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HOLOMORPHIC SERIES EXPANSION OF FUNCTIONS OF CARLEMAN TYPE ON THE INTERVAL $[-1, 1]$

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Abstract. We characterize the functions of some Carleman classes on the unit interval $[-1, 1]$ as sums of holomorphic functions in specific neighborhoods of $[-1, 1]$. As an application of our main theorem, we perform an alternative construction of the Dyn'kin's pseudoanalytic extension for these Carleman classes on $[-1, 1]$.

1. Introduction

In 1926 [**8**] Carleman raised the problem of the representation of the functions of a quasianalytic class in terms of their successive derivatives at a given point. He noticed that this problem should be solved by a decomposition method. This problem was also raised by Julia in 1925 [**13, 14, 15**], while he was looking for an algorithmic generalization of the classical Borel process which generates classes of quasi-analytic functions from sequences of complex numbers converging to 0. In 1962 [**2**] Badalyan gave, by his theory of quasi-powers (a generalization to quasianalytic Carleman classes of Taylor series expansion) the complete solution to Carleman's problem. In 1970 [**3**] Badalyan generalized his theory to some non-quasianalytic classes. In 1991 [**10**, pp. 249–253] Ecalle obtained for the functions of a regular Carleman class on a segment $[a, b]$, a series expansion into holomorphic functions on specific neighborhoods of $[a, b]$. In 2004, Belghiti obtained for certain Carleman classes on arbitrary bounded convex planar domains [**4**] a similar but more explicit holomorphic expansion series. Let us observe that the approach in [**4, 10**] relies mainly on the theorem of pseudoanalytic extension due to Dyn'kin [**9**].

Improving the methods of Ecalle and Belghiti, we obtained in [**5**] a characterization of the functions of a Gevrey class on $[-1, 1]$ as sums of series of holomorphic functions in suitable neighborhoods of $[-1, 1]$, and here we generalize this method to some Carleman classes on $[-1, 1]$. As an application of our main theorem, we

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derive an alternative construction of Dyn'kin's pseudoanalytic extension for these Carleman classes.

2. Preliminary notes

Let S be a nonempty subset of $\mathbb{C}$, $f\colon S \to \mathbb{C}$ a bounded function, and $\|f\|_{\infty,S} := \sup_{z\in S} |f(z)|$. For $z \in \mathbb{C}$ we set $\rho(z, S) := \inf_{u\in S} |z - u|$. For $r > 0$, $B(z, r)$ is the usual open ball in $\mathbb{C}$ with center z and radius r. We set also

$$S_r := S + B\,(0; r) := \{z + u : z \in S,\ u \in B(0, r)\}$$

Thus we have $S_r = \{z \in \mathbb{C} : \rho(z, S) < r\}$. $\mathcal{O}(S)$ denotes the set of holomorphic functions on some neighborhood of S.

Let $\alpha := (\alpha_1, \alpha_2)$, $\beta := (\beta_1, \beta_2) \in \mathbb{N}^2$. We write $\beta \leqslant \alpha$ if: $\beta_1 \leqslant \alpha_1$ and $\beta_2 \leqslant \alpha_2$. Given a property $\mathfrak{P}(x)$, with $x \in \mathbb{R}$, we say that $\mathfrak{P}(x)$ holds ultimately if there exists $a_0 \in \mathbb{R}$ such that $\mathfrak{P}(x)$ holds for all $x \geqslant a_0$. We define an equivalence relation on the set $\mathcal{F}$ of real valued functions which are defined on a real half-line by writing $f =_\infty g$ if we have $f(x) = g(x)$ ultimately. We denote by $[f]$ the class of equivalence of a function $f \in \mathcal{F}$ for the equivalence relation $=_\infty$. The quotient set $\mathcal{G} := \mathcal{F}/=_\infty$ is endowed with operations of addition and multiplication induced by those of $\mathcal{F}$ making $\mathcal{G}$ into a commutative ring. The classes of equivalence for this relations are called the germs of functions at $+\infty$. To simplify the writing we will identify the germ $[f]$ with its representant f. We consider the field $\mathbb{R}$ of real numbers as a subring of $\mathcal{G}$, by identifying a real number a with the germ of the function $x \mapsto a$.

We denote by $\mathcal{G}_1$ the subring of $\mathcal{G}$ consisting of the germs of functions that are ultimately of class C^1. A subring $\aleph$ of $\mathcal{G}_1$ is called a Hardy field if $\aleph$ is a field which is stable by derivation. Functions belonging to a Hardy field $\aleph$ have the following properties: they are ultimately strictly monotone unless they are ultimately constant, they are ultimately of constant sign unless they are ultimately identically vanishing. It follows that for every $f \in \aleph$ the limit $\lim_{x\to+\infty} f(x)$ exists in $\mathbb{R} \cup \{+\infty, -\infty\}$, and that for every $f,\ g \in \aleph$ we have ultimately one of the following cases $f(x) < g(x)$, $g(x) < f(x)$, $f(x) = g(x)$.

We say that an element f of $\aleph$ is bounded if $\lim_{x\to+\infty} f(x) \in \mathbb{R}$, infinitesimal if $\lim_{x\to+\infty} f(x) = 0$, and infinite if $\lim_{x\to+\infty} |f(x)| = +\infty$.

If $f, g \in \aleph$ and g is infinite and ultimately positive, then $f \circ g \in \mathcal{G}_1$ is by definition the germ in $\mathcal{G}_1$ such that ultimately $(f \circ g\ )(x) = f(g(x))$.

In our work we will need the following results.

THEOREM 2.1. [**1, 19**]. *Let f be an infinite and ultimately positive element of a Hardy field $\aleph$. Then there exists a Hardy field $\mathcal{H}$ and a germ $g \in \mathcal{H}$ such that g is an infinite and ultimately positive element of the Hardy field $\mathcal{H}$ and $(f \circ g)(x) = x$ ultimately. g is called the inverse of f and is denoted by $g^{\langle -1\rangle}$.*

THEOREM 2.2. [**1, 18, 20**]. *Let $F(Y), G(Y) \in \aleph[Y]$ and $y \in \mathcal{G}_1$ be such that $G(y) \neq 0$ and $y'\ G(y) = F(y)$ (in $\mathcal{G}_1$). Then the ring of germs $\aleph[y]$ is an integral domain with fraction field $\aleph(y) \subset \mathcal{G}_1$, and $\aleph(y)$ is a Hardy field.*

As a consequence of this theorem, it follows that a Hardy field $\aleph$ can be enlarged to a Hardy field $\aleph_0$ containing the germ Id of the identity function and the germ ln of the logarithmic function. The following theorem provides a strong generalization of this remark.

THEOREM 2.3. [**7**] *Let $\aleph$ be a Hardy field. There exists a Hardy field $\aleph_1$ containing $\aleph$ such that the germs at $+\infty$ of the functions $\exp\circ f$, $\ln\circ|f|$ belong to $\aleph_1$ for every function $f\in\aleph_1$ which is not ultimately identically vanishing.*

A positive function measurable fuction f defined on some neighborhood of $+\infty$ is said to be regularly varying with index $\tau\in\mathbb{R}$ if $\lim_{x\to+\infty}\frac{f(Cx)}{f(x)}=C^\tau$, $C>0$. We set $\mathfrak{I}(f):=\tau$. If $\mathfrak{I}(f)=0$, then we will say that the function f is slowly varying.

If f is regularly varying with index τ, then there exists a slowly varying function L such that for $f(x)=x^\tau L(x)$, for sufficiently large values of x.

Let f be a function defined on an interval of the form $[a,+\infty[$ such that f is strictly positive and belongs as a germ at $+\infty$ to a Hardy field. Then according to [**12**], the function f is regularly varying if and only if $\lim_{x\to+\infty}\frac{xf'(x)}{f(x)}\in\mathbb{R}$. Then we have $\mathfrak{I}(f)=\lim_{x\to+\infty}\frac{xf'(x)}{f(x)}$.

THEOREM 2.4 (Potter's bounds, [**6**]). *Let f be a regularly varying function of index τ. For every $\varepsilon>0$, we have ultimately $(1-\varepsilon)x^{\tau-\varepsilon}\leqslant f(x)\leqslant(1+\varepsilon)x^{\tau+\varepsilon}$.*

Let $\mu\colon\mathbb{R}_+^*\to\mathbb{R}$ be a function of class C^2 on $\mathbb{R}_*^+$ which belongs, as a germ at $+\infty$, to a Hardy field $\aleph$ containing the germ at $+\infty$ of the function $x\mapsto\ln x$. Since the function μ belongs as a germ at $+\infty$ to the Hardy field $\aleph$, it follows that the limit $\sigma(\mu):=\lim_{t\to+\infty}\frac{\ln(t)}{\mu(t)}$ exists in $\mathbb{R}_+\cup\{+\infty\}$. $\sigma(\mu)$ is called the order of the function μ. We assume that $0<\sigma(\mu)<+\infty$. It follows then that we have

$$\lim_{t\to+\infty}\mu(t)=+\infty,\quad \mu(t)=\underset{t\to+\infty}{O}(t)$$

Furthermore, we have by virtue of L'Hopital's rule, $\lim_{t\to+\infty}t\mu'(t)=\frac{1}{\sigma(\mu)}$. Thence we have $\lim_{t\to+\infty}\frac{t\mu'(t)}{\mu(t)}=0$. Consequently the function μ is slowly varying.

Consider the function $\mathcal{M}_\mu$ defined on $]0,+\infty[$ by $\mathcal{M}_\mu(t):=t^te^{t\mu(t)}$, $t>0$. The functions Ω_μ and H_μ are defined on $\mathbb{R}_+^*$ by

$$\Omega_\mu(x):=\inf_{t>0}\left[\frac{\mathcal{M}_\mu(t)}{x^t}\right],\ x>0,\qquad H_\mu(x)=\inf_{t>0}\left[\frac{\mathcal{M}_\mu(t)}{t^tx^t}\right],\ x>0$$

We consider also the sequence $M_\mu:=(M_n)_{n\in\mathbb{N}^*}$ defined by $M_n:=\mathcal{M}_\mu(n)$, $n\in\mathbb{N}^*$.

Let W be a nontrivial interval of $\mathbb{R}$. The Carleman class $C_{M_\mu}(W)$ is the set of functions f of class C^∞ on W such that $\sup_{x\in W}|f^{(n)}(x)|\leqslant C\rho^nM_n$, $n\in\mathbb{N}^*$ where $C,\rho>0$ are real constants.

We denote by Λ_{M_μ} the set of sequences $(a_n)_{n\in\mathbb{N}}$ of complex numbers such that $|a_n|\leqslant C\rho^nM_n$, $n\in\mathbb{N}^*$ where $C,\rho>0$ are constants. We denote by ω_μ and h_μ the functions defined by $\omega_\mu(x):=-\ln[\Omega_\mu(x)]$ and $h_\mu(x):=-\ln[H_\mu(x)]$, respectively.

Let γ_μ denote the function ultimately defined by the system

$$x=t^2\mu'(t),\quad \gamma_\mu(x)=\mu(t)+t\mu'(t), \tag{2.1}$$

the parameter t being uniquely determined by x. We denote then t by $t_0(x)$.

Let φ_μ denote the function defined by $\varphi_\mu(x) := \omega_\mu(x) - x\omega'_\mu(x)$ for sufficiently large values of x.

The following propositions will play a crucial role in the proof of our main result.

PROPOSITION 2.1. 1. *The function ω_μ is ultimately well defined by the system*

$$x = et\exp[\mu(t) + t\mu'(t)], \quad \omega_\mu(x) = t + t^2\mu'(t) \tag{2.2}$$

the parameter t being ultimately uniquely determined by x. We denote then t by $t_1(x)$.

2. *The function ω_μ is ultimately strictly concave.*

3. *The function φ_μ is ultimately well defined by the system*

$$x = et\exp[\mu(t) + t\mu'(t)], \quad \varphi_\mu(x) = t^2\mu'(t) \tag{2.3}$$

the parameter t being ultimately uniquely determined by x.

4. *The function φ_μ is an increasing diffeomorphism between neighborhoods of $+\infty$. The inverse function $\mathcal{N}_\mu := \varphi_\mu^{\langle -1\rangle}$ is ultimately defined by the system*

$$x = t^2\mu'(t), \quad \mathcal{N}_\mu(x) = et\exp[\mu(t) + t\mu'(t)] \tag{2.4}$$

the parameter t being ultimately uniquely determined by x

5. *The function h_μ is ultimately well defined by the system*

$$x = \exp[\mu(t) + t\mu'(t)], \quad h_\mu(x) = t^2\mu'(t) \tag{2.5}$$

the parameter t being utimatey uniquely determined by x. Furthermore h_μ is ultimately positive and infinite so it has an inverse $h_\mu^{\langle -1\rangle}$ which belongs to a Hardy field.

6. *Each of the function ω_μ, φ_μ, h_μ, $\mathcal{N}_\mu$, γ_μ belongs to a Hardy field.*

7. *The function γ_μ is slowly varying and the function ω_μ is regularly varying of index*

$$\mathfrak{I}(\omega_\mu) = \frac{\sigma(\mu)}{1 + \sigma(\mu)}. \tag{2.6}$$

8. *The function γ_μ is ultimately positive and infinite and we have*

$$\gamma_\mu(x) - \mu(x) = \underset{x\to+\infty}{O}(1) \tag{2.7}$$

9. *We have ultimately*

$$\omega'_\mu(\mathcal{N}_\mu(x)) = \frac{e^{-\gamma_\mu(x)}}{e}, \tag{2.8}$$

$$\gamma_\mu(x)) = \ln(h_\mu^{\langle -1\rangle}(x)). \tag{2.9}$$

10. *The following relations hold for every $\alpha \in \mathbb{R}_+^*$*

$$\mu(\alpha x) - \mu(x) = \underset{x\to+\infty}{O}(1), \tag{2.10}$$

$$\lim_{x\to+\infty} \frac{e^{-\alpha\varphi_\mu(x)}}{\varphi'_\mu(x)} = 0. \tag{2.11}$$

PROOF. 1. Thanks to [**4**], the function h_μ is ultimately well defined by the system

$$x = \exp[\mu(t) + t\mu'(t)], \quad h_\mu(x) = t^2\mu'(t)$$

Consider then the function $\bar{\mu}\colon x \mapsto \mu(x) + \ln(x)$. It belongs to the Hardy field $\aleph$ and we have $\sigma(\bar{\mu}) = \frac{\sigma(\mu)}{\sigma(\mu)+1} \in]0, +\infty[$. It follows that the function $h_{\bar{\mu}}$ is ultimately well defined by the system

$$x = \exp[\bar{\mu}(t) + t\bar{\mu}'(t)], \quad h_{\bar{\mu}}(x) = t^2\bar{\mu}'(t)$$

that is by the system

$$x = et\exp[\mu(t) + t\mu'(t)], \quad h_{\bar{\mu}}(x) = t + t^2\mu'(t)$$

But we know that $h_{\bar{\mu}} = \omega_\mu$, thence the function ω_μ is ultimately well defined by the system

$$x = et\exp[\mu(t) + t\mu'(t)], \quad \omega_\mu(x) = t + t^2\mu'(t)$$

the parameter t being ultimately uniquely determined by x.

On the other hand, since

$$t^2\mu'(t) \underset{t\to+\infty}{\sim} \frac{1}{\sigma(\mu)}t, \quad \exp[\mu(t) + t\mu'(t)] \underset{t\to+\infty}{\sim} e^{\frac{1}{\sigma(\mu)}}e^{\mu(t)}$$

it follows that $\lim_{x\to+\infty} h_\mu(x) = +\infty$. Consequently h_μ is ultimately positive and infinite. Thence according to Theorem 2.1 above, the function h_μ has an inverse $h_\mu^{\langle -1\rangle}$ which belongs to a Hardy field.

2. It follows from the definition of the function ω_μ that it is ultimately of class C^1. Direct computations from the system (2.2) prove then that the function ω'_μ has ultimately the following parametrical representation

$$x = et\exp[\mu(t) + t\mu'(t)], \quad \omega'_\mu(x) = \frac{1}{e\exp[\mu(t) + t\mu'(t)]} \tag{2.12}$$

It follows that the function ω'_μ is ultimately strictly decreasing. Then that the function ω_μ is ultimately strictly concave.

3. Direct computations from system (2.2) lead to the system representing ultimately the function φ_μ.

4. It is clear that the function $F_1\colon t \to e^{\mu(t)+t\mu'(t)}$ which belongs as a germ at $+\infty$ to the Hardy field $\aleph$, is ultimately strictly increasing and satisfies $\lim\limits_{x\to+\infty} F_1(t) = +\infty$. Thence, according to Theorem 2.1, the function F_1 has an inverse g belonging to a Hardy field $\aleph_1$ which contains the identity.

The function h_μ is ultimately of the class C^1, and, according to (2.5), we have ultimately

$$h'_\mu(x) = \frac{\frac{d(t^e\mu'(t))}{dt}(g(x))}{\frac{d(e^{\mu(t)+t\mu'(t)})}{dt}(g(x))} = \frac{g(x)}{e^{\mu(g(x))+g(x)\mu'(t(x))}} = \frac{g(x)}{x}$$

Hence we have ultimately $xh'_\mu(x) = g(x)$. It follows, according to Theorem 2 above, that $\aleph_1[h_\mu]$ is an integral domain whose fraction field $\aleph_1(h_\mu)$ is a Hardy field which contains the function h_μ as a germ at $+\infty$. By a similar proof we obtain that $h_{\bar{\mu}}$

belongs to a Hardy field. Since $\omega_\mu = h_{\bar{\mu}}$ it follows then that the function ω_μ belongs as a germ at $+\infty$ to a Hardy field.

It is obvious that the function φ_μ belongs to the same Hardy field $\aleph$ as ω_μ. Furthermore direct computations, based on the system representing φ_μ on some neighborhood of $+\infty$, show that φ_μ is infinite and ultimately positive. It follows then that φ_μ is ultimately strictly increasing. Thence φ_μ is a diffeomorphism between neighborhoods of $+\infty$ whose inverse $\mathcal{N}_\mu$ belongs to a Hary field. It is clear that the function $\mathcal{N}_\mu$ is ultimately well defined by the system

$$x = t^2\mu'(t), \quad \mathcal{N}_\mu(x) = et\exp[\mu(t) + t\mu'(t)]$$

the parameter t being ultimately uniquely determined by x.

Direct computations from systems (2.1), (2.2), (2.4) prove that we have ultimately $\gamma_\mu(h_\mu(x)) = \ln(x)$, that is $\gamma_\mu(x) = \ln(h_\mu^{\langle -1\rangle}(x))$

It follows, according to Theorem 3 above, that there exists a Hardy field containing the function γ_μ. It follows also from the relation (2.9) that γ_μ is ultimately positive and infinite.

Direct computations from systems (2.1), (2.2), (2.4) prove also that relation (2.8) holds ultimately.

5. The function ω_μ belongs to a Hardy field and we have

$$\begin{aligned}\lim_{x\to+\infty}\frac{x\omega'_\mu(x)}{\omega_\mu(x)} &= \lim_{x\to+\infty}\frac{t_1(x)}{t_1(x)+t_1(x)^2\mu'(t_1(x))}\\ &= \lim_{x\to+\infty}\frac{1}{1+t_1(x)\mu'(t_1(x))} = \frac{\sigma(\mu)}{1+\sigma(\mu)}\end{aligned}$$

Thence the function ω_μ is regularly varying with index $\mathfrak{I}(\omega_\mu) = \frac{\sigma(\mu)}{1+\sigma(\mu)}$.

6. The function γ_μ belongs as a germ at $+\infty$ to a Hardy field and we have

$$\lim_{x\to+\infty}\frac{x\gamma'_\mu(x)}{\gamma_\mu(x)} = \lim_{t\to+\infty}\frac{t\mu'(t)}{\mu(t)+t\mu'(t)} = \lim_{t\to+\infty}\frac{\frac{t\mu'(t)}{\mu(t)}}{1+\frac{t\mu'(t)}{\mu(t)}} = 0$$

Thence γ_μ is slowly varying.

7. Since γ_μ is slowly varying it follows, according to Theorem 2.4 above, that we have ultimately $0 \leqslant \gamma_\mu(x) \leqslant \sqrt{x}$. It follows that $\gamma_\mu(x) = o_{x\to+\infty}(x)$.

On the other hand, according to (2.1), we have ultimately

$$\begin{aligned}\gamma_\mu(x) - \mu(x) &= \mu(t_0(x)) + t_0(x)\mu'(t_0(x)) - \mu(x)\\ &= \frac{(t_0(x)-x)}{v}v\mu'(v) + t_0(x)\mu'(t_0(x))\end{aligned}$$

where v lies between x and $t_0(x)$. Since

$$x = t_0(x)^2\mu'(t_0(x)) \underset{x\to+\infty}{\sim} \frac{1}{\sigma(\mu)}t_0(x)$$

it follows that $\frac{t_0(x)-x}{v} = O_{x\to+\infty}(1)$. Consequently we have

$$\gamma_\mu(x) - \mu(x) = \underset{x\to+\infty}{O}(1)$$

8. We have

$$\lim_{x\to+\infty}\frac{e^{-\alpha\varphi_\mu(x)}}{\varphi'_\mu(x)}=\lim_{t\to+\infty}\frac{e[1+2t\mu'(t)+t^2\mu''(t)]\exp[\mu(t)+t\mu'(t)-\alpha t^2\mu'(t)]}{2t\mu'(t)+t^2\mu''(t)}$$

Thence we have by virtue of L'Hopital's rule that

$$\lim_{t\to+\infty}-t^2\mu''(t)=\lim_{t\to+\infty}t\mu'(t)=\lim_{t\to+\infty}\frac{\mu(t)}{\ln t}=\frac{1}{\sigma(\mu)}$$

Consequently the following estimate holds

$$\frac{e[1+2t\mu'(t)+t^2\mu''(t)]\exp[\mu(t)+t\mu'(t)-\alpha t^2\mu'(t)]}{2t\mu'(t)+t^2\mu''(t)}$$
$$\underset{t\to+\infty}{\sim}e(1+\sigma(\mu))e^{\frac{1}{\sigma(\mu)}}\exp[\mu(t)-\alpha t^2\mu'(t)].$$

But $\mathfrak{I}(\mu)=0$ and $\mathfrak{I}(t\mapsto\alpha t^2\mu'(t))=1$, hence we have, according to Theorem 2.4 above, that $\mu(t)=O_{t\to+\infty}(\alpha t^2\mu'(t))$. It follows that

$$\lim_{t\to+\infty}e(1+\sigma(\mu))e^{\frac{1}{\sigma(\mu)}}\exp[\mu(t)-\alpha t^2\mu'(t)]=0.$$

Consequently we have

$$\lim_{x\to+\infty}\frac{e^{-\alpha\varphi_\mu(x)}}{\varphi'_\mu(x)}=0.$$

On the other hand, according to the mean value theorem, we have for all $x>0$

$$|\mu(\alpha x)-\mu(x)|=|\alpha-1||x\mu'(u)|=|\alpha-1|\frac{x}{u}|u\mu'(u)|$$

where u lies between αx and x. It follows that

$$|\mu(\alpha x)-\mu(x)|\leqslant|\alpha-1|\max\Big(\alpha,1/\alpha\Big)|u\mu'(u)|$$

Since $\lim_{s\to+\infty}t\mu'(t)=\frac{1}{\sigma(\mu)}<+\infty$, it follows then that

$$\mu(\alpha x)-\mu(x)=\underset{x\to+\infty}{O}(1)$$ □

PROPOSITION 2.2. *Let I be a nontrivial compact interval of $\mathbb{R}$ and $a\in I$. The so-called Borel mapping $\mathcal{T}\colon C_{M_\mu}(I)\to\Lambda_{M_\mu}$, $f\mapsto(f^{(n)}(a))_{n\in\mathbb{N}}$ is surjective.*

PROOF. Following Petzsche [**17**, p. 300], we set

$$m_p^*:=\frac{M_p}{pM_{p-1}},\quad p\in\mathbb{N}^*.$$

We have then for every $p\in\mathbb{N}^*$

$$\frac{m_{2p}^*}{m_p^*}=\frac{1}{2}\frac{M_{2p}/M_{2p-1}}{M_p/M_{p-1}}=\frac{2^{2p}p^{2p}}{2(2p-1)^{2p-1}}\frac{(p-1)^{p-1}}{p^p}$$
$$\times\exp[2p\mu(2p)-(2p-1)\mu(2p-1)-p\mu(p)-(p-1)\mu(p-1)]$$
$$=\frac{(1-\frac{1}{p})^{p-1}}{(1-\frac{1}{2p})^{2p-1}}\exp[2p\mu(2p)-(2p-1)\mu(2p-1)-p\mu(p)-(p-1)\mu(p-1)].$$

But we have

$$2p\mu(2p) - (2p-1)\mu(2p-1) = z_{2p}\mu'(z_{2p}) + \mu(z_{2p}),$$
$$p\mu(p) - (p-1)\mu(p-1) = z_p\mu'(z_p) + \mu(z_p)$$

where $z_{2p} \in [2p-1, 2p]$ and $z_p \in [p-1, p]$. It follows that there exists $w_p \in [p-1, 2p]$ such that

$$\begin{aligned} 2p\mu(2p) - (2p-1)\mu(2p-1) - [p\mu(p) - (p-1)\mu(p-1)] \\ &= z_{2p}\mu'(z_{2p}) + \mu(z_{2p}) - (z_p\mu'(z_p) + \mu(z_p)) \\ &= (z_{2p} - z_p)(w_p\mu''(w_p) + 2\mu'(w_p)) \\ &= (z_{2p} - z_p)\mu'(w_p)\Big[\frac{w_p\mu''(w_p) + \mu'(w_p)}{\mu'(w_p)} + 1\Big] \end{aligned}$$

On the other hand, the limit $\lim_{x\to+\infty} \frac{x\mu''(x))+\mu'(x)}{\mu'(x)}$ exists and we have

$$\lim_{x\to+\infty} \frac{x\mu'(x)}{\mu(x)} = 0.$$

From L'Hopital's rule, it follows

$$\lim_{x\to+\infty} \frac{x\mu''(x)) + \mu'(x)}{\mu'(x)} = 0 \tag{2.13}$$

Furthermore we have $z_{2p} - z_p \geqslant \frac{1}{2}w_p - 1$. Thence we have for large values of p

$$(z_{2p} - z_p)\mu'(w_p) \geqslant \frac{1}{2}\Big[\frac{w_p - 2}{w_p}\Big] w_p\mu'(w_p) \tag{2.14}$$

We conclude from (2.13) and (2.14) that

$$\liminf_{p\to+\infty} [2p\mu(2p) - (2p-1)\mu(2p-1)] - [p\mu(p) - (p-1)\mu(p-1)] \geqslant \frac{1}{2\sigma(\mu)}$$

It follows that

$$\liminf_{p\to+\infty} \frac{m^*_{2p}}{m^*_p} \geqslant e^{\frac{1}{2\sigma(\mu)}} > 1.$$

Thence a slight refinement of a theorem in [**17**, pp. 300 and 311] yields that the Borel mapping $\mathcal{T}$ is surjective. □

Direct computations show that μ and γ_μ can be extended to $\mathbb{R}^*_+$ in a way to be functions of class C^1 on $\mathbb{R}^*_+$ such that $-\varepsilon \leqslant \mu(x) - \gamma_\mu(x) \leqslant \varepsilon$,$x \in \mathbb{R}^*_+$ where ε is a positive constant. From now on we will do so and we will set for every $A > 0$, $n \in \mathbb{N}$ and for every nonempty subset S of $\mathbb{C}$

$$S_{\mu,A,n} := S_{Ae^{-\mu(n)}}, \quad S_{\gamma_\mu,A,n} := S_{Ae^{-\gamma_\mu(n)}}.$$

Thence the following inclusions hold for all $n \in \mathbb{N}$.

$$S_{\gamma_\mu,Ae^{-\varepsilon},n} \subset S_{\mu,A,n} \subset S_{\gamma_\mu,Ae^{\varepsilon},n}.$$

3. Statement of the main result

The main result of this paper is the following.

THEOREM 3.1. 1. *Let $f \in C_{M_\mu}([-1,1])$; then there exists constants $C > 0$, $A > 0$, $0 < \rho < 1$ and a sequence $(P_n)_{n\geqslant 1}$ of rational functions defined on $\mathbb{C}\setminus\{i,-i\}$ such that $\sum P_n$ is uniformly convergent on $[-1,1]$ to f and*

$$\|P_n\|_{\infty,[-1,1]_{\mu,A,n}} \leqslant C\rho^n, \quad n \in \mathbb{N}, \qquad f(x) = \sum_{n=1}^{\infty} P_n(x), \quad x \in [-1,1]$$

2. *Conversely, let us assume that there exist some constants $C > 0$, $A > 0$, $0 < \rho < 1$ and a sequence $f_n \in \mathcal{O}([-1,1]_{\mu,A,n})$ of holomorphic functions such that $\|f_n\|_{\infty,[-1,1]_{\mu,A,n}} \leqslant C\rho^n$, $n \in \mathbb{N}^*$. Then the function series $\sum f_n$ is uniformly convergent on $[-1,1]$ to a function f which belongs to the Carleman class $C_{M_\mu}([-1,1])$.*

4. Proof of the main result

4.1. Direct part.

PROPOSITION 4.1. *Let $g\colon [-\pi,\pi] \longrightarrow C$ be a restriction of a 2π-periodic function of class C^∞ on $\mathbb{R}$. Let us assume that $g \in C_{M_\mu}([-\pi,\pi])$; then there exist constants $A > 0$, $C > 0$, $0 < \rho < 1$ and a sequence $(g_n)_{n\geqslant 0}$ of rational functions defined on $\mathbb{C}^*$ such that*

$$\|g_n\|_{\infty,\mathcal{K}_{\gamma_\mu,A,n}} \leqslant C\rho^n, \quad n \in \mathbb{N}, \qquad g(\theta) = \sum_{n=0}^{\infty} g_n(e^{i\theta}), \quad \theta \in [-\pi,\pi]$$

where $\mathcal{K}_{\gamma_\mu,A,n} := \left\{z \in \mathbb{C},\ 1 - Ae^{-\gamma_\mu(n)} < |z| < 1 + Ae^{-\gamma_\mu(n)}\right\}$.

PROOF. The Fourier series expansion of g can be written for all $\theta \in [-\pi,\pi]$ as

$$g(\theta) = \sum_{p\in\mathbb{Z}} a_p e^{ip\theta} \quad \text{where} \quad a_p = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-ip\theta}d\theta, \quad p \in \mathbb{Z}.$$

According to [**16**], the following estimations hold

$$|a_p| \leqslant C_0 e^{-C_1\omega_\mu(|p|)}, \quad p \in \mathbb{Z} \tag{4.1}$$

with some constants $C_0, C_1 > 0$.

Let us set for all $z \in \mathbb{C}^*$ and $n \in \mathbb{N}^*$

$$g_0(z) := \sum_{|p|<\mathcal{N}_\mu(1)} a_p z^p, \quad g_n(z) := \sum_{\mathcal{N}_\mu(n)\leqslant|p|<\mathcal{N}_\mu(n+1)} a_p z^p.$$

Then for all $n \in \mathbb{N}^*$, g_n is a rational function defined on $\mathbb{C}^*$. Furthermore the following estimates hold

$$|g_n(z)| \leqslant C_0 \sum_{\mathcal{N}_\mu(n)\leqslant|p|<\mathcal{N}_\mu(n+1)} C_0 e^{-C_1\omega_\mu(p)}(|z|^p + |z|^{-p}), \quad z \in \mathbb{C}^*. \tag{4.2}$$

If $z \in \mathcal{K}_{\frac{C_1}{2e},n}$, then the estimates become

$$|g_n(z)| \leqslant C_0 \sum_{\mathcal{N}_\mu(n)\leqslant|p|<\mathcal{N}_\mu(n+1)} e^{-C_1\omega_\mu(p)}\left[\Big(1+\frac{C_1}{2e}e^{-\gamma_\mu(n)}\Big)^p+\Big(1-\frac{C_1}{2e}e^{-\gamma_\mu(n)}\Big)^{-p}\right].$$

We have for large values of n

$$\Big(1-\frac{C_1}{2e}e^{-\gamma_\mu(n)}\Big)^{-1} \leqslant 1+\frac{C_1}{e}e^{-\gamma_\mu(n)}.$$

It follows that we have for such values of n

$$\|g_n\|_{\infty\mathcal{K}_{\gamma_\mu,\frac{C_1}{2e},n}} \leqslant C_0(1+\mathcal{N}_\mu(n+1)-\mathcal{N}_\mu(n))\max_{\mathcal{N}_n\leqslant p<\mathcal{N}_{n+1}} 2\exp\left[-C_1\omega_\mu(p)+C_1p\frac{e^{-\gamma_\mu(n)}}{e}\right].$$

On the other hand we have for n sufficiently large

$$\frac{e^{-\gamma_\mu(n)}}{e} = \omega'_\mu(\mathcal{N}_\mu(n))$$

Consequently we have for such values of n

$$\|g_n\|_{\infty,\mathcal{K}_{\gamma_\mu,\frac{C_1}{2e},n}} \leqslant C_0(1+\mathcal{N}_\mu(n+1)-\mathcal{N}_\mu(n)) \max_{\mathcal{N}_\mu(n)\leqslant p<\mathcal{N}_\mu(n+1)} 2\exp\big[-C_1\big(\omega_\mu(p)-\omega'_\mu(\mathcal{N}_\mu(n))p\big)\big]$$

But by virtue of Proposition 2.1, ω_μ is ultimately strictly concave. It follows that the function $h_n\colon \mathbb{R}_+^* \to \mathbb{R}$, $x \mapsto -C_1[\omega(x)-\omega'_\mu(\mathcal{N}_\mu(n))x]$ is ultimately strictly concave, thence we have for large values of n that for all $x \in [\mathcal{N}_\mu(n),\mathcal{N}_\mu(n+1)]$ we have

$$h'_n(x) = -C_1[\omega'(x)-\omega'_\mu(\mathcal{N}_\mu(n))] < 0$$

Thence the function h_n is, for large values of n, strictly decreasing on the interval $[\mathcal{N}_\mu(n),\mathcal{N}_\mu(n+1)]$. It follows that the following estimates hold for large values of n

$$\begin{aligned}
&\|g_n\|_{\infty,\mathcal{K}_{\gamma_\mu,\frac{C_1}{2e},n}}\\
&\quad\leqslant C_0[1+\mathcal{N}_\mu(n+1)-\mathcal{N}_\mu(n)]\exp[-C_1(\omega(\mathcal{N}_\mu(n))-\mathcal{N}_\mu(n)\omega'(\mathcal{N}_\mu(n)))]\\
&\quad\leqslant C_0[1+\mathcal{N}_\mu(n+1)-\mathcal{N}_\mu(n)]\exp[-C_1\varphi_\mu(\mathcal{N}_\mu(n))]\\
&\quad\leqslant C_0[1+\mathcal{N}_\mu(n+1)-\mathcal{N}_\mu(n)]e^{-C_1n}\\
&\quad\leqslant C_0\big[e^{\frac{C_1}{2}}(\mathcal{N}_\mu(n+1)-\mathcal{N}_\mu(n))e^{-\frac{C_1}{2}(n+1)}+1\big]e^{-\frac{C_1}{2}n}
\end{aligned}$$

Since $\mathcal{N}_\mu$ is ultimately strictly convex, we can write for large values of n

$$\begin{aligned}
\|g_n\|_{\infty,\mathcal{K}_{\gamma_\mu,\frac{C_1}{2e},n}} &\leqslant C_0\big[e^{\frac{C_1}{2}}\mathcal{N}'_\mu(n+1)e^{-\frac{C_1}{2}(n+1)}+1\big]e^{-\frac{C_1}{2}n}\\
&\leqslant C_0\left[e^{\frac{C_1}{2}}\frac{e^{-\frac{C_1}{2}\varphi_\mu(\mathcal{N}_\mu(n+1))}}{\varphi'_\mu(\mathcal{N}_\mu(n+1))}+1\right]e^{-\frac{C_1}{2}n}
\end{aligned}$$

According to (2.11) we have

$$C_0\left[e^{\frac{C_1}{2}}\frac{e^{-\frac{C_1}{2}\varphi_\mu(\mathcal{N}_\mu(n+1))}}{\varphi'_\mu(\mathcal{N}_\mu(n+1))}+1\right]e^{-\frac{C_1}{2}n} \underset{n\to+\infty}{\sim} C_0e^{-\frac{C_1}{2}n}$$

Thence we have

$$\|g_n\|_{\infty,\mathcal{K}_{\gamma_\mu,\frac{C_1}{2e},n}} \leqslant C_2 e^{-\frac{C_1}{2}n}, \quad n \in \mathbb{N}$$

where $C_2 > 0$ is a constant. □

PROPOSITION 4.2. *Let $f \in C_{M_\mu}([-1,1])$; then there exists a function $F \in C_{M_\mu}(\mathbb{R})$ with support contained in the interval $[-2,2]$ and whose restriction to $[-1,1]$ is the function f.*

PROOF. According to Proposition 2.2, there exist $F_1 \in C_{M_\mu}([-3,-1]$ and $F_2 \in C_{M_\mu}([1,3])$ such that $F_1^{(n)}(-1) = f^{(n)}(-1)$, $F_2^{(n)}(1) = f^{(n)}(1)$, $n \in \mathbb{N}$. On the other hand, according to [**22**], there exists $\Phi \in C_{M_\mu}(\mathbb{R})$ with support contained in $[-2,2]$ such that $\Phi(x) = 1$, $x \in [-1,1]$. The function F defined by

$$\begin{aligned} F(x) &= f(x), && x \in [-1,1] \\ F(x) &= F_1(x)\Phi(x), && x \in [-3,-1] \\ F(x) &= F_2(x)\Phi(x), && x \in [1,3] \\ F(x) &= 0, && x \in \mathbb{R} \setminus [-3,3] \end{aligned}$$

satisfies the required conditions. □

END OF THE PROOF OF THE DIRECT PART OF THE MAIN THEOREM. Let $f \in C_{M_\mu}([-1,1])$. There exists, according to Proposition 4.2, a function $F \in C_{M_\mu}(\mathbb{R})$ whose support is contained in the interval $[-2,2]$ and whose restriction to $[-1,1]$ is the function f.

Let us consider the function g defined on the interval $[-\pi,\pi]$ by

$$\begin{aligned} g(\theta) &= F(\tan(\theta/2)), && \theta \in]-2\arctan(2), 2\arctan(2)[\\ g(\theta) &= 0, && \theta \in \mathbb{R} \setminus]-2\arctan(2), 2\arctan(2)[\end{aligned}$$

According to Cartan [**11**, Theorem III, pp. 24–27], the restriction of g to the interval $J := [-2\arctan(2), 2\arctan(2)]$ belongs to the Carleman class $C_{M_\mu}(J)$. But g is itself the restriction to $[-\pi,\pi]$ of a 2π-periodic, of class $\mathcal{C}^\infty$ which is vanishing on the set $[-\pi,\pi] \setminus J$. Thence $g \in C_{M_\mu}([-\pi,\pi])$.

According to Proposition 4.1 there exists constants $0 < A < 1$, $C > 0$, $0 < \rho < 1$ and a sequence $(g_n)_{n\geqslant 1}$ of rational functions defined on $\mathbb{C}^*$ such that

$$\|g_n\|_{\infty,\mathcal{K}_{\gamma_\mu,A,n}} \leqslant C\rho^n, \quad n \in \mathbb{N}, \quad g(\theta) = \sum_{n=0}^{\infty} g_n(e^{i\theta}), \quad \theta \in [-\pi,\pi].$$

Let $x \in [-2,2]$. There exists a unique $\theta \in [-2\arctan(2), 2\arctan(2)]$ such that $x = \tan\left(\frac{\theta}{2}\right)$, thence we have $F(x) = g(\theta) = \sum_{n=1}^{+\infty} g_n\left(\frac{i-x}{i+x}\right)$. On the other hand let $z \in \mathbb{C}$ be such that $|\operatorname{Im}(z)| < 1$ (then $z \in \mathbb{C} \setminus \{i,-i\}$). Let us set $\zeta = \frac{i-z}{i+z}$; then we have $|\operatorname{Im}(z)| \geqslant \frac{|1-|\zeta||}{1+|\zeta|}$. It follows that the following implication holds for every $A' \in]0,1[$

$$|\operatorname{Im}(z)| \leqslant A' e^{-\gamma_\mu(n)} \Rightarrow \frac{1 - A' e^{-\gamma_\mu(n)}}{1 + A' e^{-\gamma_\mu(n)}} \leqslant |\zeta| \leqslant \frac{1 + A' e^{-\gamma_\mu(n)}}{1 - A' e^{-\gamma_\mu(n)}}$$

If we choose $A' \in]0,1[$ sufficiently small, then we will obtain for every $n \in \mathbb{N}$

$$0 < 1 - Ae^{-\gamma_\mu(n)} < \frac{1 - A'e^{-\gamma_\mu(n)}}{1 + A'e^{-\gamma_\mu(n)}} \leqslant \frac{1 + A'e^{-\gamma_\mu(n)}}{1 - A'e^{-\gamma_\mu(n)}} < 1 + Ae^{-\gamma_\mu(n)}$$

Let us set $\mathcal{B}_n := \{z \in \mathbb{C} : |\operatorname{Im}(z)| < A'e^{-\gamma_\mu(n)}\}$. Thence the points i and $-i$ belong to $\mathbb{C} \setminus \mathcal{B}_n$ and we have $\frac{i-z}{i+z} \in \mathcal{K}_{\gamma_n,A,n}$, $z \in \mathcal{B}_n$. For each $n \in \mathbb{N}$, the function P_n defined on $\mathbb{C}\backslash\{i,-i\}$ by $P_n(z) = g_n(\frac{i-z}{i+z})$ is a rational function satisfying

$$\|P_n\|_{\infty,\mathcal{B}_n} \leqslant C\rho^n, \quad n \in \mathbb{N}$$

We have also for all $x \in [-2,2]$ that $F(x) = \sum_{n=1}^{\infty} P_n(x)$. But $[-1,1]_{\gamma_\mu,A',n} \subset \mathcal{B}_n$ for all $n \in \mathbb{N}$; thence we have

$$f(x) = \sum_{n=1}^{\infty} P_n(x), \;\; x \in [-1,1], \qquad \|P_n\|_{\infty,[-1,1]_{\gamma_\mu,A',n}} \leqslant C\rho^n, \;\; n \in \mathbb{N}.$$

Then, it follows

$$f(x) = \sum_{n=1}^{\infty} P_n(x), \;\; x \in [-1,1], \qquad \|P_n\|_{\infty,[-1,1]_{\mu,Ae^{-\varepsilon'},n}} \leqslant C\rho^n, \;\; n \in \mathbb{N}. \qquad \square$$

4.2. Converse part.

Proof. Let $A > 0$ and for each $n \in \mathbb{N}$, a function $f_n \colon [-1,1]_{\mu,A,n} \to \mathbb{C}$ which is holomorphic on $[-1,1]_{\mu,A,n}$ such that

$$f_n \in \mathcal{O}([-1,1]_{\mu,A,n}), \;\; n \in \mathbb{N}^*, \qquad \|f_n\|_{\infty,[-1,1]_{\mu,A,n}} \leqslant C\rho^n, \;\; n \in \mathbb{N}^*.$$

It follows that $\|f_n\|_{\infty,[-1,1]_{\gamma_\mu,Ae^{-\varepsilon},n}} \leqslant C\rho^n$, $n \in \mathbb{N}^*$. Thence the function series $\sum f_{n|[-1,1]}$ converges uniformly on $[-1,1]$ to a continuous function f.

We have $[-1,1] \subset [-1,1]_{\gamma_\mu,\frac{A}{2}e^{-\varepsilon},n} \subset [-1,1]_{\gamma_\mu,Ae^{-\varepsilon},n}$. Cauchy's inequalities allow us to write for all $p \in \mathbb{N}$

$$\|f_n^{(p)}\|_{\infty,[-1,1]} \leqslant Cp!\Big(\frac{2}{A}e^{\varepsilon}\Big)^p \exp\big[p\gamma_\mu(n) - \ln(\rho^{-1/2})n\big]\rho^{-n/2}. \tag{4.3}$$

On the other hand the supremum, for sufficiently large $p \in \mathbb{N}$, of the function $u \mapsto p\gamma_\mu(u) - \ln\big(1/\sqrt{\rho}\big)u$ on $[0,+\infty]$ is reached in the real $u_p > 0$ that satisfies $\gamma'_\mu(u_p) = \frac{1}{p}\ln\big(1/\sqrt{\rho}\,\big)$. Since for sufficiently large $p \in \mathbb{N}$, we have $\gamma'_\mu(u_p) = 1/t_0(u_p)$, it follows that $t_0(u_p) = p/\ln\big(1/\sqrt{\rho}\big)$. Consequently we can write

$$\begin{aligned} \sup_{n\in\mathbb{N}}\big[p\gamma_\mu(n) - \ln\big(1/\sqrt{\rho}\big)n\big] &\leqslant p(\gamma_\mu(u_p) - u_p\gamma'_\mu(u_p)) \\ &\leqslant p\mu(t_0(u_p)) \leqslant p\mu\big(p/\ln\big(1/\sqrt{\rho}\,\big)\big) \end{aligned} \tag{4.4}$$

Thence we have for $p \in \mathbb{N}$ sufficiently large we have for all $n \in \mathbb{N}$

$$\|f_n^{(p)}\|_{\infty,[-1,1]} \leqslant Cp!\Big(\frac{2}{A}e^{\varepsilon}\Big)^p \sqrt{\rho}^n e^{p\mu(p/\ln(1/\sqrt{\rho}))}$$

It follows that the function series $\sum f_n^{(p)}$ are for sufficiently large values of p normally convergent. Thence the function f is of class C^∞ on $[-1,1]$ and we have

$$\begin{aligned}\|f^{(p)}\|_{\infty,[-1,1]} &\leqslant \frac{2C}{A(1-\sqrt{\rho})}\Big(\frac{2}{A}\Big)^p p!\exp\big[p\big(\mu\big(p/\ln\big(1/\sqrt{\rho}\big)\big)-\mu(p)\big)\big]e^{p\mu(p)}\\ &\leqslant B^{p+1}p^p e^{p\mu(p)}\end{aligned}$$

for some constant $B>0$. Thence we have $f\in C_{M_\mu}([-1,1])$. □

5. Application: Alternative construction of Dyn'kin's pseudoanalytic extension for the Carleman class $C_{M_\mu}([-1,1])$

COROLLARY 5.1. *Let be $f\in C_{M_\mu}([-1,1])$. There exists a function $F\in C^\infty(\mathbb{C})$ with compact support such that*

$$F|_{[-1,1]}=f,\qquad |\bar\partial F(z)|\leqslant C_1 H_\mu\left[\frac{C_2}{\rho(z,[-1,1])}\right],\ \ z\in\mathbb{C}\setminus[-1,1]$$

where $C_1,C_2>0$ are constants.

PROOF. According to the main result there exist constants $A\in\,]0,1[$, $C>0$, $\rho\in\,]0,1[$, and a sequence of rational functions $(f_n)_{n\in\mathbb{N}}$ defined on some strip $B:=\{z\in\mathbb{C}:|\operatorname{Im}(z)|\leqslant A\}$ such that

$$\|f_n\|_{\infty,[-1,1]_{\mu,A,n}}\leqslant C\rho^n,\ \ n\in\mathbb{N}^*,\qquad \sum_{n=1}^{+\infty} f_{n|[-1,1]}=f.$$

It follows that $\|f_n\|_{\infty,[-1,1]_{\gamma_\mu,Ae^{-\varepsilon},n}}\leqslant C\rho^n$, $n\in\mathbb{N}^*$.

On the other hand, there exists, for each $n\in\mathbb{N}^*$, a function $\theta_n\colon\mathbb{C}\to[0,1]$ of class C^∞ on $\mathbb{C}$ ($\mathbb{C}$ is here identified with $\mathbb{R}^2$) and a family of positive constants $(L_\alpha)_{\alpha\in\mathbb{N}^2}$ [**22**] such that

$$\begin{aligned}\theta_n(z)&=1,\quad z\in[-1,1]_{\mu,\frac{A}{8},n}\\ \theta_n(z)&=0,\quad z\in\mathbb{C}\setminus[-1,1]_{\mu,\frac{A}{2},n}\\ |D^\alpha\theta_n(z)|&\leqslant L_\alpha e^{|\alpha|\mu(n)},\quad \alpha\in\mathbb{N}^2,\ z\in\mathbb{R}^2\end{aligned}$$

where $|\alpha|:=p+q$ and $D^\alpha:=\frac{\partial^{p+q}}{\partial x^p\partial y^q}$ for $\alpha=(p,q)$.

We denote by F_n the function defined by

$$\begin{aligned}F_n(z)&=\theta_n(z)f_n(z),z\in[-1,1]_{\gamma_\mu,A,n}\\ F_n(z)&=0,z\in\mathbb{C}\setminus[-1,1]_{\gamma_\mu,A,n}\end{aligned}$$

The function F_n is of class C^∞ on $\mathbb{C}$ and satisfies the condition

$$F_n|_{[-1,1]_{\mu,\frac{A}{8},n}}=f_n|_{[-1,1]_{\mu,\frac{A}{8},n}}.$$

Since $\|F_n\|_{\infty,\mathbb{C}}\leqslant C\rho^n$, $n\in\mathbb{N}$, it follows that the function series $\sum F_n$ is uniformly convergent on $\mathbb{C}$ to a continuous function F with compact support contained in $[-1,1]_A$. Furthermore it is clear that F is an extension to $\mathbb{C}$ of f.

Let $\alpha \in \mathbb{N}^2$, $n \in \mathbb{N}$ and $z \in \mathbb{C}$. If $z \in \mathbb{C} \setminus [-1,1]_{\mu,\frac{A}{2},n}$, then we have $D^\alpha F_n(z) = 0$. But when $z \in [-1,1]_{\mu,\frac{A}{8},n}$ we can write, in view of Cauchy's inequalities and inequality (4.4)

$$\begin{aligned}
|D^\alpha F_n(z)| &\leqslant \sum_{\beta\leqslant\alpha} \tbinom{\alpha}{\beta} |D^\beta \theta_n(z)||D^{\alpha-\beta} f_n(z)| \\
&\leqslant \sum_{\beta\leqslant\alpha} \tbinom{\alpha}{\beta} L_\beta e^{|\beta|\mu(n)} |D^{\alpha-\beta} f_n(z)| \\
&\leqslant \sum_{\beta\leqslant\alpha} \tbinom{\alpha}{\beta} L_\beta e^{|\beta|\varepsilon} e^{|\beta|\gamma_\mu(n)} |f_n^{(|\alpha|-|\beta|)}(z)| \\
&\leqslant \sum_{\beta\leqslant\alpha} \tbinom{\alpha}{\beta} L_\beta e^{|\beta|\varepsilon} e^{|\beta|\gamma_\mu(n)} C(4/A)^{|\alpha|-|\beta|} \\
&\quad \cdot (|\alpha|-|\beta|)! \sqrt{\rho}^n \exp\left[(|\alpha|-|\beta|)\gamma_\mu(n) - \ln\left(1\sqrt{\rho}\right)n\right] \\
&\leqslant \sum_{\beta\leqslant\alpha} \tbinom{\alpha}{\beta} L_\beta e^{|\beta|\varepsilon} e^{|\beta|\gamma_\mu(n)} C(4/A)^{|\alpha|-|\beta|} \\
&\quad \cdot (|\alpha|-|\beta|)! \sqrt{\rho}^n \exp\Big[\sup_{m\in\mathbb{N}} \left((|\alpha|-|\beta|)\gamma_\mu(m) - \ln\left(1/\sqrt{\rho}\right)m\right)\Big] \\
&\leqslant \sqrt{\rho}^n \sum_{\beta\leqslant\alpha} C\tbinom{\alpha}{\beta} e^{|\beta|\varepsilon} L_\beta (|\alpha|-|\beta|)! (4/A)^{|\alpha|-|\beta|} \\
&\quad \cdot \exp\Big[(|\alpha|-|\beta|)\mu\Big(\frac{(|\alpha|-|\beta|)}{\ln(1/\sqrt{\rho})}\Big)\Big]
\end{aligned}$$

It follows that the function series $\sum D^\alpha F_n(z)$ is for all $\alpha \in \mathbb{N}^2$ normally convergent on $\mathbb{C}$. Thence the function $F = \sum_{n=1}^{+\infty} F_n$ is of class C^∞ on $\mathbb{C}$.

Let $z \in \mathbb{C} \setminus [-1,1]$. Then we have $\bar\partial F(z) = \sum_{n=1}^{+\infty} \bar\partial F_n(z)$. On the other hand, we have

$$\bar\partial F_n(z) = 0 \ \text{ if } \ \rho(z,[-1,1]) \in \big[0, \tfrac{A}{8} e^{-\varepsilon} e^{-\gamma_\mu(n)}\big[\cup \big] A e^{-\varepsilon} e^{-\gamma_\mu(n)}, +\infty\big[.$$

If $\rho(z,[-1,1]) \in \big[\frac{A}{8} e^{-\mu(n)}, A e^{-\mu(n)}\big[$, then, again by virtue of (4.4), we have

$$\begin{aligned}
|\bar\partial F_n(z)| &= |f_n(z)||\bar\partial\theta_n(z)| \\
&\leqslant \frac{C}{2}\rho^n \Big(\Big|\frac{\partial\theta_n}{\partial x}(z)\Big| + \Big|\frac{\partial\theta_n}{\partial y}(z)\Big|\Big) \\
&\leqslant \frac{C}{2}(L_{(1,0)} + L_{(0,1)}) e^{\varepsilon} e^{\gamma_\mu(n) - \frac{1}{2}\ln(\frac{1}{\rho})n} \sqrt{\rho}^n \\
&\leqslant \frac{C}{2}(L_{(1,0)} + L_{(0,1)}) e^{\varepsilon} e^{\mu(2/\ln(1/\sqrt{\rho}))} \sqrt{\rho}^n
\end{aligned}$$

Let us set

$$A_1 := \frac{C}{2}(L_{(1,0)} + L_{(0,1)}) e^{\varepsilon} e^{\mu(2/\ln(1/\sqrt{\rho}))}, \quad \lambda := -\ln\sqrt{\rho} > 0$$

Thence the following estimate holds

$$\begin{aligned}|\bar\partial F(z)| &\leqslant \sum_{\frac{A}{8}e^{-\mu(n)}\leqslant\rho(z,[-1,1])\leqslant Ae^{-\mu(n)}} A_1 e^{-\lambda n}\\ &\leqslant A_1 \sum_{\frac{A}{8\rho(z,[-1,1])}\leqslant e^{\mu(n)}} e^{-\lambda n}\\ &\leqslant A_1 \sum_{\frac{A}{8e^{\varepsilon}\rho(z,[-1,1])}\leqslant e^{\gamma_\mu(n)}} e^{-\lambda n}\end{aligned}$$

It follows that if z is sufficiently close to $[-1,1]$, then the last estimate will become

$$\begin{aligned}|\bar\partial F(z)| &\leqslant A_1 \sum_{h_\mu(\frac{A}{8e^{\varepsilon}\rho(z,[-1,1])})\leqslant n} e^{-\lambda n}\\ &\leqslant \frac{A_1}{1-e^{-\lambda}}\exp\Big[-\lambda h_\mu\Big(\frac{A}{8e^{\varepsilon}\rho(z,[-1,1])}\Big)\Big]\end{aligned}$$

But we know that the function h_μ is regularly varying. Thence there exists a constant $A_2>0$ such that we have ultimately

$$\lambda h_\mu\Big(\frac{A}{8e^{\varepsilon}}x\Big)\geqslant h_\mu(A_2x)$$

Consequently we have for z sufficiently close to $[-1,1]$

$$|\bar\partial F(z)|\leqslant \frac{A_1}{1-e^{-\frac{\lambda}{2}}}\exp\Big[-h_\mu\Big(\frac{A_2}{\rho(z,[-1,1])}\Big)\Big]$$

Thence there exists a constant $A_3>0$ such that

$$|\bar\partial F(z)|\leqslant A_3 H_\mu\Big(\frac{A_2}{\rho(z,[-1,1])}\Big),\quad z\in\mathbb{C}$$

The proof of the corollary is complete. □

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