^\varepsilon} \varphi^\varepsilon \psi\, d\sigma(x) \;\to\; \frac{1}{|Y|} \int_{\mathbb{R}^N \times \partial T} \widehat{\varphi}(x,y)\, \psi(x)\ dx\, d\sigma(y)\ , \qquad \forall\, \psi \in H^1(\Omega)\ .$$

4. *Let φ^ε be in $H^1(\Omega^\varepsilon)$ for every ε and $\widehat{\varphi} \in H^1(\Omega)$ such that*

then

$$\mathcal{T}_\varepsilon(\varphi^\varepsilon) \rightharpoonup \widehat{\varphi} \quad \textit{weakly in}\ \ L^2_{loc}(\Omega; H^1(Y^\star))\ ,$$

$$\mathcal{T}^b_\varepsilon(\varphi^\varepsilon) \rightharpoonup \widehat{\varphi} \quad \textit{weakly in}\ \ L^2_{loc}(\Omega; H^{\frac{1}{2}}(\partial T))\ .$$

Proof: **1**. For every $\varphi \in D(\Omega)$, one has

$$\int_{\mathbb{R}^N \times \partial T} \mathcal{T}^b_\varepsilon(\varphi)(x,y)\, dx\, d\sigma(y) \;=\; \varepsilon |Y| \int_{\partial T^\varepsilon} \varphi(x)\, dx\ .$$

Using 1 of Proposition 5.6 for $g = 1$, this integral goes, when $\varepsilon \to 0$, to the following limit:

$$|Y| \frac{|\partial T|}{|Y|}\, \mathcal{M}_{\partial T}(1) \int_\Omega \varphi(x)\, dx\ ,$$

and this is exactly

$$\int_{\mathbb{R}^N \times \partial T} \widetilde{\varphi}\ dx\, d\sigma(y)\ .$$

This result stands for every $\varphi \in L^2(\Omega)$ by density.

2. We get the result by using the same arguments as in 1 of Proposition 2.6.

3. Let $\psi \in D(\Omega)$. One has successively

$$\begin{aligned}\int_{\partial T^\varepsilon} \varepsilon\varphi^\varepsilon\psi\ d\sigma(x) &= \frac{1}{|Y|}\int_{\mathbb{R}^N\times\partial T} \mathcal{T}_\varepsilon^b(\varphi^\varepsilon\psi)\ dx\,d\sigma(y)\\ &= \frac{1}{|Y|}\int_{\mathbb{R}^N\times\partial T} \mathcal{T}_\varepsilon^b(\varphi^\varepsilon)\,\mathcal{T}_\varepsilon(\psi)\ dx\,d\sigma(y)\ .\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ yields

$$\int_{\partial T^\varepsilon} \varepsilon\varphi^\varepsilon\psi\ d\sigma(x) \to \frac{1}{|Y|}\int_{\mathbb{R}^N\times\partial T} \widehat{\varphi}(x,y)\,\psi(x)\ dx\,d\sigma(y)\ .$$

The result is valid for all $\psi \in H^1(\Omega)$ by density.

4. Straightforward from the definition of the unfolding operators and the Sobolev injections. ∎

6 – Application: homogenization of a Robin problem

Hereby, we apply the periodic unfolding method to an elliptic problem with Robin boundary conditions in a perforated domain. More general Robin boundary conditions will be treated in a forecoming paper.

We start by defining the following space:

$$V_\varepsilon = \Big\{\varphi \in H^1(\Omega^\varepsilon) \mid \varphi = 0 \text{ on } \partial\Omega^\varepsilon \backslash \partial T^\varepsilon_{int}\Big\}\ ,$$

where T^ε_{int} is the set of holes that do not intersect the boundary $\partial\Omega$.

Consider the problem

$$\text{(6)}\qquad \begin{cases} -\mathrm{div}(A^\varepsilon\nabla u^\varepsilon) = f & \text{in } \Omega^\varepsilon\\ A^\varepsilon\nabla u^\varepsilon\cdot n + h\,\varepsilon u^\varepsilon = \varepsilon g^\varepsilon & \text{on } \partial T^\varepsilon_{int}\\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon\backslash\partial T^\varepsilon_{int}\end{cases}$$

where

1. h is a real positive number,

2. A^ε is a matrix defined by

$$A^\varepsilon(x) = \big(a^\varepsilon_{ij}(x)\big)_{1\le i,j\le N} \quad \text{a.e. on } \Omega\ ,$$

such that

- A^ε is measurable and bounded in $L^\infty(\Omega)$,
- for every $\lambda \in \mathbb{R}^N$, one has

$$(A^\varepsilon(x)\lambda, \lambda) \geq \alpha|\lambda|^2$$

where $\alpha > 0$ is a constant independent of ε,
- there exists a constant $\beta > 0$ such that

$$|A^\varepsilon(x)\lambda| \leq \beta|\lambda| \ , \qquad \forall \lambda \in \mathbb{R}^{\mathbb{N}} \ ,$$

3. $f \in L^2(\Omega)$,
4. $g^\varepsilon(x) = g\Big(\dfrac{x}{\varepsilon}\Big)$ where g is a Y-periodic function in $L^2(\partial T)$,
5. n is the exterior unit normal to Ω^ε.

Let us suppose that

(H_1) If $h = 0$ and $g \equiv 0$, we have the uniform (with respect to ε) Poincaré inequality in V_ε.
(H_2) If $h \neq 0$ or $g \not\equiv 0$, T has a Lipschitz boundary.

Observe that these hypotheses are weaker than the ones normally made when using other homogenization methods.

Remark 6.1. Assumption (H_2) is needed for writing integrals on the boundary of the holes. It also implies (H_1) since it guarantees the existence of a uniform extension operator (see [3], [9] for details). □

Remark 6.2. Under these hypotheses we can treat the case of some fractal holes like the two dimensional snowflake (see [16]). □

Remark 6.3. Assumption (H_1) is essential in order to give a priori estimates in $H^1(\Omega^\varepsilon)$. If we add a zero order term in the equation (6)-1 we do not need it. □

The variational formulation of (6) is

$$(7) \qquad \begin{cases} \text{Find } \ u^\varepsilon \in V^\varepsilon \ \text{ solution of} \\[2mm] \displaystyle\int_{\Omega^\varepsilon} A^\varepsilon \, \nabla u^\varepsilon \, \nabla v \; dx \, + \, h\varepsilon \int_{\partial T^\varepsilon} u^\varepsilon \, v \; d\sigma(x) \;\; = \\[2mm] \displaystyle\qquad\qquad = \int_{\Omega^\varepsilon} f v \, dx \, + \, \varepsilon \int_{\partial T^\varepsilon} g^\varepsilon v \, d\sigma(x) \qquad \text{for every } \ v \in V^\varepsilon \ . \end{cases}$$

Theorem 6.4. *Let u^ε be the solution of (6). Under the assumptions listed above, we suppose that*

$$\mathcal{T}_\varepsilon(A^\varepsilon) \to A \quad \text{a.e. in } \Omega\times Y^\star . \tag{8}$$

Then, there exists $u^0 \in H_0^1(\Omega)$ such that, up to a subsequence

$$\widetilde{u^\varepsilon} \rightharpoonup \theta u^0 \quad \text{weakly in } L^2(\Omega) , \tag{9}$$

where u^0 is the unique solution of the problem

$$\begin{cases} -\operatorname{div}\big(A^0(x)\,\nabla u^0\big) + h\,\dfrac{|\partial T|}{|Y|}\,u^0 \;=\; \theta f + \dfrac{|\partial T|}{|Y|}\,\mathcal{M}_{\partial T}(g) & \text{in } \Omega \\[2ex] u^0 = 0 & \text{on } \partial\Omega \end{cases} \tag{10}$$

and $A^0(x) = (a^0_{ij}(x))_{1\le i,j\le N}$ is the constant matrix defined by

$$a^0_{ij}(x) \;=\; \frac{1}{|Y|}\sum_{k=1}^{N}\int_{Y^\star}\Big(a_{ij}(x,y) - a_{ik}(x,y)\,\frac{\partial\widehat{\chi}^j}{\partial y_k}(y)\Big)\,dy\ . \tag{11}$$

The correctors $\widehat{\chi}^j$, $j = 1,\cdots,N$, are the solutions of the cell problem

$$\begin{cases} \displaystyle\int_{Y^\star} A(x,y)\,\nabla(\widehat{\chi}^j - y_j)\,\nabla\varphi\;dy \;=\; 0 \qquad \forall\varphi\in H^1_{per}(Y^\star) \\[2ex] \widehat{\chi}^j \quad Y\text{-periodic} \\[1ex] \mathcal{M}_{Y^\star}(\widehat{\chi}^j) = 0\ . \end{cases} \tag{12}$$

Furthermore, there exists $\widehat{u} \in L^2(\Omega; H^1_{per}(Y^\star))$ such that, up to subsequences

$$\mathcal{T}_\varepsilon(u^\varepsilon) \rightharpoonup u^0 \quad \text{weakly in } L^2_{loc}(\Omega; H^1(Y^\star)) , \tag{13}$$

$$\frac{1}{\varepsilon}\,\mathcal{T}_\varepsilon(\mathcal{R}_\varepsilon u^\varepsilon) \rightharpoonup \widehat{u} \quad \text{weakly in } L^2_{loc}(\Omega; H^1(Y^\star)) , \tag{14}$$

$$\mathcal{T}_\varepsilon(\nabla u^\varepsilon) \rightharpoonup \nabla_x u^0 + \nabla_y\widehat{u} \quad \text{weakly in } L^2_{loc}(\Omega; L^2(Y^\star)) , \tag{15}$$

where $(u^0, \widehat{u})$ is the unique solution of the problem

$$\begin{cases} \forall\varphi\in H^1_0(\Omega),\quad \forall\psi\in L^2(\Omega; H^1_{per}(Y^\star)) \\[1ex] \displaystyle\int_{\Omega\times Y^\star} A(x,y)\,(\nabla_x u^0 + \nabla_y\widehat{u})\,\big(\nabla_x\varphi(x) + \nabla_y\psi(x,y)\big)\,dx\,dy \;+\; h|\partial T|\int_\Omega u^0\varphi\,dx \;= \\ \displaystyle\qquad\qquad = \;|Y^\star|\int_\Omega f\varphi\,dx \;+\; \int_\Omega\varphi\,dx\int_{\partial T} g\,d\sigma(y)\ . \end{cases} \tag{16}$$

Remark 6.5. Observe that both f and g appear in the limit problem. □

Proof of Theorem 6.4: We proceed in five steps.

First step. We start by establishing a priori estimates of u^ε, solution to problem (6). Considering u^ε as a test function in (7), one has

$$\|\nabla u^\varepsilon\|^2_{(L^2(\Omega^\varepsilon))^N} + h\,\varepsilon\|u^\varepsilon\|^2_{L^2(\partial T^\varepsilon)} \le \|f\|_{L^2(\Omega)}\,\|\nabla u^\varepsilon\|_{(L^2(\Omega^\varepsilon))^N} + \varepsilon\left|\int_{\partial T^\varepsilon} g^\varepsilon u^\varepsilon\, d\sigma(x)\right| .$$

Then, by using the uniform Poincaré inequality (H_1) and Proposition 5.4, we derive

$$\|\nabla u^\varepsilon\|^2_{(L^2(\Omega^\varepsilon))^N} + h\,\varepsilon\|u^\varepsilon\|^2_{L^2(\partial T^\varepsilon)} \le C\left(1+\varepsilon+|\mathcal{M}_{\partial T}(g)|\right)\|\nabla u^\varepsilon\|_{(L^2(\Omega^\varepsilon))^N} .$$

We deduce that

$$\|u^\varepsilon\|_{H^1(\Omega^\varepsilon)} \le C . \tag{17}$$

Thus, there exists $U^0 \in H^1(\Omega)$ such that

$$\widetilde{u^\varepsilon} \rightharpoonup U^0 \quad \text{weakly in } \; L^2(\Omega) .$$

Second step. In view of 2,3 and 4 of Theorem 3.2, there exists some $u^0 \in H^1_0(\Omega)$ and $\widehat{u} \in L^2(\Omega; H^1_{per}(Y^\star))$ such that

- $\mathcal{T}_\varepsilon(u^\varepsilon) \rightharpoonup u^0$ weakly in $L^2_{loc}(\Omega; H^1(Y^\star))$,
- $\dfrac{1}{\varepsilon}\,\mathcal{T}_\varepsilon(\mathcal{R}_\varepsilon(u^\varepsilon)) \rightharpoonup \widehat{u}$ weakly in $L^2_{loc}(\Omega; H^1(Y^\star))$,
- $\mathcal{T}_\varepsilon(\nabla_x(u^\varepsilon)) \rightharpoonup \nabla_x u^0 + \nabla_y \widehat{u}$ weakly in $L^2_{loc}(\Omega\times Y^\star)$.

To identify U^0, for $\varphi \in D(\Omega)$, we have successively

$$\int_\Omega \widetilde{u^\varepsilon}\varphi\, dx = \int_{\Omega^\varepsilon} u^\varepsilon\varphi\, dx = \frac{1}{|Y|}\int_{\Omega\times Y^\star} \mathcal{T}^\varepsilon(u^\varepsilon)\,\mathcal{T}^\varepsilon(\varphi)\, dx\, dy .$$

The former convergences yield

$$\int_\Omega \widetilde{u^\varepsilon}\varphi\, dx \;\to\; \frac{1}{|Y|}\int_{\Omega\times Y^\star} u^0(x)\,\varphi(x)\, dx\, dy = \frac{|Y^\star|}{|Y|}\int_\Omega u^0\varphi\, dx .$$

But $\displaystyle\int_\Omega \widetilde{u^\varepsilon}\varphi\, dx \to \int_\Omega U^0\varphi\, dx$ when ε goes to 0. Consequently

$$U^0 = \theta\, u^0 .$$

We also deduce that u^0 is a function of x only.

Third step. We now prove (16). With $v = \varphi$, $\varphi \in D(\Omega)$, as a test function in (7) we have

$$\int_{\Omega^\varepsilon} A^\varepsilon \nabla u^\varepsilon \nabla \varphi \, dx + h\varepsilon \int_{\partial T^\varepsilon} u^\varepsilon \varphi \, d\sigma(x) = \int_{\Omega^\varepsilon} f\varphi \, dx + \varepsilon \int_{\partial T^\varepsilon} g^\varepsilon \varphi \, d\sigma(x) .$$

By unfolding, and using Propositions 2.5 and 5.2, we get

$$\begin{aligned}\int_{\widetilde{\Omega}^\varepsilon \times Y^\star} \mathcal{T}_\varepsilon(A^\varepsilon)\, \mathcal{T}_\varepsilon(\nabla u^\varepsilon)\, \mathcal{T}_\varepsilon(\nabla \varphi) \, dx\, dy + h \int_{\widetilde{\Omega}^\varepsilon \times \partial T} \mathcal{T}^b_\varepsilon(u^\varepsilon)\, \mathcal{T}^b_\varepsilon(\varphi) \, dx\, d\sigma(y) &= \\ = \int_{\widetilde{\Omega}^\varepsilon \times Y^\star} \mathcal{T}_\varepsilon(f)\, \mathcal{T}_\varepsilon(\varphi) \, dx\, dy + \int_{\widetilde{\Omega}^\varepsilon \times \partial T} \mathcal{T}^b_\varepsilon(g^\varepsilon)\, \mathcal{T}^b_\varepsilon(\varphi) \, dx\, d\sigma(y) &.\end{aligned}$$

In view of (8), Proposition 2.6(1) and Proposition 5.7(2,4), we obtain when passing to the limit

$$\begin{aligned}\int_{\Omega \times Y^\star} A(x,y)\, (\nabla_x u^0 + \nabla_y \widehat{u})\, \nabla \varphi(x) \, dx\, dy + h \int_{\Omega \times \partial T} u^0 \varphi \, dx\, d\sigma(y) &= \\ = \int_{\Omega \times Y^\star} f\varphi \, dx\, dy + \int_{\Omega \times \partial T} g\varphi \, dx\, d\sigma(y) &.\end{aligned}$$

Since u^0, f and φ are functions of x only, we actually have

$$\begin{aligned}\int_{\Omega \times Y^\star} A(x,y)\, (\nabla_x u^0 + \nabla_y \widehat{u})\, \nabla \varphi(x) \, dx\, dy + h|\partial T| \int_\Omega u^0 \varphi \, dx &= \\ = |Y^\star| \int_\Omega f\varphi \, dx + \int_\Omega \varphi \, dx \int_{\partial T} g \, d\sigma(y) &,\end{aligned} \tag{18}$$

By density, this result is still valid for every $\varphi \in H^1_0(\Omega)$.

We take now as a test function in (7) the function v^ε defined by

$$v^\varepsilon(x) = \varepsilon\, \varphi(x)\, \xi\Big(\frac{x}{\varepsilon}\Big) ,$$

where

$$\varphi \in D(\Omega) \quad \text{and} \quad \xi \in H^1_{per}(Y^\star) .$$

First of all, observe that

$$\mathcal{T}_\varepsilon(v^\varepsilon) = \varepsilon\, \mathcal{T}_\varepsilon(\varphi)\, \xi ,$$

$$\nabla v^\varepsilon = \varepsilon\, \nabla \varphi\, \xi\Big(\frac{\cdot}{\varepsilon}\Big) + \varphi\, \nabla_y \xi\Big(\frac{\cdot}{\varepsilon}\Big) .$$

Hence

$$\mathcal{T}_\varepsilon(v^\varepsilon) \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega; H^1(Y^\star)) ,$$

and

$$\mathcal{T}_\varepsilon(\nabla v^\varepsilon) \rightharpoonup \varphi \nabla_y \xi \quad \text{weakly in } \ L^2(\Omega \times Y^\star) \ .$$

By unfolding, one obtains

$$\int_{\widetilde{\Omega}^\varepsilon \times Y^\star} \mathcal{T}_\varepsilon(A^\varepsilon)\, \mathcal{T}_\varepsilon(\nabla u^\varepsilon)\, \mathcal{T}_\varepsilon(\nabla v^\varepsilon)\ dx\, dy \ + \ h \int_{\widetilde{\Omega}^\varepsilon \times \partial T} \mathcal{T}^b_\varepsilon(u^\varepsilon)\, \mathcal{T}^b_\varepsilon(v^\varepsilon)\ dx\, d\sigma(y) \ = \\ = \int_{\widetilde{\Omega}^\varepsilon \times Y^\star} \mathcal{T}_\varepsilon(f)\, \mathcal{T}_\varepsilon(v^\varepsilon)\ dx\, dy \ + \int_{\widetilde{\Omega}^\varepsilon \times \partial T} \mathcal{T}^b_\varepsilon(g^\varepsilon)\, \mathcal{T}^b_\varepsilon(v^\varepsilon)\ dx\, d\sigma(y)\ ,$$

which gives by passing to the limit

$$\int_{\Omega \times Y^\star} A(x,y)\, (\nabla_x u^0 + \nabla_y \widehat{u})\, \varphi(x)\, \nabla_y \xi(y)\ dx\, dy \ = \ 0\ .$$

By density, we get

$$\int_{\Omega \times Y^\star} A(x,y)\, (\nabla_x u^0 + \nabla_y \widehat{u})\, \nabla_y \psi(x,y)\ dx\, dy \ = \ 0\ , \tag{19}$$

for every $\psi \in L^2(\Omega; H^1_{per}(Y^\star))$.

Finally, by summing (18) (for $\varphi \in H^1_0(\Omega)$) and (19), we obtain (16).

Fourth step. The proof of the fact that u^0 is a solution to (10) follows along the lines of the proof in [5, Chapter 9]. Taking successively $\varphi = 0$ and $\psi = 0$ in (16) yields (see [5] for details)

$$\widehat{u}(x,y) \ = \ -\sum_{j=1}^N \widehat{\chi}^j(y)\, \frac{\partial u_0}{\partial x_j}(x) + \widetilde{u}_1(x)\ , \tag{20}$$

where $\widetilde{u}_1$ is independent of y and $\widehat{\chi}^j$ is the solution to (12).

Replacing $\widehat{u}$ by its value found in (20), and using a simple change of indices, yield

$$-\sum_{i,j,k=1}^N \left[\frac{\partial}{\partial x_i} \int_{Y^\star} \left(a_{ik}(x,y) - a_{ij}(x,y)\, \frac{\partial \widehat{\chi}^k}{\partial y_j} \right) dy \right] \frac{\partial u_0}{\partial x_k} \ + \ h|\partial T| u^0 \ = \\ = \ |Y^\star| f \ + \int_{\partial T} g\, d\sigma(y)\ ,$$

which can be written in the form (10) with a^0_{ij} defined by (11).

Fifth step. By a standard argument (cf. [2], [5]), it is easily seen that the matrix A^0 is elliptic. Then, the uniqueness of u^0 as solution of (10) is a consequence of the Lax–Milgram theorem. ∎

We end this paper with a corrector result, which makes use of the operator $\mathcal{U}_\varepsilon$ introduced in Section 4.

Proposition 6.6. *Under the same hypotheses as in Theorem 6.4, if there exists an extension operator $\mathcal{P}^\varepsilon \in \mathcal{L}(V_\varepsilon; H_0^1(\Omega))$ verifying*

$$\|\nabla \mathcal{P}^\varepsilon u^\varepsilon\|_{(L^2(\Omega))^N} \le C\, \|\nabla u^\varepsilon\|_{(L^2(\Omega^\varepsilon))^N} .$$

Then,

1. $\mathcal{P}^\varepsilon u^\varepsilon \rightharpoonup u^0$ *weakly in* $H_0^1(\Omega)$,
2. $\mathcal{T}_\varepsilon(\nabla u^\varepsilon) \to \nabla_x u^0 + \nabla_y \widehat{u}$ *strongly in* $L^2_{loc}(\Omega; L^2(Y^\star))$,
3. $\widetilde{\nabla u^\varepsilon} - \nabla_x u^0 - \mathcal{U}_\varepsilon(\nabla_y \widehat{u}) \to 0$ *strongly in* $L^2_{loc}(\Omega; L^2(Y^\star))$.

Proof: **1**. Standard arguments give the result.

2. First, we prove this result in the case $h = 0$ and $g \equiv 0$.
Let $w \subset\subset \Omega$, and $v \in D(\mathbb{R}^{\mathrm{N}})$ such that

$$v \ge 0 \quad \text{and} \quad v = 1 \ \text{ on } \ w .$$

By using in (16) the functions

$$\varphi(x) = v(x)\, u^0(x) \quad \text{and} \quad \psi(x,y) = v(x)\, \widehat{u}(x,y) ,$$

one gets

$$\begin{aligned} |Y^\star| \int_\Omega f u^0 v \, dx &= \int_{\Omega\times Y^\star} A\left(\nabla_x u^0 + \nabla_y \widehat{u}\right)\left(\nabla_x(u^0 v) + \nabla_y(v\widehat{u})\right) dx\, dy \\ &= \int_{\Omega\times Y^\star} A\left(\nabla_x u^0 + \nabla_y \widehat{u}\right)\left(u^0\, \nabla_x v + (\nabla_x u^0 + \nabla_y \widehat{u})\, v\right) dx\, dy \\ &= \int_{\Omega\times Y^\star} A\left(\nabla_x u^0 + \nabla_y \widehat{u}\right) u^0\, \nabla_x v \; dx\, dy \\ &\quad + \int_{\Omega\times Y^\star} A\left(\nabla_x u^0 + \nabla_y \widehat{u}\right)\left(\nabla_x u^0 + \nabla_y \widehat{u}\right) v \; dx\, dy . \end{aligned} \tag{21}$$

On the other hand, using (7) and (9)

$$\begin{aligned} |Y^\star| \int_\Omega f u^0 v \, dx &= |Y^\star| \lim_{\varepsilon\to 0} \frac{1}{\theta} \int_{\Omega^\varepsilon} f u^\varepsilon v \, dx \\ &= |Y| \lim_{\varepsilon\to 0} \int_{\Omega^\varepsilon} f u^\varepsilon v \, dx \; = \end{aligned} \tag{22}$$

$$= |Y| \lim_{\varepsilon\to 0} \int_{\Omega^\varepsilon} A^\varepsilon \nabla_x u^\varepsilon \nabla_x (u^\varepsilon v)\, dx$$
$$= |Y| \lim_{\varepsilon\to 0} \left(\int_{\Omega^\varepsilon} A^\varepsilon \nabla_x u^\varepsilon\, u^\varepsilon \nabla_x v\, dx + \int_{\Omega^\varepsilon} A^\varepsilon \nabla_x u^\varepsilon \nabla_x u^\varepsilon v\, dx \right) .$$

From 1 and Proposition 2.6(3), we deduce that

$$\mathcal{T}_\varepsilon(\chi^\varepsilon \mathcal{P}^\varepsilon u^\varepsilon) = \mathcal{T}_\varepsilon(u^\varepsilon) \to u^0 \quad \text{strongly in } L^2_{loc}(\Omega; H^1(Y^\star)) .$$

Hence, using (8), (15) and Proposition 2.5, we have

$$|Y| \lim_{\varepsilon\to 0} \int_{\Omega^\varepsilon} A^\varepsilon \nabla_x u^\varepsilon\, u^\varepsilon \nabla_x v\, dx = \lim_{\varepsilon\to 0} \int_{\Omega} A\, \mathcal{T}_\varepsilon(\nabla_x u^\varepsilon)\, \mathcal{T}_\varepsilon(u^\varepsilon)\, \nabla_x v\, dx\, dy$$
$$= \int_{\Omega\times Y^\star} A \left(\nabla_x u^0 + \nabla_y \widehat{u}\right) u^0\, \nabla_x v\, dx\, dy .$$

This, with (21) and (22), gives

$$|Y| \lim_{\varepsilon\to 0} \int_{\Omega^\varepsilon} A^\varepsilon \nabla_x u^\varepsilon \nabla_x u^\varepsilon v\, dx = \int_{\Omega\times Y^\star} A \left(\nabla_x u^0 + \nabla_y \widehat{u}\right) \left(\nabla_x u^0 + \nabla_y \widehat{u}\right) v\, dx\, dy , \tag{23}$$

which means, by using 5 of Proposition 2.5, that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N\times Y^\star} A\, \mathcal{T}_\varepsilon(\nabla_x u^\varepsilon)\, \mathcal{T}_\varepsilon(\nabla_x u^\varepsilon)\, \mathcal{T}_\varepsilon(v)\, dx\, dy =$$
$$= \int_{\Omega\times Y^\star} A\left(\nabla_x u^0 + \nabla_y \widehat{u}\right) \left(\nabla_x u^0 + \nabla_y \widehat{u}\right) v\, dx\, dy .$$

Finally, using (15) and the ellipticity of A, and passing to the limit as $\varepsilon \to 0$, yield

$$\int_{w\times Y^\star} \left(\mathcal{T}_\varepsilon(\nabla_x u^\varepsilon) - \widetilde{\nabla_x u^0} - \nabla_y \widehat{u} \right)^2 dx\, dy \le$$
$$\le \int_{\mathbb{R}^N\times Y^\star} v \left(\mathcal{T}_\varepsilon(\nabla_x u^\varepsilon) - \widetilde{\nabla_x u^0} - \nabla_y \widehat{u} \right)^2 dx\, dy$$
$$\le \frac{1}{\alpha} \int_{\mathbb{R}^N\times Y^\star} v\, A \left(\mathcal{T}_\varepsilon(\nabla_x u^\varepsilon) - \widetilde{\nabla_x u^0} - \nabla_y \widehat{u} \right) \left(\mathcal{T}_\varepsilon(\nabla_x u^\varepsilon) - \widetilde{\nabla_x u^0} - \nabla_y \widehat{u} \right) dx\, dy \;\to\; 0 .$$

If $h \neq 0$ or $g \not\equiv 0$, boundary terms appear in (21) and (22). They can be treated as in the proof of Theorem 6.4, to obtain (23). Then, we argue as in the previous case to obtain the result.

3. Combining 2 and Theorem 4.5(2), we have

$$\nabla u^\varepsilon - \mathcal{U}_\varepsilon(\nabla_x u^0) - \mathcal{U}_\varepsilon(\nabla_y \widehat{u}) \to 0 \quad \text{strongly in } \; L^2_{loc}(\Omega; L^2(Y^\star)) \; .$$

Then, by using 1 of Proposition 4.4 we obtain the desired result. ∎

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