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WILLMORE SPACELIKE SUBMANIFOLDS IN AN INDEFINITE SPACE FORM $N_q^{n+p}(c)$

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Abstract. Let $N_q^{n+p}(c)$ be an $(n+p)$-dimensional connected indefinite space form of index q $(1 \leqslant q \leqslant p)$ and of constant curvature c. Denote by $\varphi : M \to N_q^{n+p}(c)$ the n-dimensional spacelike submanifold in $N_q^{n+p}(c)$, $\varphi : M \to N_q^{n+p}(c)$ is called a Willmore spacelike submanifold in $N_q^{n+p}(c)$ if it is a critical submanifold to the Willmore functional $W(\varphi) = \int_M \rho^n dv = \int_M (S - nH^2)^{\frac{n}{2}} dv$, where S and H denote the norm square of the second fundamental form and the mean curvature of M and $\rho^2 = S - nH^2$. If $q = p$, in [**14**], we proved some integral inequalities of Simons' type and rigidity theorems for n-dimensional Willmore spacelike submanifolds in a Lorentzian space form $N_p^{n+p}(c)$. In this paper, we continue to study this topic and prove some integral inequalities of Simons' type and rigidity theorems for n-dimensional Willmore spacelike submanifolds in an indefinite space form $N_q^{n+p}(c)$ $(1 \leqslant q < p)$.

1. Introduction

Let $N_q^{n+p}(c)$ be an $(n+p)$-dimensional connected indefinite space form of index q $(1 \leqslant q \leqslant p)$ and of constant curvature c. If $c > 0$, $c = 0$ or $c < 0$, it is denoted by $S_q^{n+p}(c)$, $\mathbf{R}_q^{n+p}$ or $H_q^{n+p}(c)$. A submanifold M in $N_q^{n+p}(c)$ is said to be spacelike if the induced metric on M from that of the ambient space is positive definite. Let $\varphi\colon M \to N_q^{n+p}(c)$ be an n-dimensional spacelike submanifold in $N_q^{n+p}(c)$. If $q = p$ and M is a complete maximal spacelike submanifold in $N_p^{n+p}(c)$, from [**6**], we know that M is totally geodesic for $c \geqslant 0$, thus the class of all such submanifolds is very small. If $0 \leqslant q < p$, from [**1**] and [**4**], we know that if M is a complete minimal submanifold in sphere $S^m(c)$ $m > n$, which is embeded in $S_q^{m+q}(c)$ as a totally geodesic spacelike submanifold such that $m - n + q = p$, then M is a complete maximal spacelike submanifold in $S_q^{n+p}(c)$, thus, we see that the class of complete maximal spacelike submanifold in $S_q^{n+p}(c)$ is very large. Therefore, if $0 \leqslant q < p$, the topic of studying spacelike submanifold in $S_q^{n+p}(c)$ is also interesting and

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important. But as far as we know, the results of this topic are less well established. In [**1**], Alias and Romero studied compact maximal spacelike submanifold M in $S_q^{n+p}(c)$ and proved that if the Ricci curvature of M satisfying $\mathrm{Ric}(M) \geqslant (n-1)c$, then M is totally geodesic. Cheng–Ishikawa [**4**] also studied compact maximal spacelike submanifold in $S_q^{n+p}(c)$ and obtained some important results in terms of the pinching conditions on the scalar curvature, sectional curvature and Ricci curvature, respectively.

Denote by h_{ij}^{α}, S, $\vec{H}$ and H the second fundamental form, the norm square of the second fundamental form, the mean curvature vector and the mean curvature of M and denote by ρ^2 the nonnegative function $\rho^2 = S - nH^2$, we define the Willmore functional (see [**2, 8, 11**]):

$$W(\varphi) = \int_M \rho^n dv = \int_M (S - nH^2)^{\frac{n}{2}} dv,$$

which vanishes if and only if M is a totally umbilical spacelike submanifold. It was shown in [**9**] that this functional is an invariant under the conformal transformations of a conformal space. The points of M are called the critical points of Willmore functional $W(\varphi)$ if $W'(\varphi) = 0$. If the critical points of $W(\varphi)$ are submanifolds in $N_q^{n+p}(c)$, we call them Willmore spacelike submanifolds. Obviously, we notice that the totally umbilical spacelike submanifold is a Willmore spacelike submanifold, but, conversely, it is not true.

Since any minimal submanifold in a unit sphere $S^{n+p}(c)$ is not necessarily Willmore submanifold, due to their backgrounds in mathematics, we know that Willmore submanifolds in a unit sphere have been extensively studied in recent years (see [**8, 13**]). In indefinite or Lorentzian geometry, we also see that any maximal spacelike submanifold in $N_q^{n+p}(c)$ $(1 \leqslant q \leqslant p)$ is not necessarily Willmore spacelike submanifold, thus the study of Willmore spacelike submanifold in $N_q^{n+p}(c)$ $(1 \leqslant q \leqslant p)$ is also interesting and important. In [**14**], if $q = p$, we proved some integral inequalities of Simons' type and rigidity theorems for n-dimensional Willmore spacelike submanifolds in a Lorentzian space form $N_p^{n+p}(c)$. In this paper, we shall continue to study this topic and prove some integral inequalities of Simons' type and rigidity theorems for n-dimensional Willmore spacelike submanifolds in an indefinite space form $N_q^{n+p}(c)$ $(1 \leqslant q < p)$.

Denote by K and Q the functions which assign to each point of M the infimum of the sectional curvature and the Ricci curvature at the point, we obtain the following:

THEOREM 1.1. *Let* $\varphi\colon M \to N_q^{n+p}(c)$ *be an* $n(n \geqslant 2)$*-dimensional compact Willmore spacelike submanifold in the indefinite space form* $N_q^{n+p}(c)$*,* $c > 0$ *and* $1 \leqslant q < p$*.*

(1) *If* $p - q = 1$*, then*

$$\int_M \rho^n \Big\{ n(c - H^2) - \Big(2 - \frac{1}{p}\Big)\rho^2 \Big\} dv \leqslant 0.$$

In particular, if

$$\rho^2 \leqslant \frac{n}{2-\frac{1}{p}}(c-H^2),$$

then M is totally umbilical or M lies in the totally geodesic spacelike submanifold $S^{n+1}(c)$ of $S_q^{n+q+1}(c)$ and is isometric to the Clifford torus $S^k(\frac{1}{\sqrt{2}}c)\times S^k(\frac{1}{\sqrt{2}}c)$;

(2) *If $p-q>1$, then*

$$\int_M \rho^n\Big\{n(c-H^2)-\frac{3}{2}\rho^2\Big\}dv \leqslant 0.$$

In particular, if

$$\rho^2 \leqslant \frac{2n}{3}(c-H^2),$$

then M is totally umbilical or M lies in the totally geodesic spacelike submanifold $S^4(c)$ of $S_q^{4+q}(c)$ and is isometric to the Veronese surface.

Theorem 1.2. *Let $\varphi\colon M\to N_q^{n+p}(c)$ be an $n(n\geqslant 2)$-dimensional compact Willmore spacelike submanifold in the indefinite space form $N_q^{n+p}(c)$ $(1\leqslant q<p)$. Then the following integral inequality holds*

$$\int_M \rho^n\Big\{K-\frac{n-2}{\sqrt{n(n-1)}}H\rho-\frac{1}{n}\Big(1-\frac{1}{p-q}\Big)\rho^2\Big\}dv \leqslant 0.$$

In particular, if

$$K \geqslant \frac{n-2}{\sqrt{n(n-1)}}H\rho+\frac{1}{n}\Big(1-\frac{1}{p-q}\Big)\rho^2,$$

then M is totally umbilical or M is a maximal spacelike submanifold in $N_q^{n+p}(c)$ with parallel second fundamental form.

Theorem 1.3. *Let $\varphi\colon M\to N_q^{n+p}(c)$ be an $n(n\geqslant 2)$-dimensional compact Willmore spacelike submanifold in the indefinite space form $N_q^{n+p}(c)$ $(1\leqslant q<p)$. Then the following integral inequality holds*

$$\int_M \rho^n\Big\{Q-(n-2)c-nH^2-\frac{1}{n}\Big(3-\frac{p+q}{(p-q)q}\Big)\rho^2\Big\}dv \leqslant 0.$$

In particular, if

$$Q \geqslant (n-2)c+nH^2+\frac{1}{n}\Big(3-\frac{p+q}{(p-q)q}\Big)\rho^2,$$

then M is totally umbilical or M is a maximal spacelike submanifold in $N_q^{n+p}(c)$ with parallel second fundamental form.

2. Preliminaries

Let $N_q^{n+p}(c)$ be an $(n+p)$-dimensional indefinite space form with index $q(1\leqslant q\leqslant p)$. Let M be an n-dimensional connected spacelike submanifold immersed in $N_q^{n+p}(c)$. We choose a local field of semi-Riemannian orthonormal frames $e_1,\dots,e_{n+p}$ in $N_q^{n+p}(c)$ such that at each point of M, $e_1,\dots,e_n$ span the tangent space of M and form an orthonormal frame there. We use the following convention on the range of indices:

$$1 \leqslant A, B, C, \ldots \leqslant n+p, \quad 1 \leqslant i, j, k, \ldots \leqslant n, \quad n+1 \leqslant \alpha, \beta, \gamma, \ldots \leqslant n+p.$$

Let $\omega_1, \ldots, \omega_{n+p}$ be its dual frame field so that the semi-Riemannian metric of $N_q^{n+p}(c)$ is given by $d\bar{s}^2 = \sum_A \varepsilon_A \omega_A^2$, where $\varepsilon_A = 1$ for $1 \leqslant A \leqslant n+p-q$ and $\varepsilon_A = -1$ for $n+p-q+1 \leqslant A \leqslant n+p$. Then the structure equations of $N_q^{n+p}(c)$ are given by

$$\begin{aligned} d\omega_A &= -\sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \qquad \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} &= -\sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \varepsilon_C \varepsilon_D K_{ABCD} \omega_C \wedge \omega_D, \\ K_{ABCD} &= c \varepsilon_A \varepsilon_B (\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD}). \end{aligned}$$

If we restrict this form to M, then $\omega_\alpha = 0$, $n+1 \leqslant \alpha \leqslant n+p$ and

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \qquad h_{ij}^\alpha = h_{ji}^\alpha.$$

The second fundamental form II, the mean curvature vector $\vec{H}$ of M are defined by

$$II = \sum_{\alpha,i,j} \varepsilon_\alpha h_{ij}^\alpha \omega_i \omega_j e_\alpha, \quad \vec{H} = \sum_\alpha \varepsilon_\alpha H^\alpha e_\alpha, \quad H^\alpha = \frac{1}{n} \sum_k h_{kk}^\alpha. \tag{2.1}$$

The norm square of the second fundamental form and the mean curvature of M are defined by

$$S = |II|^2 = \sum_{i,j,\alpha} (\varepsilon_\alpha h_{ij}^\alpha)^2 = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2, \quad H = |\vec{H}| = \frac{1}{n} \sqrt{\sum_\alpha \left(\sum_k h_{kk}^\alpha \right)^2}.$$

The Gauss equations are

$$R_{ijkl} = c(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + \sum_\alpha \varepsilon_\alpha (h_{il}^\alpha h_{jk}^\alpha - h_{ik}^\alpha h_{jl}^\alpha), \tag{2.2}$$

$$R_{jk} = (n-1)c\delta_{jk} + \sum_\alpha \varepsilon_\alpha \left(\sum_i h_{ii}^\alpha h_{jk}^\alpha - \sum_i h_{ik}^\alpha h_{ji}^\alpha \right). \tag{2.3}$$

Defining the first and the second covariant derivatives of h_{ij}^α, say h_{ijk}^α and h_{ijkl}^α by

$$\sum_k h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha - \sum_k h_{ik}^\alpha \omega_{kj} - \sum_k h_{jk}^\alpha \omega_{ki} - \sum_\beta \varepsilon_\beta h_{ij}^\beta \omega_{\beta\alpha},$$

$$\begin{aligned} \sum_l h_{ijkl}^\alpha \omega_l = dh_{ijk}^\alpha &- \sum_m h_{mjk}^\alpha \omega_{mi} - \sum_m h_{imk}^\alpha \omega_{mj} \\ &- \sum_m h_{ijm}^\alpha \omega_{mk} - \sum_\beta \varepsilon_\beta h_{ijk}^\beta \omega_{\beta\alpha}, \end{aligned}$$

we have the Codazzi equations and the Ricci identities

$$h^\alpha_{ijk} = h^\alpha_{ikj}, \tag{2.4}$$

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = -\sum_m h^\alpha_{im} R_{mjkl} - \sum_m h^\alpha_{jm} R_{mikl} - \sum_\beta \varepsilon_\beta h^\beta_{ij} R_{\beta\alpha kl}. \tag{2.5}$$

The Ricci equations are

$$R_{\alpha\beta ij} = -\sum_m (h^\alpha_{im} h^\beta_{mj} - h^\alpha_{jm} h^\beta_{mi}). \tag{2.6}$$

The Laplacian of h^α_{ij} is defined by $\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$. From (2.5), we obtain for any α, $n+1 \leqslant \alpha \leqslant n+p$,

$$\Delta h^\alpha_{ij} = \sum_k h^\alpha_{kkij} - \sum_{k,m} h^\alpha_{km} R_{mijk} - \sum_{k,m} h^\alpha_{im} R_{mkjk} - \sum_{k,\beta} \varepsilon_\beta h^\beta_{ik} R_{\beta\alpha jk}.$$

For the fix index $\alpha(n+1 \leqslant \alpha \leqslant n+p)$, we introduce an operator $\Box^\alpha$ due to Cheng-Yau [**3**] by

$$\Box^\alpha f = \sum_{i,j} (nH^\alpha \delta_{ij} - h^\alpha_{ij}) f_{i,j}. \tag{2.7}$$

Since M is compact, the operator $\Box^\alpha$ is self-adjoint (see [**3**]) if and only if

$$\int_M (\Box^\alpha f) g \, dv = \int_M f(\Box^\alpha g) \, dv, \tag{2.8}$$

where f and g are any smooth functions on M. We need the following Lemma (see [**12**]):

LEMMA 2.1. *Let A, B be symmetric $n \times n$ matrices satisfying $AB = BA$ and $\operatorname{tr} A = \operatorname{tr} B = 0$. Then*

$$|\operatorname{tr} A^2 B| \leqslant \frac{n-2}{\sqrt{n(n-1)}} (\operatorname{tr} A^2)(\operatorname{tr} B^2)^{1/2},$$

and the equality holds if and only if $(n-1)$ of the eigenvalues x_i of B and the corresponding eigenvalues y_i of A satisfy $|x_i| = (\operatorname{tr} B^2)^{1/2}/\sqrt{n(n-1)}$, $x_i x_j \geqslant 0$, $y_i = (\operatorname{tr} A^2)^{1/2}/\sqrt{n(n-1)}$.

By the same method as in the proof of [**8**, Lemma 4.2], we also have the following:

LEMMA 2.2. *Let $\varphi\colon M \to N^{n+p}_q(c)$ be an n-dimensional $(n \geqslant 2)$ spacelike submanifold in $N^{n+p}_q(c)(1 \leqslant q \leqslant p)$. Then we have*

$$|\nabla h|^2 \geqslant \frac{3n^2}{n+2} |\nabla^\perp \vec{H}|^2,$$

where $|\nabla h|^2 = \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2$, $|\nabla^\perp \vec{H}|^2 = \sum_{i,\alpha} (H^\alpha_{,i})^2$.

3. Euler–Lagrange equation of Willmore spacelike submanifolds

From [**9**, Theorem 4.1], we know that the Euler–Lagrange equation of Willmore spacelike submanifolds in terms of invariants of conformal metric g is stated as following: a spacelike submanifold is a Willmore spacelike submanifold if and only if

$$\sum_{i,j,k,l,\beta} g_{\alpha\beta} g^{ik} g^{jl} \Big(B^{\beta}_{ij,kl} + A_{ij} B^{\beta}_{kl} + \sum_{r,q,\gamma,\nu} g_{\gamma\nu} g^{rq} B^{\beta}_{ir} B^{\gamma}_{qj} B^{\nu}_{kl} \Big) = 0, \quad \forall \alpha, \tag{3.1}$$

where $1 \leqslant i,j,k,l,r,q \leqslant n$, $n+1 \leqslant \alpha,\beta,\gamma,\nu \leqslant n+p$, $(g_{ij}) = (I_n)$, $(g_{\alpha\beta}) = (I_{p-q}) \oplus (-I_q)$, $(g_{ij}) = (g^{ij})^{-1}$ and $(g_{\alpha\beta}) = (g^{\alpha\beta})^{-1}$ (see [**9**]). From [**9**, (3.23)], we have $(1-n)C^{\alpha}_i = \sum_k B^{\alpha}_{ik,k}$. Thus, by a simply calculation, we may rewrite (3.1) as

$$(1-n)\sum_i C^{\alpha}_{i,i} + \sum_{i,j} A_{ij} B^{\alpha}_{ij} + \sum_{\beta}\sum_{i,j,k} \varepsilon_{\beta} B^{\alpha}_{ik} B^{\beta}_{kj} B^{\beta}_{ij} = 0, \quad \forall \alpha, \tag{3.2}$$

where $\varepsilon_{\beta} = g_{\beta\beta}$ and $\varepsilon_{\beta} = 1$ for $n+1 \leqslant \beta \leqslant n+p-q$ and $\varepsilon_{\beta} = -1$ for $n+p-q+1 \leqslant \beta \leqslant n+p$.

From [**9**] or [**10**], we have the following relations of the connections of the conformal metric $e^{2\tau} du \cdot du$ and induced metric $du \cdot du$

$$\omega_i = e^{\tau}\theta_i, \quad \omega_{ij} = \theta_{ij} + \tau_i\theta_j - \tau_j\theta_i, \quad \omega_{\alpha\beta} = \theta_{\alpha\beta}, \tag{3.3}$$

where $e^{2\tau} = \frac{n}{n-1}(S - nH^2)$. We know that the relations of the conformal invariants and the induced invariants are

$$e^{2\tau} C_i = H^{\alpha}\tau_i - H^{\alpha}_{,i} - \sum_j h^{\alpha}_{ij}\tau^j, \tag{3.4}$$

$$e^{2\tau} A_{ij} = \tau_i\tau_j - \tau_{i,j} - \sum_{\alpha} H^{\alpha} h^{\alpha}_{ij} - \frac{1}{2}\Big(\sum_k \tau^k\tau_k - H^2 - c\Big) I_{ij}, \tag{3.5}$$

$$e^{\tau} B^{\alpha}_{ij} = h^{\alpha}_{ij} - H^{\alpha} I_{ij}, \tag{3.6}$$

where $\tau_{i,j}$ is the Hessian of τ with respect to the first fundamental form I, $\tau^i = \sum_j I^{ij}\tau_j$, $(I^{ij}) = (I_{ij})^{-1}$, $H^{\alpha}_i = e_i(H^{\alpha})$ and $c = 0$ for $\mathbf{R}^{n+p}_q(c)$, $c > 0$ for $S^{n+p}_q(c)$ and $c < 0$ for $H^{n+p}_q(c)$ (see [**10**])

From (3.3) and (3.4), by a similar calculation of Li [**8**], we have

$$\begin{aligned} \sum_j e^{\tau} C^{\alpha}_{i,j}\theta_j &= \sum_j C^{\alpha}_{i,j}\omega_j = dC^{\alpha}_i + \sum_j C^{\alpha}_j \omega_{ji} + \sum_{\beta} C^{\beta}_i \omega_{\beta\alpha} \\ &= dC^{\alpha}_i + \sum_j C^{\alpha}_j\theta_{ji} + \sum_j C^{\alpha}_j(\tau_j\theta_i - \tau_i\theta_j) + \sum_{\beta} C^{\beta}_i\theta_{\beta\alpha}, \end{aligned}$$

therefore, we have

$$\begin{aligned} e^{\tau} C^{\alpha}_{i,j} = e^{-2\tau}\Big(& -2H^{\alpha}\tau_i\tau_j + 2\tau_j\sum_k h^{\alpha}_{ik}\tau_k + 2\tau_j H^{\alpha}_{,i} + H^{\alpha}_{,j}\tau_i + H^{\alpha}\tau_{i,j} \\ & - \sum_k h^{\alpha}_{ik,j}\tau_k - \sum_k h^{\alpha}_{ik}\tau_{k,j} - H^{\alpha}_{,ij}\Big) + \sum_k C^{\alpha}_k\tau_k\delta_{ij} - \tau_i C^{\alpha}_j. \end{aligned} \tag{3.7}$$

From (3.7), we see that

$$e^{3\tau}\sum_i C^\alpha_{i,i} = (n-3)\Big(H^\alpha|\nabla\tau|^2 - \sum_{i,k} h^\alpha_{ik}\tau_k\tau_i\Big) - 2(n-2)\sum_i H^\alpha_{,i}\tau_i + H^\alpha\Delta\tau - \sum_{i,k} h^\alpha_{ik}\tau_{k,i} - \Delta^\perp H^\alpha. \tag{3.8}$$

From (3.5) and (3.6), we have

$$\begin{aligned}
&e^{3\tau}\Big(\sum_{i,j} A_{ij}B^\alpha_{ij} + \sum_\beta\sum_{i,j,k}\varepsilon_\beta B^\alpha_{ik}B^\beta_{kj}B^\beta_{ij}\Big)\\
&= \sum_{i,j}\Big[\tau_i\tau_j - \tau_{i,j} - \sum_\beta H^\beta h^\beta_{ij} - \frac{1}{2}\Big(\sum_k \tau^k\tau_k - H^2 - c\Big)I_{ij}\Big]\big(h^\alpha_{ij} - H^\alpha I_{ij}\big)\\
&\quad + \sum_\beta\sum_{i,j,k}\varepsilon_\beta\big(h^\alpha_{ik} - H^\alpha I_{ik}\big)\big(h^\beta_{kj} - H^\beta I_{kj}\big)\big(h^\beta_{ij} - H^\beta I_{ij}\big)\\
&= \sum_{i,j} h^\alpha_{ij}\big(\tau_i\tau_j - \tau_{i,j}\big) + H^\alpha\Big[\Delta\tau - |\nabla\tau|^2 + n\sum_\beta(1+2\varepsilon_\beta)(H^\beta)^2\Big]\\
&\quad + \sum_\beta\sum_{i,j,k}\varepsilon_\beta h^\alpha_{ik}h^\beta_{kj}h^\beta_{ij} - \sum_{\beta,i,j}(1+2\varepsilon_\beta)H^\beta h^\beta_{ij}h^\alpha_{ij} - H^\alpha\sum_{\beta,i,j}\varepsilon_\beta(h^\beta_{ij})^2.
\end{aligned} \tag{3.9}$$

From (3.2), (3.8) and (3.9), we see that

$$\begin{aligned}
&(n-2)^2 bigg(\sum_{i,j} h^\alpha_{ij}\tau_i\tau_j - H^\alpha|\nabla\tau|^2\Big) + 2(n-1)(n-2)\sum_i H^\alpha_{,i}\tau_i\\
&\quad + (n-2)\Big(\sum_{i,j} h^\alpha_{ij}\tau_{i,j} - H^\alpha\Delta\tau\Big) + (n-1)\Delta^\perp H^\alpha\\
&\quad - H^\alpha\sum_{\beta,i,j}\varepsilon_\beta(h^\beta_{ij})^2 - \sum_{\beta,i,j}(1+2\varepsilon_\beta)H^\beta h^\beta_{ij}h^\alpha_{ij}\\
&\quad + nH^\alpha\sum_\beta(1+2\varepsilon_\beta)(H^\beta)^2 + \sum_\beta\sum_{i,j,k}\varepsilon_\beta h^\alpha_{ik}h^\beta_{kj}h^\beta_{ij} = 0.
\end{aligned} \tag{3.10}$$

Putting $\rho^2 = S - nH^2$, we have $e^{2\tau} = \frac{n}{n-1}(S - nH^2) = \frac{n}{n-1}\rho^2$. Thus $e^\tau = \sqrt{\frac{n}{n-1}}\rho$ and $\tau = \ln(\sqrt{\frac{n}{n-1}}\rho)$. From (3.10), we see that

$$\begin{aligned}
&\frac{\rho^{n-2}}{n-1}\Big\{-H^\alpha\sum_{\beta,i,j}\varepsilon_\beta(h^\beta_{ij})^2 - \sum_{\beta,i,j}(1+2\varepsilon_\beta)H^\beta h^\beta_{ij}h^\alpha_{ij}\\
&\qquad + \sum_\beta\sum_{i,j,k}\varepsilon_\beta h^\alpha_{ik}h^\beta_{kj}h^\beta_{ij} + nH^\alpha\sum_\beta(1+2\varepsilon_\beta)(H^\beta)^2\Big\}\\
&\quad + \rho^{n-2}\Delta^\perp H^\alpha + \frac{n-2}{n-1}\rho^{n-2}\sum_{i,j}(\ln\rho)_{i,j}\big(h^\alpha_{ij} - H^\alpha\delta_{ij}\big)
\end{aligned} \tag{3.11}$$

$$+ 2(n-2)\rho^{n-2}\sum_i (\ln\rho)_i H^\alpha_{,i}$$

$$+ \frac{(n-2)^2}{n-1}\rho^{n-2}\sum_{i,j}(\ln\rho)_i(\ln\rho)_j\big(h^\alpha_{ij} - H^\alpha\delta_{ij}\big) = 0.$$

It can be easily checked that

$$\begin{aligned}
(3.12)\qquad & \rho^{n-2}\Delta^\perp H^\alpha + \frac{n-2}{n-1}\rho^{n-2}\sum_{i,j}(\ln\rho)_{i,j}\big(h^\alpha_{ij} - H^\alpha\delta_{ij}\big)\\
&+ 2(n-2)\rho^{n-2}\sum_i(\ln\rho)_i H^\alpha_{,i}\\
&+ \frac{(n-2)^2}{n-1}\rho^{n-2}\sum_{i,j}(\ln\rho)_i(\ln\rho)_j\big(h^\alpha_{ij} - H^\alpha\delta_{ij}\big)\\
&= -\frac{1}{n-1}\sum_{i,j}(\rho^{n-2})_{i,j}\big(nH^\alpha\delta_{ij} - h^\alpha_{ij}\big) + \rho^{n-2}\Delta^\perp H^\alpha\\
&+ 2\sum_i(\rho^{n-2})_i H^\alpha_{,i} + H^\alpha\Delta(\rho^{n-2}).
\end{aligned}$$

From (3.11) and (3.12), we may obtain the Euler–Lagrange equation of Willmore spacelike submanifolds in $N^{n+p}_q(c)$ in terms of the induced invariants:

THEOREM 3.1. *Let $\varphi\colon M \to N^{n+p}_q(c)$ be an n-dimensional spacelike submanifold in $N^{n+p}_q(c)$. Then M is an n-dimensional Willmore spacelike submanifold if and only if for $n+1 \leqslant \alpha, \beta \leqslant n+p$,*

$$\begin{aligned}
(3.13)\qquad & \rho^{n-2}\Big\{ - H^\alpha\sum_{\beta,i,j}\varepsilon_\beta(h^\beta_{ij})^2 - \sum_{\beta,i,j}(1+2\varepsilon_\beta)H^\beta h^\beta_{ij}h^\alpha_{ij}\\
&+ \sum_\beta\sum_{i,j,k}\varepsilon_\beta h^\alpha_{ik}h^\beta_{kj}h^\beta_{ij} + nH^\alpha\sum_\beta(1+2\varepsilon_\beta)(H^\beta)^2\Big\}\\
&+ (n-1)\rho^{n-2}\Delta^\perp H^\alpha + 2(n-1)\sum_i(\rho^{n-2})_i H^\alpha_{,i}\\
&+ (n-1)H^\alpha\Delta(\rho^{n-2}) - \Box^\alpha(\rho^{n-2}) = 0.
\end{aligned}$$

where $\Delta(\rho^{n-2}) = \sum_i(\rho^{n-2})_{i,i}$, $\Box^\alpha(\rho^{n-2}) = \sum_{i,j}(\rho^{n-2})_{i,j}(nH^\alpha\delta_{ij} - h^\alpha_{ij})$ and $(\rho^{n-2})_{i,j}$ is the Hessian of ρ^{n-2} with respect to the induced metric.

REMARK 3.1. In the proof of (3.13), since we denote $e^{2\tau} = \frac{n}{n-1}(S - nH^2) = \frac{n}{n-1}\rho^2$, it follows that $\rho^2 \neq 0$, that is, (3.13) holds only for $\rho^2 \neq 0$. But, if $\rho^2 = 0$, we should notice that (3.13) also holds. Thus, in the following discussion, we agree that the Euler–Lagrange equation of Willmore spacelike submanifolds (3.13) holds for all ρ^2. But, if $n = 3$ and $n = 5$, we need to assume that M has no umbilical points to guarantee $(\rho^{n-2})_{i,j}$ to be continuous on M.

PROPOSITION 3.1. *Every maximal spacelike surface $\varphi\colon M \to N^{2+p}_q(c)$ in $N^{2+p}_q(c)$ is a Willmore spacelike surface.*

In fact, if $n=2$, since $H=0$, from (2.1), we see that $H^\alpha=0$ and $\sum_k h^\alpha_{kk}=0$. On the other hand, since $R_{ij}=\frac{R}{2}\delta_{ij}$, from Gauss equation (2.3), we have $\sum_{\beta,j}\varepsilon_\beta h^\beta_{jk}h^\beta_{ij}=c\delta_{ik}+\sum_{\beta,j}\varepsilon_\beta h^\beta_{jj}h^\beta_{ik}-R_{ik}$, thus

$$\begin{aligned}\sum_\beta\sum_{i,j,k}\varepsilon_\beta h^\alpha_{ik}h^\beta_{kj}h^\beta_{ij}&=\sum_{i,k}h^\alpha_{ik}\Big(c\delta_{ik}+\sum_{\beta,j}\varepsilon_\beta h^\beta_{jj}h^\beta_{ik}-R_{ik}\Big)\\&=\Big(c-\frac{R}{2}\Big)\sum_i h^\alpha_{ii}+\sum_{\beta,i,k}\varepsilon_\beta h^\alpha_{ik}h^\beta_{ik}\Big(\sum_j h^\beta_{jj}\Big)=0,\end{aligned}$$

it follows that (3.13) holds and Proposition 3.1 is concluded.

Example 3.1. If $0\leqslant q<p=q+1$, since we know that the Clifford torus $S^k\Big(\sqrt{\frac{k}{n}c}\Big)\times S^{n-k}\Big(\sqrt{\frac{n-k}{n}c}\Big)$ is a complete minimal hypersurface in sphere $S^{n+1}(c)$ which is embeded in $S^{n+1+q}_q(c)$ as a totally geodesic spacelike submanifold such that $1+q=p$, then $S^k\Big(\sqrt{\frac{k}{n}c}\Big)\times S^{n-k}\Big(\sqrt{\frac{n-k}{n}c}\Big)$ is a complete maximal spacelike submanifold in $S^{n+q+1}_q(c)$, where $1\leqslant k\leqslant n-1$. Since $S^k\Big(\sqrt{\frac{k}{n}c}\Big)\times S^{n-k}\Big(\sqrt{\frac{n-k}{n}c}\Big)$ lies in the totally geodesic spacelike submanifold $S^{n+1}(c)$ of $S^{n+q+1}_q(c)$, we know that $h^\alpha_{ij}=0$ for $\alpha=n+2,\dots,n+q+1$. Thus, if and only if $n=2k$ then

$$\begin{aligned}\sum_\beta\sum_{i,j,k}\varepsilon_\beta h^\alpha_{ik}h^\beta_{kj}h^\beta_{ij}&=\sum_{i,j,k}h^{n+1}_{ik}h^{n+1}_{kj}h^{n+1}_{ij}=\sum_i\lambda_i^3\\&=k\bigg(\sqrt{\frac{n-k}{k}c}\bigg)^3+(n-k)\bigg(-\sqrt{\frac{k}{n-k}c}\bigg)^3=0,\end{aligned}$$

where $h^{n+1}_{ij}=\lambda_i\delta_{ij}$, $\sqrt{\frac{n-k}{k}c}$ and $-\sqrt{\frac{k}{n-k}c}$ are the two distinct principal curvatures of $S^k\Big(\sqrt{\frac{k}{n}c}\Big)\times S^{n-k}\Big(\sqrt{\frac{n-k}{n}c}\Big)\subset S^{n+1}(c)$ with multiplicities k and $n-k$, respectively. We also see that $\rho^2=S-nH^2=\sum_i\lambda_i^2=nc$ is constant. Thus, (3.13) holds if and only if $n=2k$, that is the Clifford torus $S^k\big(\frac{1}{\sqrt2}c\big)\times S^k\big(\frac{1}{\sqrt2}c\big)$, $1\leqslant k\leqslant n-1$, is a maximal Willmore spacelike submanifold in $S^{n+q+1}_q(c)$.

Example 3.2. From [**5**] and [**1**], we know that the Veronese surface is a minimal surface in $S^4(c)$ which is embeded in $S^{4+q}_q(c)$ as a totally geodesic spacelike submanifold such that $2+q=p$, then the Veronese surface is a maximal spacelike surface in $S^{2+p}_q(c)$, where $p=2+q$. From Proposition 3.1, we know that it is a Willmore spacelike surface in $S^{4+q}_q(c)$.

4. Basic integral equalities

Define tensors

$$\tilde h^\alpha_{ij}=h^\alpha_{ij}-H^\alpha\delta_{ij},\tag{4.1}$$

$$\tilde\sigma_{\alpha\beta}=\sum_{i,j}\tilde h^\alpha_{ij}\tilde h^\beta_{ij},\qquad \sigma_{\alpha\beta}=\sum_{i,j}h^\alpha_{ij}h^\beta_{ij}.\tag{4.2}$$

Then the $(p \times p)$-matrix $(\tilde{\sigma}_{\alpha\beta})$ is symmetric and can be assumed to be diagonized for a suitable choice of $e_{n+1}, \dots, e_{n+p}$. We set

$$\tilde{\sigma}_{\alpha\beta} = \tilde{\sigma}_\alpha \delta_{\alpha\beta}. \tag{4.3}$$

By a direct calculation, we have

$$\sum_k \tilde{h}^\alpha_{kk} = 0, \quad \tilde{\sigma}_{\alpha\beta} = \sigma_{\alpha\beta} - nH^\alpha H^\beta, \quad \rho^2 = \sum_\alpha \tilde{\sigma}_\alpha = S - nH^2, \tag{4.4}$$

$$\begin{aligned} &-H^\alpha \sum_{\beta,i,j} \varepsilon_\beta (h^\beta_{ij})^2 - \sum_{\beta,i,j} (1+2\varepsilon_\beta) H^\beta h^\beta_{ij} h^\alpha_{ij} \\ &\qquad + \sum_\beta \sum_{i,j,k} \varepsilon_\beta h^\alpha_{ik} h^\beta_{kj} h^\beta_{ij} + nH^\alpha \sum_\beta (1+2\varepsilon_\beta)(H^\beta)^2 \\ &\qquad\qquad = \sum_\beta \sum_{i,j,k} \varepsilon_\beta \tilde{h}^\alpha_{ik} \tilde{h}^\beta_{kj} \tilde{h}^\beta_{ij} - \sum_{i,j,\beta} H^\beta \tilde{h}^\beta_{ij} \tilde{h}^\alpha_{ij}. \end{aligned} \tag{4.5}$$

From (4.1), (4.4) and (4.5), Euler–Lagrange equation (3.13) can be rewritten as

PROPOSITION 4.1. *Let $\varphi\colon M \to N^{n+p}_q(c)$ be an n-dimensional spacelike submanifold in $N^{n+p}_q(c)$. Then M is a Willmore spacelike submanifold if and only if for $n+1 \leqslant \alpha \leqslant n+p$*

$$\begin{aligned} \square^\alpha(\rho^{n-2}) &= (n-1)\rho^{n-2}\Delta^\perp H^\alpha + 2(n-1)\sum_i (\rho^{n-2})_i H^\alpha_{,i} \\ &\quad + (n-1)H^\alpha \Delta(\rho^{n-2}) \\ &\quad + \rho^{n-2}\Bigg(\sum_\beta \sum_{i,j,k} \varepsilon_\beta \tilde{h}^\alpha_{ik}\tilde{h}^\beta_{kj}\tilde{h}^\beta_{ij} - \sum_{i,j,\beta} H^\beta \tilde{h}^\beta_{ij}\tilde{h}^\alpha_{ij}\Bigg). \end{aligned} \tag{4.6}$$

Setting $f = nH^\alpha$ in (2.7), we have

$$\begin{aligned} \square^\alpha(nH^\alpha) &= \sum_{i,j} (nH^\alpha \delta_{ij} - h^\alpha_{ij})(nH^\alpha)_{i,j} \\ &= \sum_i (nH^\alpha)(nH^\alpha)_{i,i} - \sum_{i,j} h^\alpha_{ij}(nH^\alpha)_{i,j}. \end{aligned} \tag{4.7}$$

We also have

$$\begin{aligned} \frac{1}{2}\Delta(nH)^2 &= \frac{1}{2}\Delta \sum_\alpha (nH^\alpha)^2 = \frac{1}{2}\sum_\alpha \Delta(nH^\alpha)^2 \\ &= \frac{1}{2}\sum_{\alpha,i} [(nH^\alpha)^2]_{i,i} = \sum_{\alpha,i} [(nH^\alpha)_{,i}]^2 + \sum_{\alpha,i} (nH^\alpha)(nH^\alpha)_{i,i} \\ &= n^2|\nabla^\perp \vec{H}|^2 + \sum_{\alpha,i} (nH^\alpha)(nH^\alpha)_{i,i}. \end{aligned} \tag{4.8}$$

Therefore, from (4.7), (4.8), we get

$$\sum_\alpha \square^\alpha(nH^\alpha) = \frac{1}{2}\Delta(nH)^2 - n^2|\nabla^\perp \vec{H}|^2 - \sum_{i,j,\alpha} h^\alpha_{ij}(nH^\alpha)_{i,j} \tag{4.9}$$

$$= \frac{1}{2}\Delta[n(n-1)H^2 - \rho^2 + S] - n^2|\nabla^\perp \vec{H}|^2 - \sum_{i,j,\alpha} h_{ij}^\alpha (nH^\alpha)_{i,j}$$
$$= \frac{1}{2}\Delta S + \frac{1}{2}n(n-1)\Delta H^2 - \frac{1}{2}\Delta\rho^2 - n^2|\nabla^\perp \vec{H}|^2 - \sum_{i,j,\alpha} h_{ij}^\alpha (nH^\alpha)_{i,j}.$$

From (2.4) and (2.5), we have

$$\begin{aligned} \frac{1}{2}\Delta S &= \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha \\ &= |\nabla h|^2 + \sum_{i,j,\alpha} h_{ij}^\alpha (nH^\alpha)_{i,j} - \sum_\alpha \sum_{i,j,k,l} h_{ij}^\alpha h_{kl}^\alpha R_{lijk} \\ &\quad - \sum_\alpha \sum_{i,j,k,l} h_{ij}^\alpha h_{li}^\alpha R_{lkjk} - \sum_{\alpha,\beta} \sum_{i,j,k} \varepsilon_\beta h_{ij}^\alpha h_{ki}^\beta R_{\beta\alpha jk}. \end{aligned} \tag{4.10}$$

Putting (4.10) into (4.9), we have

$$\begin{aligned} \sum_\alpha \Box^\alpha (nH^\alpha) &= |\nabla h|^2 - n^2|\nabla^\perp \vec{H}|^2 + \frac{1}{2}n(n-1)\Delta H^2 - \frac{1}{2}\Delta\rho^2 \\ &\quad - \sum_\alpha \sum_{i,j,k,l} h_{ij}^\alpha (h_{kl}^\alpha R_{lijk} + h_{li}^\alpha R_{lkjk}) - \sum_{\alpha,\beta} \sum_{i,j,k} \varepsilon_\beta h_{ij}^\alpha h_{ki}^\beta R_{\beta\alpha jk}. \end{aligned} \tag{4.11}$$

Multiplying (4.11) by ρ^{n-2} and taking integration, using (2.8), we have

$$\begin{aligned} \sum_\alpha \int_M (nH^\alpha)\Box^\alpha(\rho^{n-2})dv &= \int_M \rho^{n-2}(|\nabla h|^2 - n^2|\nabla^\perp \vec{H}|^2)dv \\ &\quad + \frac{1}{2}n(n-1)\int_M \rho^{n-2}\Delta H^2 dv - \frac{1}{2}\int_M \rho^{n-2}\Delta\rho^2 dv \\ &\quad - \int_M \rho^{n-2} \sum_\alpha \sum_{i,j,k,l} h_{ij}^\alpha (h_{kl}^\alpha R_{lijk} + h_{li}^\alpha R_{lkjk}) dv \\ &\quad - \int_M \rho^{n-2} \sum_{\alpha,\beta} \sum_{i,j,k} \varepsilon_\beta h_{ij}^\alpha h_{ki}^\beta R_{\beta\alpha jk} dv. \end{aligned} \tag{4.12}$$

Substituting Willmore equation (4.6) into (4.12) and making use of the following:

$$\begin{aligned} \int_M \rho^{n-2} \sum_\alpha H^\alpha \triangle^\perp H^\alpha dv &= \frac{1}{2}\int_M \rho^{n-2} \sum_\alpha \Delta^\perp (H^\alpha)^2 dv - \int_M \rho^{n-2} \sum_{i,\alpha} (H_{,i}^\alpha)^2 dv \\ &= \frac{1}{2}\int_M \rho^{n-2}\Delta H^2 dv - \int_M \rho^{n-2}|\nabla \vec{H}|^2 dv, \end{aligned}$$

$$\begin{aligned} \int_M H^2 \Delta(\rho^{n-2})dv &= \int_M \sum_\alpha (H^\alpha)^2 \sum_i (\rho^{n-2})_{i,i} dv \\ &= \sum_{\alpha,i} \int_M (H^\alpha)^2 (\rho^{n-2})_{i,i} dv = -\sum_{\alpha,i} \int_M (\rho^{n-2})_i ((H^\alpha)^2)_{,i} dv \end{aligned}$$

$$= -2\int_M \sum_\alpha H^\alpha \sum_i (\rho^{n-2})_i H^\alpha_{,i} dv,$$

$$-\frac{1}{2}\int_M \rho^{n-2}\Delta\rho^2 dv = -\frac{1}{2}\sum_i \int_M \rho^{n-2}(\rho^2)_{i,i} dv$$
$$= \frac{1}{2}\sum_i \int_M (\rho^2)_i(\rho^{n-2})_i dv = (n-2)\int_M \rho^{n-2}|\nabla\rho|^2 dv,$$

we have, by a direct calculation, the following:

PROPOSITION 4.2. *Let $\varphi\colon M \to N^{n+p}_q(c)$ be an n-dimensional spacelike submanifold in $N^{n+p}_q(c)$. Then*

$$\begin{aligned}\int_M \rho^{n-2}(|\nabla h|^2 - n|\nabla^\perp \vec{H}|^2)dv + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2 dv \\ - \int_M \rho^{n-2}\sum_{\alpha,\beta} nH^\alpha\Big(\sum_{i,j,k}\varepsilon_\beta \tilde{h}^\alpha_{ik}\tilde{h}^\beta_{kj}\tilde{h}^\beta_{ij} - H^\beta\tilde{\sigma}_{\alpha\beta}\Big)dv \\ - \int_M \rho^{n-2}\sum_\alpha\sum_{i,j,k,l} h^\alpha_{ij}(h^\alpha_{kl}R_{lijk} + h^\alpha_{li}R_{lkjk})dv \\ - \int_M \rho^{n-2}\sum_{\alpha,\beta}\sum_{i,j,k}\varepsilon_\beta h^\alpha_{ij}h^\beta_{ki}R_{\beta\alpha jk}dv = 0.\end{aligned} \tag{4.13}$$

In general, for a matrix $A = (a_{ij})$ we denote by $N(A)$ the square of the norm of A, that is, $N(A) = \mathrm{trace}(A\cdot A^t) = \sum_{i,j}(a_{ij})^2$. Clearly, $N(A) = N(T^tAT)$ for any orthogonal matrix T. From (2.6), we have

$$\begin{aligned} -\sum_{\alpha,\beta}\sum_{i,j,k}\varepsilon_\beta h^\alpha_{ij}h^\beta_{ki}R_{\beta\alpha jk} &= -\sum_{\alpha,\beta}\sum_{i,j,k,l}\varepsilon_\beta h^\alpha_{ij}h^\beta_{ki}(h^\beta_{kl}h^\alpha_{lj} - h^\beta_{jl}h^\alpha_{lk}) \\ &= -\frac{1}{2}\sum_{\alpha,\beta,j,k}\varepsilon_\beta\Big(\sum_l h^\beta_{kl}h^\alpha_{lj} - \sum_l h^\alpha_{kl}h^\beta_{lj}\Big)^2 \\ &= -\frac{1}{2}\sum_{\alpha,\beta,j,k}\varepsilon_\beta\Big(\sum_l \tilde{h}^\beta_{kl}\tilde{h}^\alpha_{lj} - \sum_l \tilde{h}^\alpha_{kl}\tilde{h}^\beta_{lj}\Big)^2 \\ &= -\frac{1}{2}\sum_{\alpha,\beta}\varepsilon_\beta N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha),\end{aligned} \tag{4.14}$$

where $\tilde{A}_\alpha := (\tilde{h}^\alpha_{ij}) = (h^\alpha_{ij} - H^\alpha\delta_{ij})$.

By use of (2.2), (2.6), (4.1), (4.2), (4.4) and (4.14), we conclude that

$$\begin{aligned} -\sum_\alpha\sum_{i,j,k,l} h^\alpha_{ij}(h^\alpha_{kl}R_{lijk} + h^\alpha_{li}R_{lkjk}) = nc\rho^2 - \sum_{\alpha,\beta}\varepsilon_\beta\sigma^2_{\alpha\beta} \\ + n\sum_{\alpha,\beta}\sum_{i,j,k}\varepsilon_\beta H^\beta h^\beta_{kj}h^\alpha_{ij}h^\alpha_{ik} - \sum_{\alpha,\beta,i,j,k}\varepsilon_\beta h^\alpha_{ji}h^\beta_{ik}R_{\beta\alpha jk}\end{aligned} \tag{4.15}$$

$$
\begin{aligned}
&= nc\rho^2 - \sum_{\alpha,\beta}\varepsilon_\beta\tilde{\sigma}^2_{\alpha\beta} - 2n\sum_{\alpha,\beta}\sum_{i,j}\varepsilon_\beta H^\alpha H^\beta \tilde{h}^\alpha_{ij}\tilde{h}^\beta_{ij}\\
&\quad - n^2\sum_\alpha (H^\alpha)^2\sum_\beta \varepsilon_\beta (H^\beta)^2 + n\sum_{\alpha,\beta}\sum_{i,j,k}\varepsilon_\beta H^\beta \tilde{h}^\beta_{kj}\tilde{h}^\alpha_{ij}\tilde{h}^\alpha_{ik}\\
&\quad + n\rho^2\sum_\beta \varepsilon_\beta (H^\beta)^2 + 2n\sum_{\alpha,\beta}\sum_{i,j}\varepsilon_\beta H^\alpha H^\beta \tilde{h}^\alpha_{ij}\tilde{h}^\beta_{ij}\\
&\quad + n^2\sum_\alpha (H^\alpha)^2\sum_\beta \varepsilon_\beta (H^\beta)^2 - \frac{1}{2}\sum_{\alpha,\beta}\varepsilon_\beta N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha)\\
&= nc\rho^2 - \sum_{\alpha,\beta}\varepsilon_\beta\tilde{\sigma}^2_{\alpha\beta} + n\rho^2\sum_\beta \varepsilon_\beta (H^\beta)^2\\
&\qquad + n\sum_{\alpha,\beta}\sum_{i,j,k}\varepsilon_\beta H^\beta \tilde{h}^\beta_{kj}\tilde{h}^\alpha_{ij}\tilde{h}^\alpha_{ik} - \frac{1}{2}\sum_{\alpha,\beta}\varepsilon_\beta N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha).
\end{aligned}
$$

Putting (4.14) and (4.15) into (4.13), we have the following:

PROPOSITION 4.3. *Let $\varphi\colon M \to N^{n+p}_q(c)$ be an n-dimensional spacelike submanifold in $N^{n+p}_q(c)$. Then*

$$
\begin{aligned}
(4.16)\quad &\int_M \rho^{n-2}(|\nabla h|^2 - n|\nabla^\perp \vec{H}|^2)dv + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2 dv\\
&\quad + n\int_M \rho^{n-2}\Big(\sum_{\alpha,\beta} H^\alpha H^\beta\tilde{\sigma}_{\alpha\beta} + \rho^2\sum_\beta \varepsilon_\beta (H^\beta)^2\Big)dv + nc\int_M \rho^n dv\\
&\quad - \int_M \rho^{n-2}\Big(\sum_{\alpha,\beta}\varepsilon_\beta N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) + \sum_{\alpha,\beta}\varepsilon_\beta\tilde{\sigma}^2_{\alpha\beta}\Big)dv = 0.
\end{aligned}
$$

5. Proofs of Theorems

PROOF OF THEOREM 1.1. (1) If $p-q=1$, from Lemma 2.2 and (4.16), we have

$$
\begin{aligned}
(5.1)\quad 0 &= \int_M \rho^{n-2}(|\nabla h|^2 - \frac{3n^2}{n+2}|\nabla^\perp \vec{H}|^2)dv + \int_M \rho^{n-2}(\frac{3n^2}{n+2} - n)|\nabla^\perp \vec{H}|^2 dv\\
&\quad + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2 dv\\
&\quad + n\int_M \rho^{n-2}\Big\{\sum_{\alpha,\beta} H^\alpha H^\beta\tilde{\sigma}_{\alpha\beta} + 2\rho^2(H^{n+1})^2 - H^2\rho^2\Big\}dv + nc\int_M \rho^n dv\\
&\quad + \int_M \rho^{n-2}\Big\{\sum_{\alpha=n+2}^{n+p}\sum_{\beta=n+2}^{n+p} N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) - 2\tilde{\sigma}^2_{n+1\,n+1}\\
&\qquad\qquad + \sum_{\alpha=n+1}^{n+p}\sum_{\beta=n+1}^{n+p}\tilde{\sigma}^2_{\alpha\beta}\Big\}dv\\
&\geqslant -n\int_M \rho^{n-2}H^2\rho^2 dv + nc\int_M \rho^n dv - 2\int_M \rho^{n-2}\rho^4 dv + \int_M \rho^{n-2}\frac{1}{p}\rho^4 dv
\end{aligned}
$$

$$= \int_M \rho^n \Big\{ n(c - H^2) - \Big(2 - \frac{1}{p}\Big)\rho^2 \Big\} dv,$$

where the inequality $N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha) \geqslant 0$ for any α, β, $\tilde{\sigma}^2_{n+1n+1} \leqslant \rho^4$ and $\sum_{\alpha,\beta} \tilde{\sigma}^2_{\alpha\beta} = \sum_\alpha \tilde{\sigma}^2_\alpha \geqslant \frac{1}{p}\big(\sum_\alpha \tilde{\sigma}_\alpha\big)^2 = \frac{1}{p}\rho^4$ is used.

In particular, if $\rho^2 \leqslant \frac{n}{2-\frac{1}{p}}(c - H^2)$, from (5.1), we see that $\rho^2 = 0$ and M is totally umbilical or $\rho^2 = \frac{n}{2-\frac{1}{p}}(c - H^2)$. In the latter case, we have from (5.1) that $\int_M \rho^{n-2} \sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta} dv = 0$, that is

$$\int_M \rho^{n-2} \sum_\alpha (H^\alpha)^2 \tilde{\sigma}_\alpha dv = 0. \tag{5.2}$$

If $\rho^2 = 0$, that is M is totally umbilical, otherwise, if $\rho^2 \neq 0$, it follows from (5.2) that $\sum_\alpha (H^\alpha)^2 \tilde{\sigma}_\alpha = 0$, thus $(H^\alpha)^2 \tilde{\sigma}_\alpha = 0$ for all α. Therefore, we see that $\tilde{\sigma}_\alpha = 0$ for all α (contradicts to $\rho^2 \neq 0$), or $H^\alpha = 0$ for all α. Thus, we have $H = 0$, that is, M is a compact maximal spacelike submanifold in $S^{n+p}_q(c)$, by Cheng and Ishikawa [**4**, Theorem 1] and Example 3.1, we know that M lies in the totally geodesic spacelike submanifold $S^{n+1}(c)$ of $S^{n+q+1}_q(c)$ and is isometric to the Clifford torus $S^k\big(\frac{1}{\sqrt{2}}c\big) \times S^k\big(\frac{1}{\sqrt{2}}c\big)$.

(2) If $p - q > 1$, from Lemma 2.2 and (4.16), we have

(5.3)

$$\begin{aligned}
0 &= \int_M \rho^{n-2}\Big(|\nabla h|^2 - \frac{3n^2}{n+2}|\nabla^\perp \vec{H}|^2\Big) dv \\
&\quad + \int_M \rho^{n-2}\Big(\frac{3n^2}{n+2} - n\Big)|\nabla^\perp \vec{H}|^2 dv + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2 dv \\
&\quad + n\int_M \rho^{n-2}\Big\{ \sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta} + 2\rho^2 \sum_{\beta=n+1}^{n+p-q} (H^\beta)^2 - H^2\rho^2 \Big\} dv + nc\int_M \rho^n dv \\
&\quad + \int_M \rho^{n-2}\Big\{ -\sum_{\alpha,\beta} N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha) - \sum_{\alpha,\beta} \tilde{\sigma}^2_{\alpha\beta} \\
&\qquad\qquad + 2\sum_\alpha \sum_{\beta=n+p-q+1}^{n+p} N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha) + 2\sum_\alpha \sum_{\beta=n+p-q+1}^{n+p} \tilde{\sigma}^2_{\alpha\beta} \Big\} dv \\
&\geqslant -n\int_M \rho^{n-2} H^2 \rho^2 dv + nc\int_M \rho^n dv + \int_M \rho^{n-2}\Big(-\frac{3}{2}\rho^4\Big) dv \\
&= \int_M \rho^n \Big\{ n(c - H^2) - \frac{3}{2}\rho^2 \Big\} dv,
\end{aligned}$$

where the inequality [**7**]

$$-\sum_{\alpha,\beta} N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha) - \sum_{\alpha,\beta} \tilde{\sigma}^2_{\alpha\beta} \geqslant -\frac{3}{2}\rho^4,$$

is used.

In particular, if $\rho^2 \leqslant \frac{2n}{3}(c - H^2)$, from (5.3), we see that $\rho^2 = 0$ and M is totally umbilical or $\rho^2 = \frac{2n}{3}(c - H^2)$. In the latter case, from (5.3), we also see that (5.2) holds. If $\rho^2 = 0$, that is M is totally umbilical, otherwise, if $\rho^2 \neq 0$, it follows from (5.2) that $\sum_\alpha (H^\alpha)^2 \tilde{\sigma}_\alpha = 0$. By the same argument as above, we see that $H^\alpha = 0$ and $H = 0$, that is, M is a compact maximal spacelike submanifold in $S_q^{n+p}(c)$, by Cheng and Ishikawa [**4**, Theorem 1] and Example 3.2, we know that M lies in the totally geodesic spacelike submanifold $S^4(c)$ of $S_q^{4+q}(c)$ and is isometric to the Veronese surface. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. For a fixed $\alpha, n+1 \leqslant \alpha \leqslant n+p$, we can take a local orthonormal frame field $\{e_1, \dots, e_n\}$ such that $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$, then $\tilde{h}_{ij}^\alpha = \mu_i^\alpha \delta_{ij}$ with $\mu_i^\alpha = \lambda_i^\alpha - H^\alpha$, $\sum_i \mu_i^\alpha = 0$. Thus

$$\begin{aligned} (5.4) \qquad -\sum_{\alpha,i,j,k,l} h_{ij}^\alpha (h_{kl}^\alpha R_{lijk} - h_{li}^\alpha R_{lkjk}) &= \frac{1}{2} \sum_{\alpha,i,k} (\lambda_i^\alpha - \lambda_k^\alpha)^2 R_{kiik} \\ &= \frac{1}{2} \sum_{\alpha,i,k} (\mu_i^\alpha - \mu_k^\alpha)^2 R_{kiik} \geqslant nK\rho^2, \end{aligned}$$

where K denotes the infimum of the sectional curvature of M and the equality in (5.4) holds if and only if $R_{kiik} = K$ for any $i \neq k$.

Let $\sum_i (\tilde{h}_{ii}^\beta)^2 = \tau_\beta$. Then $\tau_\beta \leqslant \sum_{i,j} (\tilde{h}_{ij}^\beta)^2 = \tilde{\sigma}_\beta$. Since $\sum_i \tilde{h}_{ii}^\beta = 0$, $\sum_i \mu_i^\alpha = 0$ and $\sum_i (\mu_i^\alpha)^2 = \tilde{\sigma}_\alpha$. We have from Lemma 2.1 that

$$\begin{aligned} (5.5) \qquad -\sum_{\alpha,\beta,i,j,k} H^\alpha \varepsilon_\beta \tilde{h}_{ik}^\alpha \tilde{h}_{kj}^\beta \tilde{h}_{ij}^\beta &= -\sum_{\alpha,i,j,k} \sum_{\beta=n+1}^{n+p-q} H^\alpha \tilde{h}_{ik}^\alpha \tilde{h}_{kj}^\beta \tilde{h}_{ij}^\beta \\ &\quad + \sum_{\alpha,i,j,k} \sum_{\beta=n+p-q+1}^{n+p} H^\alpha \tilde{h}_{ik}^\alpha \tilde{h}_{kj}^\beta \tilde{h}_{ij}^\beta \\ &= -\sum_{\alpha,i} \sum_{\beta=n+1}^{n+p-q} H^\alpha \tilde{h}_{ii}^\alpha (\mu_i^\beta)^2 + \sum_{\alpha,i} \sum_{\beta=n+p-q+1}^{n+p} H^\alpha \tilde{h}_{ii}^\alpha (\mu_i^\beta)^2 \\ &\geqslant -\frac{n-2}{\sqrt{n(n-1)}} \sum_\alpha \sum_{\beta=n+1}^{n+p-q} |H^\alpha| \tilde{\sigma}_\beta \sqrt{\tau_\alpha} \\ &\quad - \frac{n-2}{\sqrt{n(n-1)}} \sum_\alpha \sum_{\beta=n+p-q+1}^{n+p} |H^\alpha| \tilde{\sigma}_\beta \sqrt{\tau_\alpha} \\ &= -\frac{n-2}{\sqrt{n(n-1)}} \sum_\alpha |H^\alpha| \sqrt{\tau_\alpha} \Big(\sum_{\beta=n+1}^{n+p-q} \tilde{\sigma}_\beta + \sum_{\beta=n+p-q+1}^{n+p} \tilde{\sigma}_\beta \Big) \\ &\geqslant -\frac{n-2}{\sqrt{n(n-1)}} \Big(\sqrt{\sum_\alpha (H^\alpha)^2 \sum_\alpha \tilde{\tau}_\alpha} \Big) \rho^2 \geqslant -\frac{n-2}{\sqrt{n(n-1)}} H\rho^3. \end{aligned}$$

From Chen–Do Carmo–Kobayashi [**5**, Lemma 1], we see that

$$
\begin{aligned}
(5.6)\quad & -\frac{1}{2}\sum_{\alpha,\beta}\varepsilon_\beta N(\tilde{A}_\alpha\tilde{A}_\beta-\tilde{A}_\beta\tilde{A}_\alpha)\\
&=-\frac{1}{2}\sum_{\alpha}\sum_{\beta=n+1}^{n+p-q}N(\tilde{A}_\alpha\tilde{A}_\beta-\tilde{A}_\beta\tilde{A}_\alpha)+\frac{1}{2}\sum_{\alpha}\sum_{\beta=n+p-q+1}^{n+p}N(\tilde{A}_\alpha\tilde{A}_\beta-\tilde{A}_\beta\tilde{A}_\alpha)\\
&=-\frac{1}{2}\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}N(\tilde{A}_\alpha\tilde{A}_\beta-\tilde{A}_\beta\tilde{A}_\alpha)\\
&+\frac{1}{2}\sum_{\alpha=n+p-q+1}^{n+p}\sum_{\beta=n+p-q+1}^{n+p}N(\tilde{A}_\alpha\tilde{A}_\beta-\tilde{A}_\beta\tilde{A}_\alpha)\\
&\geqslant-\frac{1}{2}\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}N(\tilde{A}_\alpha\tilde{A}_\beta-\tilde{A}_\beta\tilde{A}_\alpha)\geqslant-\sum_{\alpha\neq\beta}\tilde{\sigma}_\alpha\tilde{\sigma}_\beta\\
&=-\Big(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha\Big)^2+\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha^2\geqslant-\Big(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha\Big)^2+\frac{1}{p-q}\Big(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha\Big)^2\\
&=-\Big(1-\frac{1}{p-q}\Big)\Big(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha\Big)^2\geqslant-\Big(1-\frac{1}{p-q}\Big)\rho^4.
\end{aligned}
$$

Thus, from (4.13), (4.14), (5.4), (5.5), (5.6) and Lemma 2.2, we have

$$
\begin{aligned}
(5.7)\quad 0 &\geqslant \int_M \rho^{n-2}(|\nabla h|^2-\frac{3n^2}{n+2}|\nabla^\perp\vec{H}|^2)dv+\int_M\rho^{n-2}(\frac{3n^2}{n+2}-n)|\nabla^\perp\vec{H}|^2dv\\
&+(n-2)\int_M\rho^{n-2}|\nabla\rho|^2dv-\int_M\rho^{n-2}\frac{n(n-2)}{\sqrt{n(n-1)}}H\rho^3dv\\
&+\int_M\rho^{n-2}\sum_{\alpha,\beta}nH^\alpha H^\beta\tilde{\sigma}_{\alpha\beta}+\int_M\rho^{n-2}nK\rho^2dv\\
&-\frac{1}{2}\sum_{\alpha,\beta}\varepsilon_\beta N(\tilde{A}_\alpha\tilde{A}_\beta-\tilde{A}_\beta\tilde{A}_\alpha)\\
&\geqslant\int_M\rho^n\Big\{nK-\frac{n(n-2)}{\sqrt{n(n-1)}}H\rho-\Big(1-\frac{1}{p-q}\Big)\rho^2\Big\}dv.
\end{aligned}
$$

In particular, if

$$K\geqslant\frac{n-2}{\sqrt{n(n-1)}}H\rho+\frac{1}{n}\Big(1-\frac{1}{p-q}\Big)\rho^2,$$

from (5.7), we see that $\rho^2=0$ and M is totally umbilical or $K=\frac{n-2}{\sqrt{n(n-1)}}H\rho+\frac{1}{n}\big(1-\frac{1}{p-q}\big)\rho^2$. In the latter case, from (5.7), we know that (5.2) holds. If $\rho^2=0$, that is M is totally umbilical, otherwise, if $\rho^2\neq0$, it follows from (5.2) that $\sum_\alpha(H^\alpha)^2\tilde{\sigma}_\alpha=0$. By the same argument as in the proof of Theorem 1.1, we see that $H^\alpha = 0$ and $H = 0$. It also follows from (5.7) that $|\nabla h|^2 = \frac{3n^2}{n+2}|\nabla^\perp \vec{H}|^2 = 0$, that is, the second fundamental form of M is parallel. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. From (2.3) and (4.1), we have

$$\begin{aligned} R_{kk} &= (n-1)c + (n-2)\sum_\alpha \varepsilon_\alpha H^\alpha \tilde{h}^\alpha_{kk} + (n-1)\sum_{\alpha=n+1}^{n+p-q}(H^\alpha)^2 \\ &\quad - (n-1)\sum_{\alpha=n+p-q+1}^{n+p}(H^\alpha)^2 - \sum_{i,\alpha=n+1}^{n+p-q}(\tilde{h}^\alpha_{ik})^2 + \sum_{i,\alpha=n+p-q+1}^{n+p}(\tilde{h}^\alpha_{ik})^2 \\ &\leqslant (n-1)c + (n-2)\sum_\alpha \varepsilon_\alpha H^\alpha \tilde{h}^\alpha_{kk} + (n-1)H^2 \\ &\quad - \sum_{i,\alpha=n+1}^{n+p-q}(\tilde{h}^\alpha_{ik})^2 + \sum_{i,\alpha=n+p-q+1}^{n+p}(\tilde{h}^\alpha_{ik})^2. \end{aligned}$$

Thus

$$nQ \leqslant \sum_k R_{kk} = n(n-1)c + n(n-1)H^2 - \sum_{i,k,\alpha=n+1}^{n+p-q}(\tilde{h}^\alpha_{ik})^2 + \sum_{i,k,\alpha=n+p-q+1}^{n+p}(\tilde{h}^\alpha_{ik})^2.$$

From (4.2) and (4.3), we have

$$-\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha + \sum_{\alpha=n+p-q+1}^{n+p}\tilde{\sigma}_\alpha \geqslant nQ - n(n-1)c - n(n-1)H^2. \tag{5.8}$$

From (5.8), we see that

$$\begin{aligned} &-\Big(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha\Big)^2 + \frac{1}{q}\Big(\sum_{\alpha=n+p-q+1}^{n+p}\tilde{\sigma}_\alpha\Big)^2 \\ &= -\Big(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha\Big)^2 + \Big(\sum_{\alpha=n+p-q+1}^{n+p}\tilde{\sigma}_\alpha\Big)^2 + \Big(\frac{1}{q}-1\Big)\Big(\sum_{\alpha=n+p-q+1}^{n+p}\tilde{\sigma}_\alpha\Big)^2 \\ &\geqslant \Big(-\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha + \sum_{\alpha=n+p-q+1}^{n+p}\tilde{\sigma}_\alpha\Big)\Big(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha + \sum_{\alpha=n+p-q+1}^{n+p}\tilde{\sigma}_\alpha\Big) - \Big(1-\frac{1}{q}\Big)\rho^4 \\ &\geqslant (nQ - n(n-1)c - n(n-1)H^2)\rho^2 - \Big(1-\frac{1}{q}\Big)\rho^4. \end{aligned} \tag{5.9}$$

By Chen–Do Carmo–Kobayashi [**5**, Lemma 1], we also see that

$$-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) \geqslant -2\Big(1-\frac{1}{p-q}\Big)\rho^4. \tag{5.10}$$

From Lemma 2.2, (4.3), (4.16), (5.9) and (5.10), we have

(5.11)

$$\begin{aligned}
0 =& \int_M \rho^{n-2}(|\nabla h|^2 - \frac{3n^2}{n+2}|\nabla^{\perp}\vec{H}|^2)dv \\
&+ \int_M \rho^{n-2}\Big(\frac{3n^2}{n+2} - n\Big)|\nabla^{\perp}\vec{H}|^2 dv + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2 dv \\
&+ n\int_M \rho^{n-2}\Big\{\sum_{\alpha,\beta} H^{\alpha}H^{\beta}\tilde{\sigma}_{\alpha\beta} + 2\rho^2\sum_{\beta=n+1}^{n+p-q}(H^{\beta})^2 - H^2\rho^2\Big\}dv + nc\int_M \rho^n dv \\
&+ \int_M \rho^{n-2}\Big\{-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q} N(\tilde{A}_{\alpha}\tilde{A}_{\beta} - \tilde{A}_{\beta}\tilde{A}_{\alpha}) - \sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_{\alpha}^2 + \sum_{\alpha=n+p-q+1}^{n+p}\tilde{\sigma}_{\alpha}^2\Big\}dv \\
\geqslant& -n\int_M \rho^{n-2}H^2\rho^2 dv + nc\int_M \rho^n dv + \int_M \rho^{n-2}\Big\{-2\Big(1 - \frac{1}{p-q}\Big)\rho^4\Big\}dv \\
&+ \int_M \rho^{n-2}\Big\{-\Big(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_{\alpha}\Big)^2 + \frac{1}{q}\Big(\sum_{\alpha=n+p-q+1}^{n+p}\tilde{\sigma}_{\alpha}\Big)^2\Big\}dv \\
\geqslant& -n\int_M \rho^{n-2}H^2\rho^2 dv + nc\int_M \rho^n dv + \int_M \rho^{n-2}\Big\{-2\Big(1 - \frac{1}{p-q}\Big)\rho^4\Big\}dv \\
&+ \int_M \rho^{n-2}\Big\{\Big(nQ - n(n-1)c - n(n-1)H^2\Big)\rho^2 - \Big(1 - \frac{1}{q}\Big)\rho^4\Big\}dv \\
&= \int_M n\rho^n\Big\{Q - (n-2)c - nH^2 - \frac{1}{n}\Big(3 - \frac{p+q}{(p-q)q}\Big)\rho^2\Big\}dv.
\end{aligned}$$

$$Q \geqslant (n-2)c + nH^2 + \frac{1}{n}\Big(3 - \frac{p+q}{(p-q)q}\Big)\rho^2,$$

from (5.11), we see that $\rho^2 = 0$ and M is totally umbilical or $Q = (n-2)c + nH^2 + \frac{1}{n}\big(3 - \frac{p+q}{(p-q)q}\big)\rho^2$. In the latter case, from (5.11), we know that (5.2) holds. If $\rho^2 = 0$, that is M is totally umbilical, otherwise, if $\rho^2 \neq 0$, it follows from (5.2) that $\sum_{\alpha}(H^{\alpha})^2\tilde{\sigma}_{\alpha} = 0$. By the same argument as in the proof of Theorem 1.1, we see that $H^{\alpha} = 0$ and $H = 0$. It also follows from (5.11) that $|\nabla h|^2 = \frac{3n^2}{n+2}|\nabla^{\perp}\vec{H}|^2 = 0$, that is, the second fundamental form of M is parallel. This completes the proof of Theorem 1.3. □

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