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REFINEMENTS OF SOME LIMIT HARDY-TYPE INEQUALITIES VIA SUPERQUADRACITY

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Abstract. Refinements of some limit Hardy-type inequalities are derived and discussed using the concept of superquadracity. We also proved that all three constants appearing in the refined inequalities obtained are sharp. The natural turning point of our refined Hardy inequality is $p=2$ and for this case we have even equality.

1. Introduction

Hardy stated in [**5**] and finally proved in [**6**] the following classical inequality: for any $p>1$ and any integrable function $f(x)\geqslant 0$ on $(0,\infty)$, the inequality

$$\int_0^\infty\Big(\frac{1}{x}\int_0^x f(t)\,dt\Big)^p dx\leqslant\Big(\frac{p}{p-1}\Big)^p\int_0^\infty f^p(x)\,dx, \tag{1.1}$$

holds, where the constant $(\frac{p}{p-1})^p$ is the best possible.

Since then, Hardy's inequality has been extensively studied. Consequently, a lot of information about Hardy's inequality abounds nowadays in literature, comprising both its generalizations and applications in different ways (see e.g. [**8, 9, 10**] and the references therein).

However, there exist very few Hardy-type inequalities with sharp constants in the limit case and when the interval $(0,\infty)$ is replaced by a finite one $(0,l)$, $l<\infty$. We now proceed to give some known examples of such Hardy-type inequalities.

In 1928, Hardy himself (see [**7**]) proved the following first weighted version of (1.1) as follows:

$$\int_0^\infty\Big(\frac{1}{x}\int_0^x f(t)\,dt\Big)^p x^a dx\leqslant\Big(\frac{p}{p-a-1}\Big)^p\int_0^\infty f^p(x)x^a dx \tag{1.2}$$

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holds for all measurable and non-negative functions f on $(0,\infty)$ whenever $a<p-1$, $p\geqslant 1$. He obviously thought that (1.2) is a generalization of (1.1). However, recently Persson and Samko [**12**] pointed out that this is not genuinely true since (1.2) is indeed equivalent to (1.1) through some suitable substitutions and variable transformations. In the same paper, they stated and proved the following result: Let g be a nonnegative and measurable function on $(0,l)$, $0<l\leqslant\infty$. If $p<0$ or $p\geqslant 1$, then

$$\int_0^l\Big(\frac{1}{x}\int_0^x g(y)\,dy\Big)^p\frac{dx}{x}\leqslant 1\cdot\int_0^l g^p(x)\Big(1-\frac{x}{l}\Big)\frac{dx}{x}, \tag{1.3}$$

while in the case $p<0$ we assume that $g(x)>0$, $0<x\leqslant l$. Furthermore, (1.3) is equivalent to the following sharp local variant of (1.2):

$$\int_0^l\Big(\frac{1}{x}\int_0^x f(y)\,dy\Big)^p x^a dx\leqslant\Big(\frac{p}{p-1-a}\Big)^p\int_0^1 f^p(x)\,x^a\Big(1-\Big(\frac{x}{l}\Big)^{\frac{p-a-1}{p}}\Big)dx,$$

where the constant $(\frac{p}{p-1-a})^p$ is sharp.

Throughout this paper we shall assume that log is the natural logarithm. In [**3**], Bennett proved that if $\alpha>0$, $1\leqslant p<\infty$, and f is a nonnegative measurable function on $[0,1]$, then the inequalities

$$\int_0^1\Big[\log\frac{e}{x}\Big]^{\alpha p-1}\Big(\int_0^x f(y)\,dy\Big)^p\frac{dx}{x}\leqslant\alpha^{-p}\int_0^1 x^p\Big[\log\frac{e}{x}\Big]^{(1+\alpha)p-1}f^p(x)\frac{dx}{x}, \tag{1.4}$$

$$\int_0^1\Big[\log\frac{e}{x}\Big]^{-\alpha p-1}\Big(\int_x^1 f(y)\,dy\Big)^p\frac{dx}{x}\leqslant\alpha^{-p}\int_0^1 x^p\Big[\log\frac{e}{x}\Big]^{(1-\alpha)p-1}f^p(x)\frac{dx}{x} \tag{1.5}$$

hold. These inequalities hold also with the usual modifications if $p=\infty$ see [**2**], where it was also proved that the constant α^{-p} is sharp. For $p=\infty$ you just raise both sides of (1.4) and (1.5) to power $1/p$ and let $p\to\infty$ to get the usual supremum interpretation of (1.4) and (1.5) for $p=\infty$. We refer interested readers to papers [**2, 4, 11**] for more information about the proofs and applications of inequalities (1.4) and (1.5).

Recently Barza et al. [**2**] obtained some refinements and extensions of inequalities (1.4) and (1.5). Specifically, the following inequalities are derived and proved:

$$\begin{aligned}\alpha^{p-1}\Big(\int_0^1 f(x)\,dx\Big)^p+\alpha^p\int_0^1\Big[\log\frac{e}{x}\Big]^{\alpha p-1}\Big(\int_0^x f(y)\,dy\Big)^p\frac{dx}{x}\\ \leqslant\int_0^1 x^p\Big[\log\frac{e}{x}\Big]^{(1+\alpha)p-1}f^p(x)\frac{dx}{x},\end{aligned} \tag{1.6}$$

$$\begin{aligned}\alpha^{p-1}\Big(\int_0^1 f(x)\,dx\Big)^p+\alpha^p\int_0^1\Big[\log\frac{e}{x}\Big]^{-\alpha p-1}\Big(\int_x^1 f(y)\,dy\Big)^p\frac{dx}{x}\\ \leqslant\int_0^1 x^p\Big[\log\frac{e}{x}\Big]^{(1-\alpha)p-1}f^p(x)\frac{dx}{x},\end{aligned} \tag{1.7}$$

where f is a nonnegative measurable function on $[0,1]$. Both constants α^{p-1} and α^p in (1.6) and (1.7) are sharp.

The motivation for the current paper comes from the works of Bennett [**3**] and Barza et al. [**2**]. Our aim is to obtain some further refinements and extensions of (1.6) and (1.7). These inequalities have some remarkable properties e.g., that now the natural "turning point" (the point where the inequality reverses) is $p = 2$, while all other inequalities above have turning point at $p = 1$. Another remarkable property is that our new inequalities contain three constants and all are sharp.

This paper is organized as follows: In Section 2, we present our main result (Theorem 2.1) which is a refined version of inequalities (1.4), (1.5), (1.6), and (1.7) via superquadracity argument (see Proposition 2.1). This proposition is then employed to prove our main result which have refinement terms not present in the results of Bennett [**3**] and Barza et al. [**2**]. This inequality has the remarkable property that it holds in the reversed direction for $1 < p \leqslant 2$ so that for $p = 2$ we get a new identity (cf. Remark 3.1). We also show that the constants involved are all sharp. In Section 3, we present further result and some examples and remarks.

2. Some inequalities involving superquadratic and subquadratic functions

Definition 2.1. [**1**, Definition 2.1] A function $\Phi : [0,\infty) \to \mathbb{R}$ is said to be superquadratic provided that for all $x \geqslant 0$ there exists a constant $C_x \in \mathbb{R}$ such that $\Phi(y) - \Phi(x) - C_x(y-x) - \Phi(|y-x|) \geqslant 0$ for all $y \geqslant 0$. Φ is subquadratic if $-\Phi$ is superquadratic. A function $f : [0,\infty) \to \mathbb{R}$ is superadditive provided $f(x+y) \geqslant f(x) + f(y)$ for all $x, y \geqslant 0$. If the reverse inequality holds, then f is said to be subadditive.

Lemma 2.1 (See [**1**, Lemma 2.2]). *Let $\Phi(x)\colon [0,\infty) \to \mathbb{R}$ be a superquadratic function with C_x as in Definition 2.1.*

(1) *Then $\Phi(0) \leqslant 0$.*
(2) *If $\Phi(0) = \Phi'(0) = 0$, then $C_x = \Phi'(x)$ whenever Φ is differentiable at $x > 0$.*
(3) *If $\Phi \geqslant 0$, then Φ is convex and $\Phi(0) = \Phi'(0) = 0$.*

Here and in the sequel the notation $\Phi'(0)$ means $\Phi'_+(0)$.

Lemma 2.2 (See [**1**], Lemma 3.1). *Suppose $\Phi : [0,\infty) \to \mathbb{R}$ is continuously differentiable and that $\Phi(0) \leqslant 0$. If Φ' is superadditive or $\frac{\Phi'(x)}{x}$ is nondecreasing, then Φ is superquadratic.*

Before we state our main result, we state the following Proposition which is of independent interest and very useful in the proof of our main result.

Proposition 2.1. *Let $\Phi\colon [0,\infty) \to \mathbb{R}$ be a differentiable function such that $\Phi(0) = \Phi'(0) = 0$. Then*

$$\Phi(y) - \Phi(1) - \Phi'(1)(y-1) - \Phi(|y-1|) \begin{cases} \geqslant 0 & \text{if } \Phi \text{ superquadratic} \\ \leqslant 0 & \text{if } \Phi \text{ subquadratic}, \end{cases} \tag{2.1}$$

holds for all $y \geqslant 0$. If $\Phi(x) = x^p$, then equality in (2.1) hold for all y if and only if $p = 2$.

Proof. Since $\Phi(x)$ is differentiable, it follows from Definition 2.1 and Lemma 2.1 that for $x = 1$, there exists a constant $C_1 = \Phi'(1)$ such that

$$\Phi(y) - \Phi(1) - \Phi'(1)(y-1) - \Phi(|y-1|) \geqslant 0,$$

for all $y \geqslant 0$ whenever Φ is superquadratic. The proof of the case when Φ is subquadratic is similar to the one given above, except that the inequality sign is reversed. The claimed equality case is obvious. □

Remark 2.1. By setting $\Phi(x) = x^p$ in Proposition 2.1, we get by Lemma 2.2, that x^p is superquadratic if $p \geqslant 2$ and subquadratic if $1 < p \leqslant 2$. Furthermore, Proposition 2.1 implies the following result:

Lemma 2.3. *Let $h > 0$; then*

$$h^p - p(h-1) - 1 - |h-1|^p \begin{cases} \geqslant 0 & \text{if } p \geqslant 2 \\ \leqslant 0 & \text{if } 1 < p \leqslant 2. \end{cases} \tag{2.2}$$

Equality holds for all $h > 0$ if and only if $p = 2$ and when $p \neq 2$ if and only if $h = 1$.

Proof. Apply Proposition 2.1 with $\Phi(x) = x^p$, $p > 1$. Then, we find that (2.2) holds and equality holds if $p = 2$ and when $p \neq 2$, $h = 1$. The "only if" part follows by considering the function $f(h) = h^p - p(h-1) - 1 - (h-1)^p$ and noting that $f(h)$ is increasing for $p \geqslant 2$, decreasing for $p \leqslant 2$ and $f(1) = 1$. □

Our main result reads:

Theorem 2.1. *Let $\alpha, p > 1$ and f be a nonnegative and measurable function on $[0,1]$.*

(1) *If $p \geqslant 2$, then*

$$\begin{aligned} \alpha^{p-1}\Big(\int_0^1 f(x)\,dx\Big)^p + \alpha^p \int_0^1 \Big[\log\frac{e}{x}\Big]^{\alpha p-1}\Big(\int_0^x f(y)\,dy\Big)^p \frac{dx}{x} \\ + \int_0^1 \Big|x\log\frac{e}{x}f(x) - \alpha\int_0^x f(y)\,dy\Big|^p \Big(\log\frac{e}{x}\Big)^{\alpha p-1}\frac{dx}{x} \\ \leqslant \int_0^1 x^p\Big(\log\frac{e}{x}\Big)^{(1+\alpha)p-1} f^p(x)\frac{dx}{x} \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \alpha^{p-1}\Big(\int_0^1 f(x)\,dx\Big)^p + \alpha^p \int_0^1 \Big[\log\frac{e}{x}\Big]^{-\alpha p-1}\Big(\int_x^1 f(y)\,dy\Big)^p \frac{dx}{x} \\ + \int_0^x \Big|x\log\frac{e}{x}f(x) - \alpha\int_0^x f(y)\,dy\Big|^p \Big[\log\frac{e}{x}\Big]^{-\alpha p-1}\frac{dx}{x} \\ \leqslant \int_0^1 x^p\Big[\log\frac{e}{x}\Big]^{(1-\alpha)p-1} f^p(x)\frac{dx}{x}. \end{aligned} \tag{2.4}$$

All constants α^{p-1}, α^p and 1 in front of the integrals on the left-hand side in (2.3) and (2.4) are sharp. If $p > 2$, then equality is never attained unless f is identically zero.

(2) *If $1 < p \leqslant 2$, then both* (2.3) *and* (2.4) *hold in the reverse direction and the constants in both inequalities are sharp. Also in this case all constants α^{p-1}, α^p and 1 in front of the integrals on the left-hand side of reversed inequalities* (2.3) *and* (2.4) *are sharp. If $1 < p < 2$, then equality is never attained unless f is identically zero.*

(3) *If $p = 2$ we have equality in* (2.3) *and* (2.4) *for any measurable function f and any $\alpha > 1$.*

PROOF. (a) Let $p > 1$. Suppose that f is continuous and nonnegative. Then define for $x \in (0, 1]$ the function F by

$$
\begin{aligned}
F(x;\alpha,p) := & \int_0^x y^p \Big[\log\frac{e}{y}\Big]^{(1+\alpha)p-1} f^p(y)\frac{dy}{y} \\
& - \alpha^{p-1}\Big(\log\frac{e}{x}\Big)^{\alpha p}\Big(\int_0^x f(y)\,dy\Big)^p \\
& - \alpha^p \int_0^x \Big[\log\frac{e}{y}\Big]^{\alpha p-1}\Big(\int_0^y f(s)\,ds\Big)^p\frac{dy}{y} \\
& - \int_0^x \Big|y\log\frac{e}{y}f(y) - \alpha\int_0^y f(s)\,ds\Big|^p\Big[\log\frac{e}{y}\Big]^{\alpha p-1}\frac{dy}{y}.
\end{aligned}
\tag{2.5}
$$

Differentiating (2.5) yields

$$
\begin{aligned}
\frac{d}{dx}F(x;\alpha,p) = & \, x^p\Big[\log\frac{e}{x}\Big]^{(1+\alpha)p-1} f^p(x)\frac{1}{x} \\
& + \frac{p}{x}\alpha^p\Big(\log\frac{e}{x}\Big)^{\alpha p-1}\Big(\int_0^x f(y)\,dy\Big)^p \\
& - pf(x)\alpha^{p-1}\Big(\log\frac{e}{x}\Big)^{\alpha p}\Big(\int_0^x f(y)\,dy\Big)^{p-1} \\
& - \alpha^p\Big[\log\frac{e}{x}\Big]^{\alpha p-1}\Big(\int_0^x f(s)\,ds\Big)^p\frac{1}{x} \\
& - \Big|x\log\frac{e}{x}f(x) - \alpha\int_0^x f(y)\,dy\Big|^p\Big[\log\frac{e}{x}\Big]^{\alpha p-1}\frac{1}{x}.
\end{aligned}
\tag{2.6}
$$

We assume without restriction that $f(t) > 0, t > 0$ (if not we first assume this and use a limit argument). By putting

$$h(x,\alpha) := \frac{x(\log\frac{e}{x})f(x)}{\alpha\int_0^x f(y)\,dy},$$

in (2.6) we obtain

$$
\begin{aligned}
\frac{d}{dx}F(x;\alpha,p) = & \, \alpha^p\Big[\log\frac{e}{x}\Big]^{\alpha p-1}\Big(\int_0^x f(s)\,ds\Big)^p\frac{1}{x} \\
& \times [h^p(x;\alpha) - p(h(x;\alpha)-1) - 1 - |h(x;\alpha)-1|^p].
\end{aligned}
$$

Hence, by Lemma 2.3, we get $\frac{d}{dx}F(x;\alpha,p) > 0$. That is $F(x;\alpha,p)$ is strictly increasing. In particular $F(1;\alpha,p) \geqslant \lim_{x\to 0^+} F(x;\alpha,p)$. We claim that

$$\lim_{x\to 0^+} F(x;\alpha,p) = 0.$$

To justify our claim, we use Hölder's inequality in the following form:

$$\int_0^x |f_1(y)f_2(y)|dy \leqslant \Big(\int_0^x |f_1|^p dy\Big)^{1/p}\Big(\int_0^x |f_2|^{p'} dy\Big)^{1/p'}, \tag{2.7}$$

which holds for all continuous functions f, g and for all $p, p' > 1$ such that $1/p + 1/p' = 1$. If we let

$$f_1(y) = y^{1-1/p}\Big(\log\frac{e}{y}\Big)^{\alpha+1-1/p} f(y), \quad f_2(y) = y^{-1+1/p}\Big(\log\frac{e}{y}\Big)^{-\alpha-1+1/p},$$

then with p and $p' = p/(p-1)$, we find (2.7) gives

$$\begin{aligned}\int_0^x f(y)\,dy &= \int_0^x \Big(y^{1-1/p}\Big(\log\frac{e}{y}\Big)^{\alpha+1-1/p} f(y)\Big)\Big(y^{-1+1/p}\Big(\log\frac{e}{y}\Big)^{-\alpha-1+1/p}\Big)dy\\
&\leqslant \Big(\int_0^x y^{p-1}\Big(\log\frac{e}{y}\Big)^{(1+\alpha)p-1} f^p(y)\,dy\Big)^{1/p}\\
&\quad \Big(\int_0^x \frac{1}{y}\Big(\log\frac{e}{y}\Big)^{-\alpha p/(p-1)-1} dy\Big)^{(p-1)/p}\\
&= \Big(\int_0^x y^{p-1}\Big(\log\frac{e}{y}\Big)^{(1+\alpha)p-1} f^p(y)\,dy\Big)^{1/p}\\
&\quad \Big(\frac{p-1}{\alpha p}\Big(\log\frac{e}{x}\Big)^{-\alpha p/(p-1)}\Big)^{(p-1)/p}\\
&= \Big(\int_0^x y^{p-1}\Big(\log\frac{e}{y}\Big)^{(1+\alpha)p-1} f^p(y)\,dy\Big)^{1/p}\Big(\frac{p-1}{\alpha p}\Big)^{(p-1)/p}\Big(\log\frac{e}{x}\Big)^{-\alpha}.\end{aligned}$$

Hence, we get

$$\begin{aligned}0 &< \Big(\log\frac{e}{x}\Big)^{\alpha p}\Big(\int_0^x f(y)\,dy\Big)^p\\
&\leqslant \Big(\frac{p-1}{\alpha p}\Big)^{p-1}\Big(\int_0^x y^{p-1}\Big(\log\frac{e}{y}\Big)^{(1+\alpha)p-1} f^p(y)\,dy\Big).\end{aligned} \tag{2.8}$$

Taking the limit of (2.8) as $x \to 0^+$, we obtain

$$\lim_{x\to 0^+}\Big(\Big(\log\frac{e}{x}\Big)^{\alpha p}\Big(\int_0^x f(y)\,dy\Big)^p\Big) = 0.$$

This consequently implies that $\lim_{x\to 0^+} F(x;\alpha,p) = 0$ and, in particular that,

$$F(1;\alpha,p) \geqslant \lim_{x\to 0^+} F(x;\alpha,p) = 0.$$

So, we proved that (2.3) holds for all continuous functions. By standard approximating arguments, (2.3) holds for all measurable functions.

Now we proceed to prove that the constants in the inequality (2.3) are all sharp. To this end, assume on the contrary that (2.3) holds for some constants C_1, C_2 such that $0 < C_1$, $C_2 < \infty$ and $C_2 > \alpha^p$, i.e.,

$$C_1\Big(\int_0^1 f(x)\,dx\Big)^p + C_2\int_0^1 \Big[\log\frac{e}{x}\Big]^{\alpha p-1}\Big(\int_0^x f(y)\,dy\Big)^p\frac{dx}{x} \tag{2.9}$$

$$+\int_0^1 \left|x\log\frac{e}{x}f(x)-\alpha\int_0^x f(y)\,dy\right|^p\left[\log\frac{e}{x}\right]^{\alpha p-1}\frac{dx}{x}$$
$$\leqslant \int_0^1 x^p\left[\log\frac{e}{x}\right]^{(1+\alpha)p-1}f^p(x)\frac{dx}{x}.$$

By using the test function,

$$f_\epsilon(x) := \frac{1}{x}\Big(\log\frac{e}{x}\Big)^{-(1+\epsilon+\alpha)},\quad \epsilon>0 \tag{2.10}$$

we find (after some calculations) that (2.9) yields

$$\frac{C_1}{(\epsilon+\alpha)^p}+\frac{C_2}{\epsilon p(\epsilon+\alpha)^p}+\frac{\epsilon^p}{\epsilon p(\alpha+\epsilon)^p}\leqslant\frac{1}{\epsilon p}$$

i.e., that $\epsilon p C_1 + C_2 + \epsilon^p \leqslant (\epsilon+\alpha)^p$. By letting $\epsilon \to 0_+$, we obtain that $C_2 \leqslant \alpha^p$, a contradiction. Thus, the constant $C_2 = \alpha^p$ in (2.9) is sharp. We assume now that (2.9) holds with $C_2 = \alpha^p$ for some $C_1 > \alpha^{p-1}$ and use the same test function f_ϵ in (2.10) to obtain

$$\frac{C_1}{(\epsilon+\alpha)^p}+\frac{\alpha^p}{\epsilon p(\epsilon+\alpha)^p}+\frac{\epsilon^p}{(\epsilon+\alpha)^p\epsilon p}\leqslant\frac{1}{\epsilon p},$$

i.e., $C_1 \leqslant \big((\epsilon+\alpha)^p-\epsilon^p-\alpha^p\big)/\epsilon p$. Hence, by letting $\epsilon\to 0_+$, we find

$$C_1\leqslant\lim_{\epsilon\to 0_+}\Big(\frac{(\epsilon+\alpha)^p-\epsilon^p-\alpha^p}{\epsilon p}\Big)=\frac{1}{p}\lim_{\epsilon\to 0_+}\frac{(\epsilon+\alpha)^p-\alpha^p}{\epsilon}-\frac{1}{p}\lim_{\epsilon\to 0_+}\epsilon^{p-1}=\alpha^{p-1}.$$

This contradiction shows that $C_1 = \alpha^{p-1}$ is a sharp constant in (2.3). That the constant $C_3 = 1$, in front of the third integral in (2.3), is also sharp follows in a similar way. In fact, consider (2.3) with the constants $C_1 = \alpha^{p-1}$, $C_2 = \alpha^p$ and $C_3 > 1$. Then, by using the same test functions $f_\epsilon(x)$ as above and letting $\epsilon \to 0^+$, we get a contradiction. It is clear from Lemma 2.1 that for $p > 2$, we cannot have equality in (2.3) unless f is identically zero.

The proof of (2.4) is similar. For this case we consider

$$\begin{aligned}G(x,\alpha,p) :=& \int_0^x y^p\Big[\log\Big(\frac{e}{y}\Big)\Big]^{(1-\alpha)p-1}f^p(y)\frac{dy}{y}\\ &-\alpha^{p-1}\Big(\log\Big(\frac{e}{x}\Big)\Big)^{-\alpha p}\Big(\int_0^x f(s)\,ds\Big)^p\\ &-\alpha^p\int_0^x\Big(\log\Big(\frac{e}{y}\Big)\Big)^{-\alpha p-1}\Big(\int_y^1 f(s)\,ds\Big)^p\frac{dy}{y}\\ &-\int_0^x \mid y\log\Big(\frac{e}{y}\Big)f(y)-\alpha\int_0^x f(y)\,dy\mid^p\Big[\log\Big(\frac{e}{y}\Big)\Big]^{-\alpha p-1}\frac{dy}{y}\end{aligned}$$

and argue in a similar way as above. Also, the proof of the sharpness of the constants α^{p-1},α^p and 1 and cases of equality is similar as before, so we omit the details.

(b) For the case $1 < p \leqslant 2$, the crucial inequality (2.2) holds in the reversed direction (see Lemma 2.3). Hence, the reverse of inequality (2.3) holds in this case. Moreover, the proof of the sharpness of the constants α^{p-1},α^p and 1 and the cases of equality only consist of obvious modifications of the proof above, so we leave out the details.

(c) The proof of equality for the case $p = 2$ is just an easy consequence and modification of the proof above. $\square$

3. Concluding result, remarks and examples

We put

$$I_1 = \alpha^{p-1}\Big(\int_0^1 f(x)\,dx\Big)^p + \alpha^p \int_0^1 \Big[\log\Big(\frac{e}{x}\Big)\Big]^{\alpha p-1}\Big(\int_0^x f(y)\,dy\Big)^p \frac{dx}{x},$$

$$I_2 = \int_0^1 \Big|x\log\Big(\frac{e}{x}\Big)f(x) - \alpha\int_0^x f(y)\,dy\Big|^p \Big(\log\Big(\frac{e}{x}\Big)\Big)^{\alpha p-1}\frac{dx}{x}$$

and

$$I_3 = \int_0^1 x^p\Big(\log\Big(\frac{e}{x}\Big)\Big)^{(1+\alpha)p-1} f^p(x)\frac{dx}{x}.$$

In particular, our result implies the following new information for the limit case of Hardy's inequality:

Example 3.1. (1) If $p \geqslant 2$, then the Bennett and Barza et al. estimate $I_1 \leqslant I_3$ is improved to $I_1 + I_2 \leqslant I_3$.

(2) For the case $1 < p \leqslant 2$ the Bennett and Barza et al. estimate $I_1 \leqslant I_3$ is even turned to the two-sided estimate $I_1 \leqslant I_3 \leqslant I_1 + I_2$.

(3) For the case $p = 2$ we get the remarkable new identity $I_3 = I_1 + I_2$.

Remark 3.1. The natural "turning point" (when equality sign is reversed) in the Hardy type inequalities is $p = 1$. The example above is the only example so far of a limit Hardy type inequality where the turning point is $p = 2$. A similar example of the limit Hardy inequality with turning point $p = 2$ as that in Example 3.1 can be obtained by using (1.7) and (2.4) in a similar way as above.

Remark 3.2. All four inequalities in Theorem 2.1 contain three constants in front of the integrals on the left-hand side and all are sharp. This is the only inequality in the literature on the Hardy type inequalities with this property.

Proposition 3.1. *Let f be a positive and measurable function on $(0,1)$ and let u and v be two weight functions on $(0,1)$ such that*

$$u(y) = \int_y^1 \frac{v(x)}{x^2}dx.$$

If $\Psi\colon [0,\infty) \to \Re$ is a non-negative superquadratic function, then

(3.1)
$$\int_0^1 \Psi\Big(\frac{1}{x}\int_0^x f(y)\,dy\Big)\frac{v(x)}{x}dx \leqslant \int_0^1 \Psi(f(x))u(x)\frac{dx}{x} - \int_0^1\int_y^1 \Psi\Big(\Big|f(y) - \frac{1}{x}\int_0^x f(y)\,dy\Big|\Big)\frac{v(x)}{x^2}dx\,dy.$$

If Ψ is subquadratic, then inequality (3.1) *holds in the reverse direction. Moreover, in inequality* (3.1) *and the reverse inequality for subquadratic Ψ, equality holds for $f \equiv C, C > 0$.*

Proof. By using Jensen's refined inequality of Abramovich et al. [**1**] for a superquadratic function Ψ and Fubini's theorem, we find that

$$\int_0^1 \Psi\Big(\frac{1}{x}\int_0^x f(y)\,dy\Big)\frac{v(x)}{x}dx \leqslant \int_0^1 \Psi(f(y))\Big(\int_y^1 \frac{v(x)}{x^2}dx\Big)dy - \int_0^1\int_y^1 \Psi\Big(\Big|f(y)-\frac{1}{x}\int_0^x f(y)\,dy\Big|\Big)\frac{v(x)}{x^2}dx\,dy,$$

i.e.,

$$\int_0^1 \Psi\Big(\frac{1}{x}\int_0^x f(y)\,dy\Big)\frac{v(x)}{x}dx \leqslant \int_0^1 \Psi(f(y))u(y)\frac{dy}{y} - \int_0^1\int_y^1 \Psi\Big(\Big|f(y)-\frac{1}{x}\int_0^x f(y)\,dy\Big|\Big)\frac{v(x)}{x^2}dx\,dy.$$

The proof for the case when Ψ is subquadratic follows similarly, except that the only inequality above holds in the reverse direction. By substituting $f \equiv C$ in inequality (3.1), we have after some easy calculations that equality holds in (3.1). □

Example 3.2. By using Proposition 3.1 with the superquadratic function $\Psi(x) = x^p, p \geqslant 2$, for $v(x) = x^p(\log\frac{e}{x})^{\alpha p-1}$, we find that

$$I_4 := \int_0^1 \Big(\log\frac{e}{x}\Big)^{\alpha p-1}\Big(\int_0^x f(y)\,dy\Big)^p\frac{dx}{x} \leqslant \int_0^1 f^p(x)u(x)\frac{dx}{x} - \int_0^1\int_y^1 \Big|xf(y)-\int_0^x f(y)\,dy\Big|^p\Big(\log\frac{e}{x}\Big)^{\alpha p-1}\frac{dx}{x^2}dx\,dy =: I_5$$

where

$$u(x) := x\int_x^1 x^{p-2}(\log\frac{e}{y})^{\alpha p-1}dy.$$

Remark 3.3. If we put

$$I_6 : = \alpha^{-p}\int_0^1 x^p\Big[\log\frac{e}{x}\Big]^{(1+\alpha)p-1}f^p(x)\frac{dx}{x} - \alpha^{-1}\Big(\int_0^1 f(x)\,dx\Big)^p - \alpha^{-p}\int_0^x \Big|x\log\frac{e}{x}f(x)-\alpha\int_0^x f(y)\,dy\Big|^p\Big[\log\frac{e}{x}\Big]^{\alpha p-1}\frac{dx}{x},$$

then, it follows from Example 3.2 and inequality (2.3) that we have the following strict improvement of inequality (2.3),

$$I_4 \leqslant \min(I_5, I_6),$$

where I_4, I_5 and I_6 are as defined above and $p \geqslant 2$.

References

1. S. Abramovich, G. Jameson, G. Sinnamon, *Refining of Jensen's Inequality*, Bull. Math. Soc. Sci. Math. Roum., Nouv. Sér. **47**(1–2) (2004), 3–14.
2. S. Barza, L-E. Persson, N. Samko, *Some new sharp limit Hardy-type inequalities via convexity*, J. Inequal. Appl. **2014** (2014), 2014:6.
3. C. Bennett, *Intermediate spaces and the class* $L\log^{+}L$, Ark. Mat. **11** (1973), 215–228.
4. H. Brezis, S. A. Waigner, *A note on limiting cases of Sobolev embeddings and convolution inequalities*, Commun. Partial Differ. Equations **5** (1980), 773–789 .
5. G. H. Hardy, *Notes on a theorem of Hilbert*, Math. Z. **6** (1920), 314–317.
6. ______, *Notes on some points in the integral calculus, LX. An inequality between integrals*, Messenger Math. **54** (1925), 150–156.
7. ______, *Notes on some points in the integral calculus, LXIV. Further inequalities between integrals*, Messenger Math. **57** (1928), 12–16.
8. V. Kokilashvili, A. Meskhi, L.-E. Persson, *Weighted Norm Inequalities for Integral Transforms with Product Weights*, Nova Scientific Publishers, New York, 2010.
9. A. Kufner, L. Maligranda, L.-E. Persson, *The Hardy Inequality. About its History and Some Related Results*, Vydavatelsky Servis Publishing House, Pilsen, 2007.
10. A. Kufner, L.-E. Persson, *Weighted Inequalities of Hardy type*, World Scientific, River Edge, New Jersey, 2003.
11. E. Nursultanov and S. Tikhonov, *Convolution inequalities in Lorentz spaces*, J. Fourier Anal. Appl. **17**(3) (2011), 486–505.
12. L. E. Persson, N. Samko, *What should have happened if Hardy had discovered this?*, J. Inequal. Appl. **2012** (2012), 2012:29.

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