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OPTIMAL CONTROL FOR A NONLINEAR POPULATION DYNAMICS PROBLEM

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Abstract: We consider a nonlinear age dependent and spatially structured population dynamics model. In this model the birth process is nonlocal and we investigate the existence of an optimal control which makes the density of the population as close as possible of some given density. After necessary optimality conditions in the case where the birth process is governed by the function $F(\alpha) = c\alpha^2/(b+\alpha^2)$ are investigated.

1 – Introduction

We consider a population with age dependence and spatial structure, and we assume that the population lives in a bounded domain $\Omega \subset \mathbb{R}^m$, $m = 1, 2$ or 3.

Let $y(t, a, x)$ be the distribution of individuals of age a at time t and location $x \in \Omega$, $\mu(a)$ and $\beta(a)$ respectively the natural death rate and the natural fertility rate of individuals of age a. Let $\partial/\partial\eta$ be the normal derivative oriented towards the exterior of Ω, ω be the maximal age of an individual and T a strictly positive real.

Let $J(v) = \|y(v) - z_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(\Sigma)}$, $v \in U_{ad}$, where

$$U_{ad} = \Big\{ v \in L^2(]0,T[\times]0,\omega[\times\Gamma),\ \ v \geq 0 \ \text{ a.e. in } \Sigma \Big\}$$

is the set of admissible controls, N a strictly positive real, and $y(v)$ the solution

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of the following system:

$$
(1.1)\qquad \begin{cases} \partial y/\partial t + \partial y/\partial a - \Delta y + \mu y \;=\; 0 & \text{in } Q \\ \partial y/\partial \eta \;=\; v & \text{on } \Sigma \\ y(0,a,x) \;=\; y_0(a,x) & \text{in } Q_\omega \\ y(t,0,x) \;=\; F\Big(\displaystyle\int_0^\omega \beta\, y\, da\Big) & \text{in } Q_T \end{cases}
$$

where $Q =]0,T[\times]0,\omega[\times\Omega$, $Q_T =]0,T[\times\Omega$, $Q_\omega =]0,\omega[\times\Omega$, $\Sigma =]0,T[\times]0,\omega[\times\Gamma$ and Δ the laplacian with respect to the variable x. In the sequel we shall denote by ∇ the gradient with respect to the variable x. Here, F is the function which governes the birth process. From the biological point of view, $\int_0^\omega \beta y(t,a,x)\,da$ is the distribution of newborn individuals at time t and location x, and $F(\int_0^\omega \beta y(t,a,x)\,da)$ is the proportion of those individuals which attains the minimum age of the species. In this paper, we investigate an optimal boundary control which makes the distribution of individuals as close as possible of some suitable density z_d.

In the sequel we study the following problem:

$(\mathbf{P})$: Find $\overline{v} \in U_{ad}$ such that $J(\overline{v}) = \mathrm{Inf}_{v\in U_{ad}} J(v)$.

Optimal harvesting problems for linear age-structured population has been treated extensively in the literature. See for example Barbu [4], Brokate [6], Gurtin et al [10].

Optimal harvesting problem for nonlinear initial value age-structured population was studied later by some authors like B. Ainseba et al [2] and S. Anita [3].

In [7] and [8] Chan and Zhu studied some optimal birth control problems for an age-structured population of Mc-Kendrick type but, without diffusion. In [7], they established maximum principles for problems with free end condition and fixed final horizon. In the same paper, the time optimal control problem, the problem with target set and the infinite planning horizon case are investigated. Further, in [8], Chan and Zhu studied the problem with free final time, phase constraints and min-max costs.

In [1], B. Ainseba proved exact and approximate controllability results for a linear age and space population dynamics structured model by using a derivative of Carlemann inequality.

However, these results may not be applied in our context since here, the system is nonlinear and one want to make the density of the population as close as possible of some suitable density z_d.

The remainder of the paper is organized as follows. In section 2, we give the hypotheses and we examine the existence of an optimal control. In section 3, we investigate necessary optimality conditions.

2 – Existence of an optimal control

For the sequel we assume the following hypotheses

(H_1) Ω is a bounded domain of $\mathbb{R}^m$ with a smooth boundary Γ.

(H_2) $\mu \in L^\infty(0;\omega)$, $\mu \geq 0$ a.e. in $]0;\omega[$.

(H_3) $\beta \in C^1([0;\omega])$, $\beta \geq 0$ a.e. in $[0;\omega]$.

(H_4) $v \in L^2(\Sigma)_+ = \{u \in L^2(\Sigma),\ u \geq 0 \text{ a.e. on} \Sigma\}$.

(H_5) F is a positive lipschitz function such that $F(0) = 0$.

We denote by K_F the lipschitz constant.

(H_6) $y_0 \in L^2(Q_\omega)$ and $y_0 \geq 0$ a.e. in Q_ω.

(H_7) $K_F \displaystyle\int_0^\omega \beta\, da \leq 1$.

In the sequel we shall denote by $d\sigma$ the measure on Γ .

Let $V = H^1(\Omega)$, and

$$W(T,\omega) = \Big\{\theta \in L^2(]0,T[\times]0,\omega[,V);\ \ \partial\theta/\partial t + \partial\theta/\partial a \in L^2(]0,T[\times]0,\omega[,V')\Big\}$$

where $\partial/\partial t$ and $\partial/\partial a$ indicate partial differentiation in the sense of $\mathcal{D}'(]0,T[\times]0,\omega[,V')$ (cf. [14]).

$\forall\, \theta \in W(T,\omega)$, we set:

$$\|\theta\|^2_{W(T,\omega)} = \|\theta\|^2_{L^2(]0,T[\times]0,\omega[,V)} + \|\partial\theta/\partial t + \partial\theta/\partial a\|^2_{L^2(]0,T[\times]0,\omega[,V')} \,.$$

We recall the following well known lemma (cf. [9] or [13]).

Lemma 1. *$W(T,\omega)$ is continuously embedded respectively in $C([0,T],L^2(Q_\omega))$ and in $C([0,\omega],L^2(Q_T))$.*

Let $y \in W(T,\omega)$ then, y has a trace at $t = t_0$ in $L^2(Q_\omega)$ (resp at $a = a_0$ in $L^2(Q_T)$).

Moreover, the "trace applications" are continuous in the strong and weak topology and we have the following formula: $\forall z, \widehat{z} \in W(T,\omega)$

$$\int_0^T\!\!\int_0^\omega \langle \partial z/\partial t + \partial z/\partial a, \widehat{z}\rangle \, dt\, da \; + \int_0^T\!\!\int_0^\omega \langle \partial \widehat{z}/\partial t + \partial \widehat{z}/\partial a, z\rangle \, dt\, da \; = \tag{2.1}$$
$$= \int_{Q_T} (z\widehat{z})\,(t,\omega,x)\; dt\,dx \;-\int_{Q_T} (z\widehat{z})\,(t,0,x)\; dt\,dx$$
$$+\int_{Q_\omega} (z\widehat{z})\,(T,a,x)\; da\,dx \;-\int_{Q_\omega} (z\widehat{z})\,(0,a,x)\; da\,dx\;.$$

Remark 2. Under hypotheses $H_1, ..., H_6$ one can adapt the method of [9], [13] to prove that (1.1) admits a unique solution in the following sense:

$$\langle \partial y/\partial t + \partial y/\partial a, z\rangle_{(H^1(\Omega))', H^1(\Omega)} \; + \int_\Omega [\nabla y.\nabla z + \mu y z]\, dx \;= \int_\Gamma v z_{|\Gamma}\, d\sigma\;,$$
$$\forall z\in H^1(\Omega), \qquad \text{a.e. in}\;\; U =]0,T[\times]0,\omega[\;,$$

$$y(0,a,x) = y_0(a,x) \qquad \text{a.e. in}\;\;]0,\omega[\times\Omega\;,$$
$$y(t,0,x) = F\Big(\int_0^\omega \beta\, y(t,a,x)\, da\Big) \qquad \text{a.e. in}\;\;]0,T[\times\Omega\;.$$

One can see this proof in [15].

Moreover, if $v_1 \le v_2$ a.e. on Σ then $y(v_1) \le y(v_2)$ a.e. in Q [15].

In the same manner, by using the positivity of the solution one can prove like in [9] or [15] that if $y_0^1 \le y_0^2$ (resp. $\mu^2 \le \mu^1;\; f_1 \le f_2,\; 0 \le y_1 \le y_2$) then $y^1 \le y^2$, where y^1 and y^2 are solution of:

$$\begin{cases} \partial y^1/\partial t + \partial y^1/\partial a - \Delta y^1 + \mu y^1 \;=\; f_1 & \text{in}\;\; Q \\ \partial y^1/\partial\eta \;=\; v_1 & \text{on}\;\; \Sigma \\ y^1(0,a,x) \;=\; y_0^1(a,x) & \text{in}\;\; Q_\omega \\ y^1(t,0,x) \;=\; y_1(t,x) & \text{in}\;\; Q_T \end{cases} \tag{2.2}$$

and

$$\begin{cases} \partial y^2/\partial t + \partial y^2/\partial a - \Delta y^2 + \mu y^2 \;=\; f_2 & \text{in}\;\; Q \\ \partial y^2/\partial\eta \;=\; v_2 & \text{on}\;\; \Sigma \\ y^2(0,a,x) \;=\; y_0^2(a,x) & \text{in}\;\; Q_\omega \\ y^2(t,0,x) \;=\; y_2(t,x) & \text{in}\;\; Q_T\;. \end{cases} \tag{2.3}$$

Note that, this is natural, because a strong initial distribution or a weak natural mortality rate of individuals implies a strong density of individuals. □

We have the following result.

Proposition 3. *Under hypotheses $H_1, ..., H_6$, (1.1) admits a unique solution $y(v)$. The map $v \mapsto y(v)$ is globally a lipschitz function on $L^2(\Sigma)$ onto $L^2(]0,T[\times]0,\omega[, H^1(\Omega))$.*

Moreover, under H_7 if $y_0 \in L^\infty(Q_\omega)$, $v_0 \in H^{1/2}(\Gamma)$ and $0 \le v \le v_0$ then $y(v) \in L^\infty(Q)$.

Proof: Thanks to Remark 2, we have to prove that the map $v \mapsto y(v)$ is a globally lipschitz function.

Let $\overline{y} = e^{-\lambda t} y(v)$, where $y(v)$ is the unique solution of (1.1), then $\overline{y}$ is the solution of

$$
(2.4) \qquad \begin{cases} \partial\overline{y}/\partial t + \partial\overline{y}/\partial a - \Delta\overline{y} + (\mu+\lambda)\overline{y} \;=\; 0 & \text{in } Q \\ \partial\overline{y}/\partial\eta \;=\; e^{-\lambda t}\, v & \text{on } \Sigma \\ \overline{y}(0,a,x) \;=\; y_0(a,x) & \text{in } Q_\omega \\ \overline{y}(t,0,x) \;=\; e^{-\lambda t} F\Big(\displaystyle\int_0^\omega \beta\, e^{\lambda t}\overline{y}\, da\Big) & \text{in } Q_T\ . \end{cases}
$$

Let v_1 and v_2 be two controls functions, then $z = \overline{y}(v_1) - \overline{y}(v_2)$ verifies

$$
(2.5) \begin{cases} \partial z/\partial t + \partial z/\partial a - \Delta z + (\mu+\lambda) z \;=\; 0 & \text{in } Q \\ \partial z/\partial\eta \;=\; e^{-\lambda t}(v_1 - v_2) & \text{on } \Sigma \\ z(0,a,x) \;=\; 0 & \text{in } Q_\omega \\ z(t,0,x) \;=\; e^{-\lambda t}\Big[F\Big(\displaystyle\int_0^\omega \overline{\beta}\overline{y}(v_1)\Big) - F\Big(\displaystyle\int_0^\omega \overline{\beta}\overline{y}(v_2)\Big)\Big] & \text{in } Q_T \end{cases}
$$

where $\overline{\beta} = e^{\lambda t}\beta$. Multiplying the first equation of (2.5) by z, and integrating over Q yield, after some calculations:

$$
-\frac{1}{2}\|z(.,0,.)\|^2_{L^2(Q_T)} + \|\nabla z\|^2_{(L^2(Q))^m} + \lambda\|z\|^2_{L^2(Q)} \;\le\; \int_\Sigma e^{-\lambda t}(v_1 - v_2) z\; dt\, da\, d\sigma\ .
$$

Hence,

$$
\|\nabla z\|^2_{(L^2(Q))^m} + \lambda\|z\|^2_{L^2(Q)} \;\le\; \frac{1}{2}\|z(.,0,.)\|^2_{L^2(Q_T)} + \int_\Sigma e^{-\lambda t}(v_1 - v_2) z\; dt\, da\, d\sigma\ .
$$

It is obvious that

$$
\begin{aligned}
\|z(.,0,.)\|^2_{L^2(Q_T)} \;&=\; \left\| e^{-\lambda t}\Big[F\Big(\int_0^\omega \overline{\beta}\overline{y}(v_1)\Big) - F\Big(\int_0^\omega \overline{\beta}\overline{y}(v_2)\Big)\Big]\right\|^2_{L^2(Q_T)} \\
&\le\; K_F^2 \left\|\int_0^\omega \beta z\right\|^2_{L^2(Q_T)} \\
&\le\; K_F^2\, \omega\, \|\beta\|^2_{L^\infty(0,\omega)}\, \|z\|^2_{L^2(Q)}\ .
\end{aligned}
$$

Let $\lambda_0 = \lambda - \frac{1}{2}K_F^2\omega\|\beta\|^2_{L^\infty(0,\omega)}$, with $\lambda > \frac{1}{2}K_F^2\omega\|\beta\|^2_{L^\infty(0,\omega)}$ we obtain:

$$\|\nabla z\|^2_{(L^2(Q))^m} + \lambda_0\|z\|^2_{L^2(Q)} \leq \int_\Sigma e^{-\lambda_0 t}(v_1 - v_2)z \; dt\, da\, d\sigma \;.$$

Using the continuity of the map $\varphi \mapsto \varphi_{|\Gamma}$ on $H^1(\Omega)$ onto $L^2(\Gamma)$[14], one can choose λ_0 such that:

$$\|z\|^2_{L^2(]0,T[\times]0,\omega[,H^1(\Omega))} \leq C\, \|e^{-\lambda_0 t}(v_1 - v_2)\|^2_{L^2(\Sigma)} \;.$$

This means that the map $v \mapsto y(v)$ is a lipschitz function on $L^2(\Sigma)$ onto $L^2(]0,T[\times]0,\omega[,H^1(\Omega))$.

Now, assuming that H_7 holds, $y_0 \in L^\infty(Q_\omega)$, $v_0 \in H^{1/2}(\Gamma)$ and $0 \leq v \leq v_0$ we will prove that $y(v) \in L^\infty(Q)$. For this, we introduce here a function $f \in L^\infty(\Omega)$ such that $f \geq \|y_0\|_{L^\infty(Q_\omega)}$ a.e. in Ω.

Let us consider the following system:

$$\left\{\begin{aligned} -\Delta\theta + \lambda\theta &= f && \text{in } \Omega \\ \partial\theta/\partial\eta &= v_0 && \text{on } \Gamma \;. \end{aligned}\right. \tag{2.6}$$

We note that (2.6) admits a unique solution $\theta \in H^2(\Omega)$ and we have $\theta \geq \|y_0\|_{L^\infty(Q_\omega)}$ a.e. in Ω ([5]).

Let $\widetilde{\theta}(t,a,x) = \theta(x)$, then, $\widetilde{\theta}$ is the unique solution of

$$\left\{\begin{aligned} \partial\widetilde{\theta}/\partial t + \partial\widetilde{\theta}/\partial a - \Delta\widetilde{\theta} + \lambda\widetilde{\theta} &= f && \text{in } Q \\ \partial\widetilde{\theta}/\partial\eta &= v_0 && \text{on } \Sigma \\ \widetilde{\theta}(0,a,x) &= \theta(x) && \text{in } Q_\omega \\ \widetilde{\theta}(t,0,x) &= 1/\omega \int_0^\omega \widetilde{\theta}\, da && \text{in } Q_T \;. \end{aligned}\right. \tag{2.7}$$

Let S be a functional defined on $L^2(Q)$ by the formula: $\forall\, \varphi \in L^2(Q)$, $S(\varphi) = y_\varphi$ with y_φ the solution of the following system:

$$\left\{\begin{aligned} \partial y_\varphi/\partial t + \partial y_\varphi/\partial a - \Delta y_\varphi + (\lambda+\mu)y_\varphi &= 0 && \text{in } Q \\ \partial y_\varphi/\partial\eta &= v e^{-\lambda t} && \text{on } \Sigma \\ y_\varphi(0,a,x) &= y_0(a,x) && \text{in } Q_\omega \\ y_\varphi(t,0,x) &= e^{-\lambda t}\, F\Big(\int_0^\omega e^{\lambda t}\beta\varphi\, da\Big) && \text{in } Q_T \;. \end{aligned}\right.$$

Like in [15] or [9], using the lipschitz condition on F, it follows that one can choose a judicious λ such that S admits a fixed point $\overline{y}$, solution of (2.4).

Let $(\overline{y}_n)$ be a sequence defined by:

$$\begin{cases} \overline{y}_0 = S(\widetilde{\theta}) \\ \overline{y}_{n+1} = S(\overline{y}_n) \quad \text{for } n \in N \;. \end{cases}$$

Then, with the previous choice of λ, the sequence $(\overline{y}_n)$ converges strongly to $\overline{y}$ in $L^2(Q)$.

Thanks to hypotheses H_5, H_7 and the positivity of the functions F, $\widetilde{\theta}$ and β we get $F(\int_0^\omega \beta\widetilde{\theta}\,da) \leq \widetilde{\theta} = \int_0^\omega \frac{1}{\omega}\widetilde{\theta}\,da$. Now, because $\theta \geq y_0$ a.e. in Q_ω, $v \leq v_0$, $\mu+\lambda \geq \lambda$ and $f \geq 0$, it follows from Remark 2 that $\overline{y}_0 \leq \widetilde{\theta}$.

On the other hand, using the fact that $F(\int_0^\omega \beta\overline{y}_0\,da) \leq \int_0^\omega \frac{1}{\omega}\widetilde{\theta}\,da$, one can prove easily and inductively that $\overline{y}_n \leq \widetilde{\theta}$. Then, we infer that $\overline{y} \leq \widetilde{\theta}$.

Since $m = 1, 2$ or 3 we have $\theta \in C(\overline{\Omega})$, then because $0 \leq \overline{y}$ a.e. in Q, we obtain finally that $\overline{y} \in L^\infty(Q)$. ∎

Now, we examine the existence of an optimal control.

Theorem 4. *Under assumptions H_1,..., H_6, the problem* (**P**) *admits at least one optimal control.*

Proof: Let (v_n) be a sequence such that $\lim_{n\to+\infty} J(v_n) = \mathrm{Inf}_{v\in U_{ad}} J(v)$. We deduce that (v_n) is bounded in $L^2(\Sigma)$. Using the fact that the map $v \mapsto y(v)$ is a lipschitz function, we obtain that $(y(v_n))$ is bounded in $L^2(]0,T[\times]0,\omega[, H^1(\Omega))$. We also deduce that $(\partial y(v_n)/\partial t + \partial y(v_n)/\partial a)$ is bounded in $L^2(]0,T[\times]0,\omega[, (H^1(\Omega))')$. Then, one can extract a subsequence also denoted $(y(v_n))$ such that $v_n \to v$ weakly in $L^2(\Sigma)$, $y(v_n) \to y(v)$ weakly in $L^2(]0,T[\times]0,\omega[; H^1(\Omega))$ and $\partial y(v_n)/\partial t + \partial y(v_n)/\partial a \to \partial y(v)/\partial t + \partial y(v)/\partial a$ weakly in $L^2(]0,T[\times]0,\omega[, (H^1(\Omega))')$.

It follows easily that y satisfies:

$$\text{(2.8)} \qquad \begin{cases} \partial y(v)/\partial t + \partial y(v)/\partial a - \Delta y(v) + \mu y(v) = 0 \quad \text{in } Q \\ \partial y(v)/\partial\eta = v \quad \text{on } \Sigma \;. \end{cases}$$

Applying Lemma 1 and the fact that: $y(v_n)(0,.,.) = y_0$ a.e. in Q_ω we get $y(v_n)(0,.,.) \to y(v)(0,.,.)$ weakly in $L^2(Q_\omega)$ and we deduce that:

$$\text{(2.9)} \qquad y(v)(0,.,.) = y_0 \quad \text{a.e. in } Q_\omega \;.$$

In the same manner, we get $y(v_n)(.,0,.) \to y(v)(.,0,.)$ weakly in Q_T.

Now, let us prove that:

$$\text{(2.10)} \qquad y(v)(.,0,.) = F\left(\int_0^\omega \beta\, y\, da\right) .$$

Multiplying by β the first equation of the following system:

$$\begin{cases} \partial y(v_n)/\partial t + \partial y(v_n)/\partial a - \Delta y(v_n) + \mu y(v_n) = 0 & \text{in } Q \\ \partial y(v_n)/\partial \eta = v_n & \text{on } \Sigma \\ y(v_n)(0,a,x) = y_0(a,x) & \text{in } Q_\omega \\ y(v_n)(t,0,x) = F\Big(\int_0^\omega \beta y(v_n)\, da\Big) & \text{in } Q_T \end{cases}$$

integrating over $]0;\omega[$ and letting $z_n = \int_0^\omega \beta y(v_n)\, da$ we deduce the following system

$$\begin{cases} \partial z_n/\partial t - (\beta y_n)(t,0,x) + (\beta y_n)(t,\omega,x) - \Delta z_n + \int_0^\omega (y_n \partial\beta/\partial a + \mu\beta y_n)\, da = 0 \\ \partial z_n/\partial \eta = w_n \\ z_n(0,x) = z_0(x) \end{cases} \tag{2.11}$$

where $y_n = y(v_n)$, $w_n = \int_0^\omega \beta v_n\, da$ and $z_0 = \int_0^\omega \beta y_0\, da$.

It is obvious that (2.11) can be equivalently written as

$$\begin{cases} \partial z_n/\partial t - \Delta z_n = f_n & \text{in }]0,T[\times\Omega \\ \partial z_n/\partial \eta = w_n & \text{on } \Sigma \\ z_n(0,x) = z_0(x) & \text{in } \Omega \end{cases} \tag{2.12}$$

where $f_n(t,x) = (\beta y_n)(t,0,x) - (\beta y_n)(t,\omega,x) - \int_0^\omega (y_n \partial\beta/\partial a + \mu\beta y_n) da$.

Since $\beta \in C^1([0,\omega])$ and (y_n) is bounded in $L^2(Q)$, we deduce from the continuity of "trace application" at $a = 0$ and at $a = \omega$ that f_n is bounded in $L^2(Q_T)$. Then, one gets that (2.12) admits a unique solution z_n in $L^2(0,T;H^1(\Omega))$, the sequence (z_n) and $(\partial z_n/\partial t)$ are bounded respectively in $L^2(0,T;H^1(\Omega))$ and in $L^2(0,T;(H^1(\Omega))')$.

Consequently, we obtain the compactness of (z_n) in $L^2(0,T;L^2(\Omega))$. So, one can extract a subsequence, also denoted (z_n) such that $z_n \to z$ strongly in $L^2(0,T;L^2(\Omega))$.

We recall that $y_n \to y(v)$ weakly in $L^2(Q)$, then $z = \int_0^\omega \beta y(v)\, da$. Keeping in mind that F is a lipschitz function, one obtains:

$$y_n(t,0,x) = F\Big(\int_0^\omega \beta\, y_n\, da\Big) \longrightarrow F\Big(\int_0^\omega \beta\, y(v)\, da\Big) \quad \text{in } L^2(Q_T)\ .$$

Hence, we obtain (2.10). Finally, from (2.8), (2.9), (2.10) we deduce that $y(v)$ is a solution of (1.1) and this ends the proof. ∎

Remark 5. We recall that, we have only used the globally lipschitz condition on F. In order to get more, we will make more hypotheses on F. □

3 – Necessary optimality conditions

In this section, we are concerned by necessary optimality conditions, for this, we will assume the following assumption:

(H_8): $F(\alpha) = c\alpha^2/(b+\alpha^2)$, where c and b are positive real numbers.

Remark 6. We have $F'(\alpha) = 2bc\alpha/(b+\alpha^2)^2$ and we see that F satisfies a lipschitz condition.

When the birth process is governed by the function $F(\alpha) = c\alpha^2/(b+\alpha^2)$, with c and b two strictly positive and fixed constants, the model (1.1) is said to be a "depensatory model" [11]–[12]. This reproduction function models the situation in which there is a threshold level such that if the population size falls below this threshold, the species are overwhelmed by predators and driven to extinction, but if the population size exceeds this threshold, the predators are satisfied, and the population of prey is viable. The constants c and b depend only on the populations size of species in competition.

Note that, one can get analogous result by taking another globally lipschitz birth process which verifies H_7. □

The following result give the Gateaux derivative of $y(v)$ with respect to the function v.

Proposition 7. *Let* $y_0 \in L^\infty(Q_\omega)$ *and* $U_{ad} = \{v \in L^2(]0,T[\times]0,\omega[\times\Gamma),$ $v_0 \geq v \geq 0$ a.e. *in* $\Sigma\}$, *and assume that hypotheses* H_1–H_8 *hold.*

Let $v \in U_{ad}$, $u \in U_{ad}$ *and* $s > 0$ *such that* $su + v \in U_{ad}$.

Let $z_s = [y(su+v) - y(v)]/s$, *then* (z_s) *converges strongly in* $L^2(Q)$, *as* $s \to 0$ *to a function* ζ *solution of*

$$(3.1)\quad \begin{cases} \partial\zeta/\partial t + \partial\zeta/\partial a - \Delta\zeta + \mu\zeta \;=\; 0 & \text{in } Q \\ \partial\zeta/\partial\eta \;=\; u & \text{on } \Sigma \\ \zeta(0,a,x) \;=\; 0 & \text{in } Q_\omega \\ \zeta(t,0,x) \;=\; \displaystyle\int_0^\omega \beta\zeta\,da\;\; F'\Big(\int_0^\omega \beta y\,da\Big) & \text{in } Q_T\,. \end{cases}$$

Proof: It is easy to consider the auxiliary system of the section 2. Hence, $\overline{z}_s$ satisfies the following system

$$(3.2)\qquad \begin{cases} \partial\overline{z}_s/\partial t + \partial\overline{z}_s/\partial a - \Delta\overline{z}_s + (\mu+\lambda)\overline{z}_s \;=\; 0 \\ \partial\overline{z}_s/\partial\eta \;=\; u e^{-\lambda t} \\ \overline{z}_s(0,a,x) \;=\; 0 \\ \overline{z}_s(t,0,x) \;=\; \gamma_s(t,x) \end{cases}$$

where

$$(3.3)\qquad \gamma_s(t,x) \;=\; \frac{e^{-\lambda t}}{s}\left[F\Big(\int_0^\omega \beta e^{\lambda t}\,\overline{y}(su+v)\,da\Big) - F\Big(\int_0^\omega \beta e^{\lambda t}\,\overline{y}(v)\,da\Big)\right].$$

From (3.3), we get:

$$\|\overline{z}_s(.,0,.)\|^2_{L^2(Q_T)} \;\le\; K_F^2\,\|\beta\|^2_{L^\infty(0,\omega)}\;\omega\,\|\overline{z}_s\|^2_{L^2(Q)}\;.$$

Then, as in the proof of Proposition 3, we get that the sequence $(\overline{z}_s)$ is bounded in $L^2(U,H^1(\Omega))$, and then $(\partial\overline{z}_s/\partial t+\partial\overline{z}_s/\partial a)$ is also bounded in $L^2(]0,T[\times]0,\omega[\,;(H^1(\Omega))')$. There exist a subsequence still denoted $(\overline{z}_s)$ such that $(\overline{z}_s)$ converge weakly to a function $\overline{\zeta}$ in $L^2(]0,T[\times]0,\omega[,H^1(\Omega))$ and $(\partial\overline{z}_s/\partial t+\partial\overline{z}_s/\partial a)$ converges weakly to χ.

It is known that the derivative in $\mathcal{D}(]0,T[\times]0,\omega[,(H^1(\Omega))')$ is continuous, so one gets $\chi=\partial\overline{\zeta}/\partial t+\partial\overline{\zeta}/\partial a$.

Thanks to the weak convergence of these sequences, we finally deduce that ζ verifies:

$$(3.4)\qquad \partial\zeta/\partial t+\partial\zeta/\partial a-\Delta\zeta+\mu\zeta \;=\; 0 \qquad \text{in }\; Q$$

and

$$(3.5)\qquad \partial\zeta/\partial\eta \;=\; u \qquad \text{on }\; \Sigma\;.$$

The third equation of (3.2) is a consequence of the continuity of the "trace applications" in the weak topology at $t=0$.

Let us now prove the fourth equation of (3.2).

Let $i(s)=\int_0^\omega \beta e^{\lambda t}\,\overline{y}(su+v)\,da$, $j=\int_0^\omega \beta e^{\lambda t}\overline{\zeta}\,da$, $p_s=\int_0^\omega \beta e^{\lambda t}\overline{z}_s\,da$.

It suffices to show that: $\|F(i(s))-F(i(0))/s-F'(i(0))j\|_{L^2(Q_T)}\to 0$ when $s\to 0$, $s>0$.

By using H_8 and the fact that $y(v) \in L^\infty(Q)$, we get after some calculations:

$$\begin{aligned}(3.6) \qquad & \left\| \Big(F(i(s)) - F(i(0))\Big)/s - F'(i(0))j \right\|_{L^2(Q_T)} \leq \\ & \leq \ C \left[\|p_s - j\|_{L^2(Q_T)} + \|i(0) - i(s)\|_{L^2(Q_T)} \right]\end{aligned}$$

where C is a strictly positive constant.

Because $\overline{z}_s \to \overline{\zeta}$ weakly in $L^2(Q)$, we infer that $p_s \to \int_0^\omega \beta e^{\lambda t}\overline{\zeta}\, da$ weakly in $L^2(Q_T)$, and because the map $v \mapsto y(v)$ is continuous , we deduce that $i(s) \to \int_0^\omega \beta e^{\lambda t}\overline{y}(v)\, da$.

Consequently, one obtains

$$(3.7) \qquad \gamma_s \longrightarrow \int_0^\omega \beta\overline{\zeta}\, da\ F'\Big(\int_0^\omega \beta e^{\lambda t}\overline{y}(v)\, da\Big) \quad \text{in } \ L^2(Q_T)\ .$$

Therefore, it follows that $\zeta(t,0,x) = \int_0^\omega \beta\zeta\, da\ F'(\int_0^\omega \beta y(v)\, da)$.

Let us show now that $z_s \to \zeta$ in $L^2(Q)$ when $s \to 0,\ s > 0$.

Let $\rho = z_s - \zeta,$ then ρ is solution of

$$\begin{cases} \partial\rho/\partial t + \partial\rho/\partial a - \Delta\rho + \mu\rho = 0 & \text{in } Q \\ \partial\rho/\partial\eta = 0 & \text{on } \Sigma \\ \rho(0,a,x) = 0 & \text{in } Q_\omega \\ \rho(t,0,x) = \Big(F(i(s)) - F(i(0))\Big)/s - \int_0^\omega \beta\zeta\, da\ F'\Big(\int_0^\omega \beta y\, da\Big) & \text{in } Q_T\,. \end{cases}$$

One can prove that $\rho \to 0$ strongly in $L^2(Q)$ when $s \to 0$ by multiplying the first equation of the previous system by ρ, integrating by parts and using (3.7).

From the uniqueness of the solution of (3.1), we infer that all the sequence (z_s) converge strongly to ζ in $L^2(Q)$. This achieves the proof. ∎

The necessary optimality conditions are given by the following result.

Theorem 8. *Let* $y_0 \in L^\infty(Q_\omega)$ *and* $U_{ad} = \{v \in L^2(]0,T[\times]0,\omega[\times\Gamma),$ $v_0 \geq v \geq 0$ *a.e. in* $\Sigma\}$.

Under hypotheses H_1–H_8, (**P**) *admits at least one optimal control.*

Moreover if v^* *is an optimal control then:*

$$(3.8) \qquad v^* = \begin{cases} 0 & \text{a.e. on } \ \{\theta_{|\Sigma} \geq 0\} \\ v_0 & \text{a.e. on } \ \{\theta_{|\Gamma} \leq -Nv_0\} \\ \dfrac{-1}{N}\,\theta_{|\Gamma} & \text{a.e. on } \ \{-Nv_0 < \theta_{|\Gamma} < 0\} \end{cases}$$

where θ is the solution of

$$(3.9)\begin{cases} -\partial\theta/\partial t - \partial\theta/\partial a - \Delta\theta + \mu\theta = y(v^*) - z_d + \beta(a)\, F'\Big(\int_0^\omega \beta y(v^*)\, da\Big) & \text{in } Q \\ \partial\theta/\partial\eta = 0 & \text{on } \Sigma \\ \theta(T,a,x) = 0 & \text{in } Q_\omega \\ \theta(t,\omega,x) = 0 & \text{in } Q_T\,. \end{cases}$$

Proof: By letting $t\prime = T - t$ et $a\prime = \omega - a$, one can easily prove that (3.9) admits a unique solution.

Let v^* be an optimal control function and $u \in T_{U_{ad}}(v^*)$, the tangent cone to U_{ad} at v^*.

Then:

$$\begin{aligned}\Big(J(v^*{+}su) - J(v^*)\Big)/s \;&=\; -2\Big(\Big(y(v^*{+}\,su) - y(v^*)\Big)/s,\; z_d\Big)_{L^2(Q)} \\ &\quad+\, N(u, 2v^*{+}\,su)_{L^2(\Sigma)} \\ &\quad+\, \Big(\Big(y(v^*{+}\,su) - y(v^*)\Big)/s,\; y(v^*{+}\,su) + y(v^*)\Big)_{L^2(Q)}\,.\end{aligned}$$

Applying Proposition 7, it follows when $s \to 0$: that

$$\begin{aligned} J'(v^*,u) \;&=\; -2\,(\zeta, z_d)_{L^2(Q)} + N(u, 2v^*) + 2\,(\zeta, y(v^*)) \\ &=\; 2\,(\zeta,\, y(v^*){-}z_d)_{L^2(Q)} + 2\,N(u,v^*)_{L^2(\Sigma)} \end{aligned}$$

where ζ is the solution of (3.1).

Since $v^*$is an optimal control, one gets for all $s > 0$ but small enough that

$$\Big(J(v^*{+}\,su) - J(v^*)\Big)/s \;\geq\; 0\;.$$

Consequently, $(\zeta,\, y(v^*){-}z_d)_{L^2(Q)} + N(u,v^*)_{L^2(\Sigma)} \geq 0\,.$

This last result means

$$N\int_\Sigma u v^*_{|\Sigma}\; dt\, da\, d\sigma \;+ \int_Q \zeta(y(v^*){-}z_d)\; dt\, da\, dx \;\geq\; 0\;.$$

Let us multiply (3.9) by ζ and integrate the result by parts over Q we obtain:

$$\int_\Sigma u\theta_{|\Gamma}\; dt\, da\, d\sigma \;= \int_Q \zeta(y(v^*){-}z_d)\; dt\, da\, dx\;.$$

Then,

$$\int_\Sigma u[Nv^*{+}\,\theta_{|\Gamma}]\; dt\, da\, d\sigma \;\geq\; 0\,, \qquad \forall\, u \in T_{U_{ad}}(v^*)\;.$$

Therefore,

$$-(Nv^* + \theta_{|\Gamma}) \in N_{U_{ad}}(v^*)$$

where $N_{U_{ad}}(v^*)$ is the normal cone to U_{ad} at v^*.

And then we easily deduce (3.8). ∎

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