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EXPLICIT AND ASYMPTOTIC FORMULAE FOR VASYUNIN–COTANGENT SUMS

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Abstract. For coprime numbers p and q, we consider the Vasyunin–cotangent sum

$$V(q,p)=\sum_{k=1}^{p-1}\left\{\frac{kq}{p}\right\}\cot\left(\frac{\pi k}{p}\right).$$

First, we prove explicit formula for the symmetric sum $V(p,q)+V(q,p)$ which is a new reciprocity law for the sums above. This formula can be seen as a complement to the Bettin–Conrey result [**13**, Theorem 1]. Second, we establish an asymptotic formula for $V(p,q)$. Finally, by use of continued fraction theory, we give a formula for $V(p,q)$ in terms of continued fraction of $\frac{p}{q}$.

1. Introduction and statement of results

1.1. Introduction. Let $H=L^2([0,\infty];\ t^{-2}dt)$ be the Hilbert space with the inner product

$$\langle f,g\rangle=\int_0^\infty f(t)\,\overline{g(t)}\,t^{-2}dt,\qquad f,g\in H. \tag{1.1}$$

For any real number x, $\lfloor x\rfloor$ is the integer part of x, and $\{x\}=x-\lfloor x\rfloor$ is the fractional part of x. Let p be a positive integer. Denote by e_p the function in H given by $e_p(t)=\{t/p\}$. The properties of the subspace $H_n=\mathrm{Vect}(e_1,\dots,e_n)$ spanned by the functions $e_1,\dots,e_n$ is studied in [**6**, **8**]. Consider the characteristic function

$$\chi(t):=\begin{cases}1, & \text{if } t\in[1,\infty],\\ 0, & \text{otherwise.}\end{cases}$$

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The distance d_n from χ to the subspace H_n is given by $d_n = \operatorname{dist}(\chi, H_n) = \inf_{h\in H_n} \|\chi - h\|$. In [**7**] it conjectured that

$$d_n^2 \sim \frac{2+\gamma-\log 4\pi + o(1)}{\log n}, \quad n \to +\infty. \tag{1.2}$$

Balazard and Roton proved in [**9**] that

$$d_n^2 \geqslant \frac{2+\gamma-\log 4\pi}{\log n}, \quad n \to +\infty.$$

It is well known that conjecture (1.2) implies the Riemann hypothesis [**2**–**6**, **8**]. For computation aspects of d_n, we refer to Landreau et al [**18**].

For fixed $n \geqslant 1$, from [**10**], we quote the formula

$$d_n^2 = \frac{\operatorname{Gram}(\chi, e_1, \cdots, e_n)}{\operatorname{Gram}(e_1, \cdots, e_n)}.$$

To compute this quantity, we need to evaluate two types of inner products, namely, $\langle \chi, e_p\rangle$, $\langle e_p, e_q\rangle$. The first one is given in [**6**, §1] by

$$\langle \chi, e_p\rangle = \frac{\log p + 1 - \gamma}{p}.$$

On the other hand, for two coprime numbers p, q, the second inner product is given by the Vasyunin formula [**8**, **28**]

$$\langle e_p, e_q\rangle = \frac{\log 2\pi - \gamma}{2}\Big(\frac{1}{p}+\frac{1}{q}\Big) + \frac{p-q}{2pq}\log\frac{q}{p} - \frac{\pi}{2pq}[V(q,p)+V(p,q)] \tag{1.3}$$

where

$$V(q,p) = \sum_{k=1}^{p-1}\Big\{\frac{kq}{p}\Big\}\cot\Big(\frac{\pi k}{p}\Big). \tag{1.4}$$

The Vasyunin–cotangent sum $V(p,q)$ is still curious. Recently, Bettin and Conrey [**13**, Theorem 1] proved a formula for

$$\frac{\bar{q}}{p}V(q,p) + V(\bar{p},\bar{q})$$

where $p\bar{p} \equiv 1(q)$ and $q\bar{q} \equiv 1(p)$ with $1 \leqslant \bar{p} \leqslant q$, $1 \leqslant \bar{q} \leqslant p$.

For $q = 1$, the cotangent sum (1.4) is first studied by Vasyunin [**28**]. He proved the asymptotic formula (for large p)

$$V(1,p) = -\frac{p\log p}{\pi} + \frac{p}{\pi}(\log 2\pi - \gamma) + O(\log p).$$

This formula is improved by Rassias [**26**] and Maier and Rassias [**20**] as follows

$$V(1,p) = -\frac{p\log p}{\pi} + \frac{p}{\pi}(\log 2\pi - \gamma) + O(1).$$

Let p, $n \in \mathbb{N}$, $p > 6N$, with $N = \lfloor \frac{n}{2} \rfloor + 1$. Maier and Rassias [**21**, Theorem 1.7] proved that there exist absolute real constants $A_1, A_2 \geqslant 1$ and absolute real constants E_l, $l \in \mathbb{N}$ with $|E_l| \leqslant (A_1 l)^{2l}$, such that for each n, we have

$$V(1,p) = -\frac{p\log p}{\pi} + \frac{p}{\pi}(\log 2\pi - \gamma) + \frac{1}{\pi} - \sum_{l=1}^{n} E_l p^{-l} - R_n^*(p) \tag{1.5}$$

where $|R_n^*(p)| \leqslant (n\ A_2)^{4n} p^{-(n+1)}$.

The sum $V(p,q)$ can be interpreted as the value of the Estermann zeta function [**13**, **17**] at $s = 0$

$$E_0\Big(s, \frac{p}{q}\Big) = \sum_{k\geqslant 1} \frac{\tau(k)}{n^s} \exp\Big(\frac{2\pi i k p}{q}\Big).$$

Recently, this sum is studied in [**12**, **13**] and it is proved that $V(p,q)$ satisfies the reciprocity formula for all positive coprime numbers p and q

$$\frac{\bar{q}}{p} V(q,p) + V(\bar{p},\bar{q}) = -\frac{1}{\pi p} - g\Big(\frac{\bar{q}}{p}\Big) \tag{1.6}$$

where g is an analytic function in $\mathbb{C} \smallsetminus \mathbb{R}_-$, which has the following asymptotic expansion of order $N \geqslant 2$, $x \to 0$

$$g(x) = -\frac{\log 2\pi x - \gamma}{\pi} + \frac{2}{\pi}\sum_{k=2}^{N} \frac{\zeta(k) B_k}{k} x^k + O(x^{N+1}),$$

B_k is the k^{th} Bernoulli number. Since $\zeta(2k) = (-1)^{k-1} 2^{2k-1} \pi^{2k} B_{2k}/(2k)!$, we have

$$g(x) = -\frac{\log 2\pi x - \gamma}{\pi} - \frac{1}{2}\sum_{k=1}^{\lfloor N/2\rfloor} (-1)^k \frac{4^k \pi^{2k-1} B_{2k}^2}{k(2k)!} x^k + O(x^{N+1}).$$

1.2. Statement of main results. Now consider the digamma function

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty}\Big(\frac{1}{k} - \frac{1}{k+z}\Big)$$

and the symmetric function

$$G(p,q) = \sum_{r=1}^{pq-1} \Big(\psi\Big(\frac{r+1}{pq}\Big) - \psi\Big(\frac{r}{pq}\Big)\Big)\Big\{\frac{r}{p}\Big\}\Big\{\frac{r}{q}\Big\}. \tag{1.7}$$

We state a new reciprocity formula $V(p,q)$.

Theorem 1.1. *For p, q positive coprime numbers, we have*

$$V(p,q) + V(q,p) = \frac{1}{\pi}\log p^{q-1} q^{p-1} - \frac{2}{\pi} - pg\Big(\frac{1}{p}\Big) - qg\Big(\frac{1}{q}\Big) - \frac{2}{\pi}G(p,q). \tag{1.8}$$

Corollary 1.1. *For p, q positive coprime numbers, we have*

$$\begin{aligned}\langle e_p, e_q\rangle = {}& \frac{2 + (\log 2\pi - \gamma)(p+q)}{2pq} - \frac{1}{2pq}\log p^{p-1} q^{q-1} \\ & + \frac{\pi}{2}\Big(\frac{1}{q}\ g\Big(\frac{1}{p}\Big) + \frac{1}{p}\ g\Big(\frac{1}{q}\Big)\Big) + \frac{1}{pq} G(p,q).\end{aligned} \tag{1.9}$$

Next, we state an asymptotic formula for the sum $V(\bar{a}, pa + r)$.

THEOREM 1.2. *Let $a > r$ and p be integers such that $(a, pa+r) = 1$, $\bar{a}a \equiv 1 \pmod{ap+r}$ and $\bar{r}r \equiv 1 \pmod{a}$. Then for large p, we have*

$$V(\bar{a}, pa+r) = -\Big(p+\frac{r}{a}\Big)V(\bar{r}, a) - \frac{1}{\pi a} + \frac{1}{\pi}\Big(p+\frac{r}{a}\Big)\Big(\log\frac{2\pi a}{pa+r} - \gamma\Big) + \frac{1}{2}\sum_{k=1}^{\lfloor N/2\rfloor}(-1)^k\frac{4^k\pi^{2k-1}B_{2k}^2}{k(2k)!}\Big(\frac{a}{pa+r}\Big)^{2k-1} + O\Big(\frac{1}{p^N}\Big). \tag{1.10}$$

We get the following corollary.

COROLLARY 1.2. *Let $a \geqslant 1$. For large p, we have*

$$V(p+1, ap+a-1) = -\Big(p+1-\frac{1}{a}\Big)V(1,a) - \frac{1}{\pi a} + \frac{1}{\pi}\Big(p+1-\frac{1}{a}\Big)\Big(\log\frac{2\pi a}{(p+1)a-1} - \gamma\Big) + \frac{1}{2}\sum_{k=1}^{\lfloor \frac{N}{2}\rfloor}(-1)^k\frac{4^k\pi^{2k-1}B_{2k}^2}{k(2k)!}\Big(\frac{a}{(p+1)a-1}\Big)^{2k-1} + O\Big(\frac{1}{p^N}\Big). \tag{1.11}$$

In the particular case $a = 1$, we have

$$V(1,p) = \frac{1}{\pi}\Big(\log\frac{2\pi}{p} - \gamma\Big)p - \frac{1}{\pi} - \frac{\pi}{144p} + \frac{1}{2}\sum_{k=2}^{\lfloor \frac{N}{2}\rfloor}(-1)^k\frac{4^k\pi^{2k-1}B_{2k}^2}{k(2k)!}\Big(\frac{1}{p}\Big)^{2k-1} + O\Big(\frac{1}{p^N}\Big). \tag{1.12}$$

Relation (1.12) improves asymptotic formula (1.5) proved by Maier and Rassias in [**21**, Theorem 1.7].

In what follows, we give a few examples.

EXAMPLES 1.1. For large p we have

$$V(p+1, 2p+1) = -\frac{1}{2\pi} + \frac{1}{\pi}\Big(p+\frac{1}{2}\Big)\Big(\log\frac{4\pi}{2p+1} - \gamma\Big) + \frac{1}{2}\sum_{k=1}^{\lfloor N/2\rfloor}(-1)^k\frac{4^k\pi^{2k-1}B_{2k}^2}{k(2k)!}\Big(\frac{2}{2p+1}\Big)^{2k-1} + O\Big(\frac{1}{p^N}\Big).$$

$$V(2p+1, 3p+1) = -\frac{p+\frac{1}{3}}{3\sqrt{3}} - \frac{1}{3\pi} + \frac{1}{\pi}\Big(p+\frac{1}{3}\Big)\Big(\log\frac{3\pi}{2} - \gamma\Big) + \frac{1}{2}\sum_{k=1}^{\lfloor N/2\rfloor}(-1)^k\frac{4^k\pi^{2k-1}B_{2k}^2}{k(2k)!}\Big(\frac{3}{3p+1}\Big)^{2k-1} + O\Big(\frac{1}{p^N}\Big).$$

$$V(p+1, 3p+2) = \frac{p+\frac{2}{3}}{3\sqrt{3}} - \frac{1}{3\pi} + \frac{1}{\pi}\Big(p+\frac{2}{3}\Big)\Big(\log\frac{6\pi}{5} - \gamma\Big) + \frac{1}{2}\sum_{k=1}^{\lfloor N/2\rfloor}(-1)^k\frac{4^k\pi^{2k-1}B_{2k}^2}{k(2k)!}\Big(\frac{3}{3p+2}\Big)^{2k-1} + O\Big(\frac{1}{p^N}\Big).$$

We can consider $V(q,p)$ as a function of a single rational argument, by defining $V(q/p) = V(q,p)$. That this function is well defined is clear from the conditions on q and p. By use of continued fractions [**25**, §7] and the reciprocity law [**13**, Theorem 1], we obtain

THEOREM 1.3. *Let $1 < q < p$ be coprime positive integers and $\bar{p}$ denote the inverse of p modulo q. Write $\bar{p}/q = [a_0, a_1, \ldots, a_n]$ for a simple finite continued fraction of $\bar{p}/q$ and $(p_k)_{0 \leqslant k \leqslant n+1}$ for the finite sequence given by: $p_{k-1} = a_{k-1}p_k + p_{k+1}$ with $p_0 = \bar{p}$, $p_1 = q$ and $p_{n+1} = 0$. Then we have*

$$\begin{aligned} V(p,q) = {} & \frac{1}{\pi(q[a_0, a_1, \ldots, a_{n-2}] - \bar{p})} \\ & - \sum_{k=2}^{n} (-1)^k g([0, a_{k-1}, \ldots, a_n]) \prod_{j=2}^{k} [a_{j-1}, \ldots, a_n]. \end{aligned}$$

COROLLARY 1.3. *Let $\frac{q}{p} = [b_0, \ldots, b_n]$ and $(q_k)_k$ the associated sequence: $q_{k-1} = b_{k-1}q_k + q_{k+1}$, $q_0 = q$, $q_1 = p$, $q_n = 1$. Then we have*

$$\begin{aligned} V(q,p) = {} & -\frac{1}{\pi} - pg\Big(\frac{1}{p}\Big) - \frac{1}{\pi}\log pq + \frac{1}{\pi}(\log q + (-1)^n \log q_{n-1}) \\ & - \frac{1}{\pi}\sum_{k=1}^{n-2} (-1)^k (q_k \log q_{k+1} + q_{k+1}\log q_k - 2G(q_k, q_{k+1})). \end{aligned} \tag{1.13}$$

Moreover, we have

$$\begin{aligned} V(q,p) = {} & -\frac{1}{\pi}(1 + \log f_1) - q\Bigg(f_1 g\Big(\frac{1}{qf_1}\Big) - \frac{1}{\pi}\sum_{k=1}^{n-2}(-1)^k (f_k \log f_{k+1}) + f_{k+1}\log f_k\Bigg) \\ & + \frac{(-1)^n}{\pi}\log f_{n-1} + \frac{(-1)^n - 1}{\pi}\log q + ((-1)^n f_{n-1} - f_1)\frac{q\log q}{\pi} \\ & + \frac{2}{\pi}\sum_{k=1}^{n-2}(-1)^k G(qf_k, qf_{k+1}), \quad \text{where } f_k = \prod_{j=1}^{k}[0, b_{j-1}, \ldots, b_n]. \end{aligned} \tag{1.14}$$

1.3. General case: *p,q* arbitrary. Let $\omega = \gcd(p,q) \geqslant 1$. Vasyunin formula in [**28**] is given by

$$\langle e_p, e_q\rangle = \frac{\log 2\pi - \gamma}{2}\Big(\frac{1}{p} + \frac{1}{q}\Big) + \frac{p-q}{2pq}\log\frac{q}{p} - \frac{\pi\omega}{2pq}\Big(V\Big(\frac{p}{\omega}, \frac{q}{\omega}\Big) + V\Big(\frac{q}{\omega}, \frac{p}{\omega}\Big)\Big).$$

As a consequence of Theorem 1.1, we find a new expression for $\langle e_p, e_q\rangle$:

$$\begin{aligned} \langle e_p, e_q\rangle = {} & \frac{2\omega + (\log 2\pi - \gamma)(p+q)}{2pq} \\ & - \frac{1}{2pq}((p-\omega)\log p + (q-\omega)\log q - (p+q-2\omega)\log\omega) \\ & + \frac{\pi}{2}\Big(\frac{1}{p}g\Big(\frac{\omega}{q}\Big) + \frac{1}{q}g\Big(\frac{\omega}{p}\Big)\Big) + \frac{\omega}{pq}G\Big(\frac{p}{\omega}, \frac{q}{\omega}\Big). \end{aligned}$$

2. Proof of Theorem 1.1 and Corollary 1.1

We start this section with some useful prelimary results.

2.1. Computation of $\langle v_p, v_q\rangle$. Let p be positive integer and v_p be the function given by $v_p(t) = \{\lfloor t\rfloor/p\}$; v_p is defined on $\mathbb{R}_+$ and it can be regarded as a restriction of e_p to $\mathbb{N}$. Since

$$v_p(x) = \begin{cases} 0, & \text{if } x \in [0,1], \\ \{k/p\}, & x \in [k, k+1] \end{cases}$$

we have the relation

$$v_p = e_p - \frac{1}{p}e_1. \tag{2.1}$$

Then, from definition (1.1), we get

$$\langle v_p, v_q\rangle = \sum_{k\geqslant 1} \frac{1}{k(k+1)}\Big\{\frac{k}{p}\Big\}\Big\{\frac{k}{q}\Big\}.$$

Lemma 2.1. *Let p, q be coprime numbers. Then we have*

$$\frac{2pq}{\pi}\langle v_p, v_q\rangle = V(1,p) + V(1,q) - (V(p,q) + V(q,p)) + \frac{1}{\pi}\log p^{q-1}q^{p-1}.$$

Proof. Using expression (2.1), we get

$$\langle v_p, v_q\rangle = \langle e_p - \frac{1}{p}e_1, e_q - \frac{1}{q}e_1\rangle = \langle e_p, e_q\rangle + \frac{1}{pq}\langle e_1, e_1\rangle - \frac{1}{p}\langle e_1, e_q\rangle - \frac{1}{q}\langle e_1, e_p\rangle$$

On the other hand, from Vasyunin formula (1.3), we obtain

$$\langle e_1, e_q\rangle = \frac{\log 2\pi - \gamma}{2}\Big(1 + \frac{1}{q}\Big) + \frac{1-q}{2q}\log q - \frac{\pi}{2q}V(1,q),$$

$$\langle e_1, e_p\rangle = \frac{\log 2\pi - \gamma}{2}\Big(1 + \frac{1}{p}\Big) + \frac{1-p}{2p}\log p - \frac{\pi}{2p}V(1,p),$$

$$\langle e_p, e_q\rangle = \frac{\log 2\pi - \gamma}{2}\Big(\frac{1}{p} + \frac{1}{q}\Big) + \frac{p-q}{2pq}\log\frac{q}{p} - \frac{\pi}{2pq}[V(q,p) + V(p,q)].$$

Therefore we deduce

$$\frac{2pq}{\pi}\langle v_p, v_q\rangle = V(1,p) + V(1,q) - V(p,q) - V(q,p) + \frac{1}{\pi}(q\log p + p\log q - \log pq). \quad \square$$

The series

$$\sum_{k\geqslant 1} \frac{1}{k(k+1)}\Big\{\frac{k}{p}\Big\}\Big\{\frac{k}{q}\Big\} \tag{2.2}$$

is convergent. Let us rewrite it in another form. For integer k we set $k \equiv r(pq)$, $1 \leqslant r \leqslant pq-1$ and then $\big\{\frac{k}{p}\big\}\big\{\frac{k}{q}\big\} = \big\{\frac{r}{p}\big\}\big\{\frac{r}{q}\big\}$. Series (2.2) is equal to

$$\sum_{r=1}^{pq-1}\sum_{i\geqslant 0} \frac{1}{(ipq+r)(ipq+r+1)}\Big\{\frac{r}{p}\Big\}\Big\{\frac{r}{q}\Big\}. \tag{2.3}$$

Finally, to estimate $\langle v_p, v_q \rangle$ we need to compute the sums

$$\sum_{i \geqslant 0} \frac{1}{(ipq+r)(ipq+r+1)}.$$

LEMMA 2.2. *For a, b two distinct positive numbers, we have*

$$\sum_{k=0}^{\infty} \frac{1}{(k+a)(k+b)} = \frac{\psi(a)-\psi(b)}{a-b}.$$

PROOF. We write

$$\psi(a)-\psi(b) = \frac{1}{b} - \frac{1}{a} + \sum_{k \geqslant 1} \Big(\frac{1}{k+b} - \frac{1}{k+a} \Big) = (a-b) \sum_{k=0}^{\infty} \frac{1}{(k+a)(k+b)}. \qquad \square$$

COROLLARY 2.1. *Let $\alpha > 0$. We have*

$$\sum_{k=0}^{\infty} \frac{1}{(\alpha k+a)(\alpha k+b)} = \frac{\psi(a/\alpha)-\psi(b/\alpha)}{\alpha(a-b)}. \tag{2.4}$$

From [**19**], we have the integral representation

$$\psi(a)-\psi(b) = \int_0^{\infty} \frac{e^{-bt}-e^{-at}}{1-e^{-t}}\, dt.$$

Thus, we can get

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{a-b}{(k+a)(k+b)} &= \sum_{k=0}^{\infty} \Big(\frac{1}{k+b} - \frac{1}{k+a} \Big) = \sum_{k=0}^{\infty} \int_0^1 (x^{k+b-1} - x^{x+a-1}) dx \\
&= \int_0^1 \frac{x^{b-1}-x^{a-1}}{1-x}\, dx = \int_0^{\infty} \frac{e^{-bt}-e^{-at}}{1-e^{-t}}\, dt.
\end{aligned}$$

Hence, the special case $\alpha = pq$ of relations (1.7), (2.3), and (2.4) gives the result.

COROLLARY 2.2. *For p, q arbitrary, we have*

$$\sum_{k \geqslant 1} \frac{pq}{k(k+1)} \Big\{ \frac{k}{p} \Big\} \Big\{ \frac{k}{q} \Big\} = G(p,q). \tag{2.5}$$

2.2. Proof of Theorem 1.1 and Corollary 1.1. From Lemma 2.1 and Corollary 2.2, we deduce that

$$2G(p,q) = V(1,p) + V(1,q) - (V(p,q)+V(q,p)) + \frac{1}{\pi} \log p^{q-1} q^{p-1}.$$

Therefore, we get (1.8) from the expression

$$V(1,p) = -\frac{1}{\pi} - pg\Big(\frac{1}{p}\Big). \tag{2.6}$$

We can obtain formula (1.9) by use of reciprocity formula (1.8) and Vasyunin formula (1.3).

3. Proof of Theorem 1.2 and Corollary 1.2

From reciprocity law (1.6), we have

$$\frac{a}{pa+r}V(\bar{a},pa+r)+V(\bar{r},a)=-\frac{1}{\pi(pa+r)}-g\Big(\frac{a}{pa+r}\Big)$$

and then we obtain

$$V(\bar{a},pa+r)=-\frac{pa+r}{a}V(\bar{r},a)-\frac{1}{\pi a}-\frac{pa+r}{a}g\Big(\frac{a}{pa+r}\Big).$$

Moreover, for large p we have

$$g\Big(\frac{a}{pa+r}\Big)=-\frac{1}{\pi}\Big(\log\frac{2\pi a}{pa+r}-\gamma\Big)-\frac{1}{2}\sum_{k=1}^{\lfloor N/2\rfloor}(-1)^k\frac{4^k\pi^{2k-1}B_{2k}^2}{k(2k)!}\Big(\frac{a}{pa+r}\Big)^k+O\Big(\frac{1}{p^{N+1}}\Big)$$

Taking $r=a-1$, one remarks that for $a\geqslant 1$ $(p+1)a\equiv 1 \pmod{pa+a-1}$ and $(a-1)^2\equiv 1(a)$. Then $\bar{a}=p+1$ and $\bar{r}=r=a-1$. Since $V(a-1,a)=V(1,a)$ and thanks to relation (1.10) of Theorem 1.2, we obtain relation (1.11). Finally from relation (1.11) we can get (1.12).

4. Proof of Theorem 1.3 and Corollary 1.3

4.1. Continued fractions – an overview. In this subsection we quote some facts about continued fractions, that will be useful later. For more details we refer to [**25**, §7]. For real number x let

$$x=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cdots}}}=[a_0,a_1,a_3,\dots]$$

be its continued fraction expansion with partial quotients $a_0\in\mathbb{Z}$, $a_k\in\mathbb{N}\smallsetminus\{0\}$, $k\geqslant 1$. In fact, we can determine a_0, $a_1,\dots,a_n$ via the *algorithm*:

- $x=\lfloor x\rfloor+\{x\}$, $a_0=\lfloor x\rfloor$, $\xi_0=\{x\}$, if $\xi_0=0$, then x is represented by $x=[a_0]$.
- if $\xi_0\neq 0$ then $\lfloor\frac{1}{\xi_0}\rfloor>1$, $r_1=\frac{1}{\xi_0}$ we obtain $x=[a_0,r_1]$, $a_1=\lfloor r_1\rfloor$ and $\xi_1=r_1-a_1$; if $\xi_1=0$ then $x=a_0+\frac{1}{r_1}=a_0+\frac{1}{a_1}=[a_0,a_1]$.
- Otherwise we take $r_2=\frac{1}{\xi_1}$ and iterate the process.

We then get the sequence $a_0,a_1,a_2,\dots$. This sequence is finite if and only if x is a rational number. In the rational case, this algorithm is the *Euclidian algorithm.* Let us express a/b as a continued fraction of the form $a/b=[a_0,a_1,\dots,a_n]$, with $a_{n+1}=0$. We can determine $a_0,a_1,\dots,a_n$ by the Euclidean algorithm

$$p_{k-1}=a_{k-1}p_k+p_{k+1},\quad p_0=a,\quad p_1=b,\quad p_{n+1}=0. \tag{4.1}$$

Then $p_n=\gcd(p,q)=1$. We quote from [**25**, §7] the following elementary properties of continued fractions which are needed in this paper. We omit their proofs.

LEMMA 4.1. *Let* $[a_0, a_1, \ldots, a_n]$ *be continued fraction. Then*

$$[a_0, a_1, \ldots, a_n] = \Big[a_0, a_1, \ldots, a_{n-1} + \frac{1}{a_n}\Big],$$
$$[a_0, a_1, \ldots, a_n] = a_0 + \frac{1}{[a_1, \cdots, a_n]}.$$

We get this lemma from the continued fraction definition. Let s_k, t_k be two sequences of integers

$$s_k = \begin{cases} 0, & \text{if } k = -2 \\ 1, & \text{if } k = -1 \\ a_k s_{k-1} + s_{k-2}, & \text{if } k \geqslant 0 \end{cases} \qquad t_k = \begin{cases} 1, & \text{if } k = -2 \\ 0, & \text{if } k = -1 \\ a_k t_{k-1} + t_{k-2}, & \text{if } k \geqslant 0 \end{cases}$$

The sequences p_k, s_k and t_k have the following properties.

LEMMA 4.2.

$$[a_0, a_1, \ldots, a_k] = \frac{s_k}{t_k}, \quad k \geqslant 0, \tag{4.2}$$
$$[a_k, a_{k-1}, \ldots, a_1] = \frac{t_k}{t_{k-1}}, \quad k \geqslant 1,$$
$$[0, a_{k-1}, \ldots, a_n] = \frac{p_k}{p_{k-1}}, \quad k \geqslant 0, \tag{4.3}$$
$$[a_k, a_{k-1}, \ldots, a_0] = \frac{s_k}{s_{k-1}}, \; k \geqslant 0.$$

This lemma represents the classical properties of the continued fractions. The quotient $\frac{s_k}{t_k}$ is called the k^{th}-*convergent* of the continued fraction of $\frac{a}{b}$.

COROLLARY 4.1. *Let* $\frac{a}{b} = [a_0, a_1, \ldots, a_n]$*; then*

$$p_k = b \prod_{j=2}^{k} [0, a_{j-1}, \ldots, a_n], \quad k \geqslant 2, \tag{4.4}$$
$$b = p_k t_{k-1} + p_{k+1} t_{k-2}, \quad k \geqslant 1.$$

LEMMA 4.3. *The sequences* s_k, t_k *and* p_k *satisfy*

$$t_k s_{k-1} - t_{k-1} s_k = (-1)^k,$$
$$p_k = (-1)^k (a t_{k-2} - b s_{k-2}). \tag{4.5}$$

PROPOSITION 4.1. *Under the hypothesis of Corollary* 4.1*, we have*

$$\sum_{k=1}^{n-1} \frac{(-1)^k}{p_k p_{k+1}} = \frac{1}{b^2 [a_0, a_1, \ldots, a_{n-2}] - ab}. \tag{4.6}$$

PROOF. We have from Lemma 4.3

$$\sum_{k=1}^{n-1} \frac{(-1)^k}{p_k p_{k+1}} = \frac{1}{b} \sum_{k=1}^{n-1} \Big(\frac{(-1)^k t_{k-1}}{p_{k+1}} + \frac{(-1)^k t_{k-2}}{p_k} \Big)$$

$$= \frac{1}{b}\sum_{k=1}^{n-1}\frac{(-1)^k t_{k-1}}{p_{k+1}} + \frac{1}{b}\sum_{k=1}^{n-1}\frac{(-1)^k t_{k-2}}{p_k}$$
$$= \frac{1}{b}\sum_{k=2}^{n}\frac{(-1)^{k-1} t_{k-2}}{p_k} + \frac{1}{b}\sum_{k=1}^{n-1}\frac{(-1)^k t_{k-2}}{p_k}$$
$$= \frac{(-1)^{n-1}t_{n-2}}{p_n\ b} - \frac{t_{-1}}{b\ p_1} = \frac{(-1)^{n-1}t_{n-2}}{p_n\ b}.$$

From (4.5) we deduce $p_n = (-1)^n(at_{n-2} - bs_{n-2})$ and then

$$\frac{(-1)^{n-1}p_n}{t_{n-2}} = \Big(b\frac{s_{n-2}}{t_{n-2}} - a\Big). \tag{4.7}$$

Relation (4.2) implies that

$$\frac{s_{n-2}}{t_{n-2}} = [a_0, a_1, \ldots, a_{n-2}] \tag{4.8}$$

From relations (4.7) and (4.8) we have

$$\frac{(-1)^{n-1}t_{n-2}}{p_n b} = \frac{1}{b^2[a_0, a_1, \ldots, a_{n-2}] - ab}.$$

Therefore we obtain

$$\sum_{k=1}^{n-1}\frac{(-1)^k}{p_k p_{k+1}} = \frac{1}{b^2[a_0, a_1, \ldots, a_{n-2}] - ab}.$$ □

4.2. Application: Computation of $V(p,q)$. Using reciprocity formula (1.6) and continued fractions properties, we can get an explicit formula for $V(p,q)$.

LEMMA 4.4. *Let p and q be coprime positive numbers. We have*

$$V(p,q) = \frac{q}{\pi}\sum_{k=1}^{n-1}\frac{(-1)^k}{p_k p_{k+1}} - q\sum_{k=2}^{n}\frac{(-1)^k}{p_k} g\Big(\frac{p_k}{p_{k-1}}\Big). \tag{4.9}$$

PROOF. From the reciprocity law (1.6), with $\bar{p} = a$, $q = b$ and the sequence $(p_k)_k$ defined in relation (4.1), we have

$$\frac{1}{p_{k-1}}V(\bar{p}_k, p_{k-1}) + \frac{1}{p_k}V(\bar{p}_{k-1}, p_k) = \frac{1}{p_k}\Big(\frac{1}{\pi p_{k-1}} + g\Big(\frac{p_k}{p_{k-1}}\Big)\Big). \tag{4.10}$$

Observe that

$$p_{k-1} = a_{k-1}p_k + p_{k+1}, \quad \bar{p}_{k-1} = \bar{p}_{k+1}, \quad V(\bar{p}_{k-1}, p_k) = V(\bar{p}_{k+1}, p_k),$$

then relation (4.10) becomes

$$\frac{1}{p_{k-1}}V(\bar{p}_k, p_{k-1}) + \frac{1}{p_k}V(\bar{p}_{k+1}, p_k) = -\frac{1}{p_k}\Big(\frac{1}{\pi p_{k-1}} + g\Big(\frac{p_k}{p_{k-1}}\Big)\Big).$$

We have the computation

$$\sum_{k=1}^{n}\Big[\frac{(-1)^k}{p_{k-1}}V(\bar{p}_k, p_{k-1}) + \frac{(-1)^k}{p_k}V(\bar{p}_{k+1}/p_k)\Big]$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \frac{(-1)^k}{p_{k-1}} V(\bar{p}_k, p_{k-1}) + \sum_{k=1}^{n} \frac{(-1)^k}{p_k} V(\bar{p}_{k+1}, p_k) \\
&= \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{p_k} V(\bar{p}_{k+1}, p_k) + \sum_{k=1}^{n} \frac{(-1)^k}{p_k} V(\bar{p}_{k+1}, p_k) \\
&= -\frac{1}{p_0} V(\bar{p}_1, p_0) + \frac{(-1)^n}{p_n} V(\bar{p}_{n+1}, p_n).
\end{aligned}
$$

Since $p_{n+1} = 0$, the relation $V(\bar{p}_{n+1}, p_n) = 0$ implies

$$
\begin{aligned}
&\sum_{k=1}^{n} \left[\frac{(-1)^k}{p_{k-1}} V(\bar{p}_k, p_{k-1}) + \frac{(-1)^k}{p_k} V(\bar{p}_{k+1}, p_k) \right] = -\frac{1}{\bar{p}} V(\bar{q}, \bar{p}) \\
&= -\frac{1}{\bar{p}} V(\bar{q}, \bar{p}) - \frac{1}{q} V(p, q) + \sum_{k=2}^{n} \left[\frac{(-1)^k}{p_{k-1}} V(\bar{p}_k, p_{k-1}) + \frac{(-1)^k}{p_k} V(\bar{p}_{k+1}, p_k) \right].
\end{aligned}
$$

Moreover,

$$
\frac{1}{q} V(p, q) = \sum_{k=2}^{n} \left[\frac{(-1)^k}{p_{k-1}} V(\bar{p}_k, p_{k-1}) + \frac{(-1)^k}{p_k} V(\bar{p}_{k+1}, p_k) \right]
$$

then we deduce

$$
\frac{1}{q} V(p, q) = -\sum_{k=2}^{n} \frac{(-1)^k}{p_k} \left(\frac{1}{\pi p_{k-1}} + g\left(\frac{p_k}{p_{k-1}} \right) \right) = \frac{1}{\pi} \sum_{k=1}^{n-1} \frac{(-1)^k}{p_k p_{k+1}} - \sum_{k=2}^{n} \frac{(-1)^k}{p_k} g\left(\frac{p_k}{p_{k-1}} \right). \quad \square
$$

4.3. Proof of Theorem 1.3 and Corollary 1.3. First, we prove Theorem 1.3. Taking $a = \bar{p}$ and $b = q$, by virtue of relation (4.6) we obtain

$$
\frac{q}{\pi} \sum_{k=1}^{n-1} \frac{(-1)^k}{p_k p_{k+1}} = \frac{1}{q[a_0, a_1, \ldots, a_{n-2}] - \bar{p}}. \tag{4.11}
$$

From (4.4) we get

$$
\frac{1}{p_k} = \frac{1}{q} \prod_{j=2}^{k} [a_{j-1}, \ldots, a_n], \quad k \geqslant 2,
$$

and from (4.3) we obtain

$$
\sum_{k=2}^{n} \frac{(-1)^k}{p_k} g\left(\frac{p_k}{p_{k-1}} \right) = \sum_{k=2}^{n} \frac{1}{q} g\left([a_{k-1}, \cdots, a_n] \right) \prod_{j=2}^{k} [a_{j-1}, \cdots, a_n]. \tag{4.12}
$$

Substituting quantities (4.11) and (4.12) into (4.9) we get Theorem 1.3.

To prove Corollary 1.3, note that for $p \equiv r(q)$ we have

$$
V(p, q) = \begin{cases} 0, & \text{if } r = 0 \\ V(r, q), & \text{otherwise.} \end{cases} \tag{4.13}
$$

Applying several times reciprocity formula (1.8) to the sequence q_k, we obtain

$$\sum_{k=1}^{n-1}(-1)^k\theta(q_{k-1},q_k)=\sum_{k=1}^{n-1}(-1)^k[V(q_{k-1},q_k)+V(q_k,q_{k-1})]$$

where

$$\begin{aligned}\theta(q_{k-1},q_k)&=\frac{1}{\pi}(q_{k-1}\log q_k+q_k\log q_{k-1}-\log q_{k-1}q_k)\\&\quad-\frac{2}{\pi}-q_{k-1}g\Big(\frac{1}{q_{k-1}}\Big)-q_kg\Big(\frac{1}{q_k}\Big)-\frac{2}{\pi}G(q_{k-1},q_k).\end{aligned}$$

From (4.13) we have $V(q_{k+1},q_k)=V(q_{k-1},q_k)$ and $V(q_{n-1},q_n)=0$, and then

$$\begin{aligned}\sum_{k=1}^{n-1}(-1)^k\theta(q_{k-1},q_k)&=\sum_{k=1}^{n-1}(-1)^kV(q_{k-1},q_k)+\sum_{k=1}^{n-1}(-1)^kV(q_k,q_{k-1})\\&=\sum_{k=1}^{n-1}(-1)^kV(q_{k+1},q_k)+\sum_{k=0}^{n-2}(-1)^{k+1}V(q_{k+1},q_k)\\&=-V(p,q)-(-1)^nV(q_n,q_{n-1}).\end{aligned}$$

In addition we have

$$\begin{aligned}\sum_{k=1}^{n-1}(-1)^k\theta(q_{k-1},q_k)&=\frac{1}{\pi}\sum_{k=1}^{n-1}(-1)^k[(q_{k-1}\log q_k+q_k\log q_{k-1}-\log q_{k-1}q_k)]\\&=\ -\frac{2}{\pi}\sum_{k=1}^{n-1}(-1)^k[1+G(q_{k-1},q_k)]\\&\quad-\sum_{k=1}^{n-1}(-1)^k\Big[q_{k-1}g\Big(\frac{1}{q_{k-1}}\Big)+q_kg\Big(\frac{1}{q_k}\Big)\Big]\end{aligned}$$

and we push this computation we arrive to

$$\begin{aligned}\sum_{k=1}^{n-1}(-1)^k\Big[q_{k-1}g\Big(\frac{1}{q_{k-1}}\Big)+q_kg\Big(\frac{1}{q_k}\Big)\Big]&=\sum_{k=1}^{n-1}(-1)^kq_{k-1}g\Big(\frac{1}{q_{k-1}}\Big)+\sum_{k=1}^{n-1}(-1)^kq_kg\Big(\frac{1}{q_k}\Big)\\&=\sum_{k=0}^{n-2}(-1)^{k+1}q_kg\Big(\frac{1}{q_k}\Big)+\sum_{k=1}^{n-1}(-1)^kq_kg\Big(\frac{1}{q_k}\Big)\\&=-qg\Big(\frac{1}{q}\Big)+(-1)^{n-1}q_{n-1}g\Big(\frac{1}{q_{n-1}}\Big).\end{aligned}$$

We have also

$$\begin{aligned}\sum_{k=1}^{n-1}(-1)^k\log q_{k-1}q_k&=\sum_{k=1}^{n-1}(-1)^k(\log q_{k-1}+\log q_k)\\&=\sum_{k=1}^{n-1}(-1)^k\log q_{k-1}+\sum_{k=1}^{n-1}(-1)^k\log q_k\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-2} (-1)^{k+1} \log q_k + \sum_{k=1}^{n-1} (-1)^k \log q_k \\
&= -\log q - (-1)^n \log q_{n-1}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
-V(p,q) - (-1)^n V(q_n, q_{n-1}) = {}& \frac{1}{\pi} \sum_{k=1}^{n-1} (-1)^k (q_{k-1} \log q_k + q_k \log q_{k-1}) + qg\Big(\frac{1}{q}\Big) \\
& + (-1)^n q_{n-1} g\Big(\frac{1}{q_{n-1}}\Big) - \frac{2}{\pi} \sum_{k=1}^{n-1} (-1)^k [1 + G(q_{k-1}, q_k)] \\
& + \frac{1}{\pi} (\log q + (-1)^n \log q_{n-1})
\end{aligned}
$$

and

$$
\begin{aligned}
-V(p,q) &= (-1)^n V(q_n, q_{n-1}) + (-1)^n q_{n-1} g\Big(\frac{1}{q_{n-1}}\Big) - \frac{1}{\pi} (q \log p + p \log q) \\
&\quad + qg\Big(\frac{1}{q}\Big) + \frac{2}{\pi}(1 + G(q,p)) + \frac{1}{\pi}(\log q + (-1)^n \log q_{n-1}) \\
&\quad - \frac{1}{\pi} \sum_{k=1}^{n-2} (-1)^k [q_k \log q_{k+1} + q_{k+1} \log q_k - 2G(q_k, q_{k+1}) - 2] \\
&= \frac{(-1)^{n+1}}{\pi} - V(p,q) - V(q,p) - \frac{1}{\pi} \log pq - pg\Big(\frac{1}{p}\Big) \\
&\quad + \frac{1}{\pi}(\log q + (-1)^n \log q_{n-1}) \\
&\quad - \frac{1}{\pi} \sum_{k=1}^{n-2} (-1)^k [q_k \log q_{k+1} + q_{k+1} \log q_k - 2G(q_k, q_{k+1}) - 2].
\end{aligned}
$$

Then we complete the proof of relation (1.13)

$$
\begin{aligned}
V(q,p) = {}& -\frac{1}{\pi} - pg\Big(\frac{1}{p}\Big) - \frac{1}{\pi} \log pq + \frac{1}{\pi}(\log q + (-1)^n \log q_{n-1}) \\
& - \frac{1}{\pi} \sum_{k=1}^{n-2} (-1)^k [q_k \log q_{k+1} + q_{k+1} \log q_k - 2G(q_k, q_{k+1})].
\end{aligned}
$$

We observe that for any integer $1 \leqslant k \leqslant n$ we have $q_k = q \prod_{j=1}^{k} [0, b_{j-1}, \ldots, b_n]$, so that we can get (1.14) from (1.13).

5. Further identities on $V(p,q)$ and computation

In this section we relate the sums $V(p,q)$ to some interesting and well-known convergent series. The functions ψ and cotangent are related by the reflection formula [**1**, §6.3.7]

$$\psi(1-z) - \psi(z) = \pi \cot(\pi z). \tag{5.1}$$

Moreover, ψ can be written in terms of harmonic function $H_n(z) = \sum_{k=0}^{n} \frac{1}{(k+z)}$ and the n-th harmonic number $H_n = \sum_{k=1}^{n} \frac{1}{k}$ as follows.

LEMMA 5.1. *Let $z \in \mathbb{C}$ such that $\operatorname{Re}(z) > 0$. Then we have*

$$\psi(z) = \lim_{n\to\infty} (\log n - H_n(z))$$

at $z = n$ positive integer we have

$$\psi(n+1) = -\gamma + H_n. \tag{5.2}$$

PROOF. It is is easy to see that

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty} \Big(\frac{1}{k} - \frac{1}{k+z}\Big) = \lim_{n\to\infty} \Big(\log n - \sum_{k=0}^{n} \frac{1}{k+z}\Big).$$

Then

$$\begin{aligned}\psi(n+1) &= \lim_{m\to\infty} \left(\log m - \sum_{k=0}^{m} \frac{1}{k+n+1}\right) = \lim_{m\to\infty} \left(\log m - \sum_{k=n}^{m+n} \frac{1}{k+1}\right)\\ &= \lim_{m\to\infty} \left(\log m - \left(\sum_{k=0}^{m+n} \frac{1}{k+1} - \sum_{k=0}^{n-1} \frac{1}{k+1}\right)\right)\\ &= \lim_{m\to\infty} \left(\log(m+n) - \sum_{k=0}^{m+n} \frac{1}{k+1}\right) + H_n = -\gamma + H_n. \qquad \square\end{aligned}$$

PROPOSITION 5.1. *For $x = \bar{p}/q$ with $(p,q) = 1$, we have*

$$V(p,q) = \frac{1}{\pi q} \sum_{k\geqslant 0} \sum_{r=1}^{q-1} \frac{r(1-2rx)}{(k+1-rx)(k+rx)}. \tag{5.3}$$

PROOF. We have

$$V(p,q) = \sum_{r=1}^{q-1} \Big\{\frac{rp}{q}\Big\} \cot\Big(\frac{\pi r}{q}\Big) = \sum_{r=1}^{q-1} \frac{r}{q} \cot\Big(\frac{\pi r\bar{p}}{q}\Big).$$

By reflection formula (5.1) we have

$$\cot\Big(\frac{\pi r\bar{p}}{q}\Big) = \frac{1}{\pi}\Big(\psi\Big(\frac{q-r\bar{p}}{q}\Big) - \psi\Big(\frac{r\bar{p}}{q}\Big)\Big).$$

From relations (1.7) and (2.5) we have

$$\psi\Big(\frac{q-r\bar{p}}{q}\Big) - \psi\Big(\frac{r\bar{p}}{q}\Big) = q \sum_{k\geqslant 0} \frac{q-2r\bar{p}}{(q(k+1)-r\bar{p})(qk+r\bar{p})}.$$

Then

$$V(p,q) = \frac{1}{\pi} \sum_{r=1}^{q-1} \sum_{k\geqslant 0} \frac{r(q-2r\bar{p})}{(q(k+1)-r\bar{p})(qk+r\bar{p})},$$

and we obtain

$$V(p,q) = \frac{1}{\pi q}\sum_{r=1}^{q-1}\sum_{k\geqslant 0}\frac{r(1-2rx)}{(k+1-rx)(k+rx)}. \qquad \square$$

As an immediate consequence we derive

COROLLARY 5.1. *Let q be a positive integer; we have*

$$V(1,q) = -\frac{q(\psi(q)+\gamma-2)+2}{\pi} + \frac{1}{\pi}\sum_{k\geqslant 1}\sum_{r=1}^{q-1}\frac{r(q-2r)}{(q(k+1)-r)(qk+r)}, \tag{5.4}$$

$$V(1,q) = -\frac{q(\psi(q)+\gamma-2)+2}{\pi} - \frac{1}{\pi}\sum_{k\geqslant 1}\sum_{r=1}^{\lfloor q/2\rfloor}\frac{(q-2r)^2}{(q(k+1)-r)(qk+r)}.$$

PROOF. From (5.3), special case $p=1$, we can get

$$V(1,q) = \frac{1}{\pi q}\sum_{k\geqslant 0}\sum_{r=1}^{q-1}\frac{r(1-2r/q)}{(k+1-r/q)(k+r/q)}$$

then we have

$$V(1,q) = \frac{1}{\pi}\sum_{k\geqslant 0}\sum_{r=1}^{q-1}\frac{r(q-2r)}{(q(k+1)-r)(qk+r)}$$

and

$$\begin{aligned} V(1,q) &= \frac{1}{\pi}\sum_{r=1}^{q-1}\frac{q-2r}{q-r} - \frac{1}{\pi}\sum_{k\geqslant 1}\sum_{r=1}^{q-1}\frac{r(q-2r)}{(q(k+1)-r)(qk+r)} \\ &= \frac{1}{\pi}(q(2-H_{q-1})-2) + \frac{1}{\pi}\sum_{k\geqslant 1}\sum_{r=1}^{q-1}\frac{r(q-2r)}{(q(k+1)-r)(qk+r)}. \end{aligned}$$

From (5.2) we obtain (5.4). To end the proof we use that for $t+r=q$, we have

$$\frac{1}{(q(k+1)-r)(qk+r)} = \frac{1}{(q(k+1)-t)(qk+t)},$$
$$r(q-2r)+t(q-2t) = (q-2r)^2. \qquad \square$$

PROPOSITION 5.2. *The sum $V(1,q)$ has the following integral representation.*

$$V(1,q) = -\frac{1}{\pi}\int_0^1 \frac{(q-2)x^q - qx^{q-1} + qx - q + 2}{(x-1)^2(x^q-1)}\,dx. \tag{5.5}$$

REMARK 5.1. Note that from (5.5) and (2.6) we deduce that

$$g\Big(\frac{1}{q}\Big) = \frac{1}{\pi q}\int_0^1 \Big(\frac{(q-2)x^q - qx^{q-1} + qx - q + 2}{(x-1)^2(x^q-1)} - 1\Big)dx.$$

PROOF. From the relation (5.4) we have

$$V(1,q)=\frac{1}{\pi}\sum_{r=1}^{q-1} r\sum_{k\geqslant 0}\frac{q-2r}{(q(k+1)-r)(qk+r)},$$

$$\frac{q-2r}{(q(k+1)-r)(qk+r)}=\frac{1}{qk+r}-\frac{1}{qk+q-r}=\int_0^1 (x^{qk+r-1}-x^{qk+q-r-1})dx.$$

Then

$$\sum_{k\geqslant 0}\frac{q-2r}{(q(k+1)-r)(qk+r)}=\int_0^1 \frac{x^{r-1}-x^{q-r-1}}{1-x^q}\,dx$$

which gives

$$V(1,q)=-\frac{1}{\pi}\int_0^1 \frac{\sum_{r=1}^{q-1} r(x^{r-1}-x^{q-r-1})}{x^q-1}\,dx.$$

On the other hand

$$\sum_{r=1}^{q-1} rx^{r-1}=\Big(\frac{x^q-1}{x-1}\Big)'=\frac{(q-1)x^q-qx^{q-1}+1}{(x-1)^2}. \qquad \square$$

Thanks to Proposition 5.2 we give a few values of $V(1,p)$

EXAMPLE 5.1. $V(1,2)=0$,

$$V(1,3)=-\frac{1}{\pi}\int_0^1 \frac{dx}{x^2+x+1}=-\frac{1}{3\sqrt{3}},\quad V(1,4)=-\frac{2}{\pi}\int_0^1 \frac{dx}{x^2+1}=-\frac{1}{2},$$

$$V(1,6)=-\frac{1}{\pi}\int_0^1 \Big(\frac{3}{x^2-x+1}+\frac{1}{x^2+x+1}\Big)dx=-\frac{7}{3\sqrt{3}}.$$

6. Another estimation of $V(p,q)+V(q,p)$

In this section we establish another asymptotic formula for the sum $V(p,q)+V(q,p)$. For this we prove the lemma

LEMMA 6.1. *Let p be a positive integer. We have*

$$\frac{\log p}{p}=\sum_{k\geqslant 1}\frac{1}{k(k+1)}\Big\{\frac{k}{p}\Big\}, \tag{6.1}$$

$$\sum_{r=1}^{pq-1}\Big\{\frac{r}{p}\Big\}\Big\{\frac{r}{q}\Big\}=\frac{(p-1)(q-1)}{4}.$$

PROOF. We have

$$\begin{aligned}\sum_{k\geqslant 1}\frac{1}{k(k+1)}\Big\{\frac{k}{p}\Big\}&=\sum_{k\geqslant 1}\Big\{\frac{k}{p}\Big\}\int_0^1 (x^{k-1}-x^k)dx\\ &=\sum_{r=1}^{p-1}\sum_{i\geqslant 0}\Big\{\frac{r}{p}\Big\}\int_0^1 (x^{ip+r-1}-x^{ip+r})dx\end{aligned}$$

$$= \frac{1}{p}\int_0^1 \frac{\sum_{r=1}^{p-1} r x^{r-1}}{1+x+\cdots+x^{p-1}}\,dx = \frac{\log p}{p}.$$

Thus we obtain (6.1). Let us write

$$\sum_{r=1}^{pq-1}\left\{\frac{r}{p}\right\}\left\{\frac{r}{q}\right\} = \sum_{i=0}^{p-1}\sum_{t=1}^{q-1}\left\{\frac{iq+t}{p}\right\}\left\{\frac{t}{q}\right\} = \frac{1}{pq}\sum_{t=1}^{q-1} t\Big(\sum_{i=0}^{p-1} p\Big\{\frac{iq+t}{p}\Big\}\Big)$$

and we take $r = p\{\frac{iq+t}{p}\}$, observe that r is integer and runs from 1 to $p-1$. Then, we have

$$\sum_{r=1}^{pq-1}\left\{\frac{r}{p}\right\}\left\{\frac{r}{q}\right\} = \frac{1}{pq}\sum_{t=0}^{q-1}\sum_{r=1}^{p-1} tr = \frac{(p-1)(q-1)}{4}. \qquad \square$$

REMARK 6.1. Another proof of relation (6.1) is given in [**10**].

The geometry of $\psi(\frac{r}{p})$ is studied in [**14, 24**]. In the next corollary we write $\log p$ as a linear combination of $\psi(\frac{r}{p})$ for $1 \leqslant r \leqslant p-1$.

COROLLARY 6.1. *Let p be a positive integer. Then we have*

$$\log p = \frac{1}{p}\sum_{r=1}^{p-1} r\Big(\psi\Big(\frac{r+1}{p}\Big) - \psi\Big(\frac{r}{p}\Big)\Big).$$

PROOF. We start with the equalities

$$\frac{\log p}{p} = \sum_{k\geqslant 1}\frac{1}{k(k+1)}\left\{\frac{k}{p}\right\} = \sum_{r=1}^{p-1}\Big(\sum_{i\geqslant 0}\frac{1}{(ip+r)(ip+r+1)}\Big)\left\{\frac{r}{p}\right\}$$

and from (2.4) we have

$$\sum_{i=0}^{\infty}\frac{1}{(pi+r)(pi+r+1)} = \frac{\psi(r+1)/p - \psi(r/p)}{p}$$

thus the result follows. $\square$

PROPOSITION 6.1. *Let p, q be two coprime integers and put*

$$\Delta(p,q) = \frac{2}{\pi}\Big(1 + \max\Big\{\frac{(p-1)(q-1)}{4}\psi\Big(\frac{1}{pq}\Big), \min\{q-\gamma-\psi(q), p-\gamma-\psi(p)\}\Big\}\Big).$$

Then we have

$$\begin{aligned}(6.2)\qquad \Delta(p,q) &< \frac{1}{\pi}\log p^{q-1}q^{p-1} - pg\Big(\frac{1}{p}\Big) - qg\Big(\frac{1}{q}\Big) - V(p,q) - V(q,p)\\ &\leqslant \frac{2+2\sqrt{pq\log p\log q}}{\pi}.\end{aligned}$$

PROOF. From equality (6.1) we remark that

$$\langle v_p, v_q\rangle \leqslant \sum_{k\geqslant 1}\frac{1}{k(k+1)}\left\{\frac{k}{p}\right\} = \frac{\log p}{p} \quad \text{and} \quad \langle v_p, v_q\rangle \leqslant \sum_{k\geqslant 1}\frac{1}{k(k+1)}\left\{\frac{k}{q}\right\} = \frac{\log q}{q}$$

we deduce that $\langle v_p, v_q\rangle \leqslant \sqrt{\log p \log q/pq}$. For $q<p$ we have

$$\langle v_p, v_q\rangle > \frac{1}{pq}\sum_{k=1}^{q-1}\frac{k}{k+1}.$$

Similarly if $p<q$ we have $\langle v_p, v_q\rangle > \frac{1}{pq}\sum_{k=1}^{p-1}\frac{k}{k+1}$. Therefore we have

$$\langle v_p, v_q\rangle > \frac{1}{pq}\Big(q-\sum_{k=1}^{q-1}\frac{1}{k+1}\Big) \quad \text{or} \quad \langle v_p, v_q\rangle > \frac{1}{pq}\Big(p-\sum_{k=1}^{p-1}\frac{1}{k+1}\Big).$$

From (5.2) we have

$$\langle v_p, v_q\rangle > \frac{1}{pq}(q-\gamma-\psi(q)) \quad \text{or} \quad \langle v_p, v_q\rangle > \frac{1}{pq}(p-\gamma-\psi(p))$$

and then $\langle v_p, v_q\rangle > \frac{1}{pq}\min\{q-\gamma-\psi(q), p-\gamma-\psi(p)\}$. Finally we obtain

$$\frac{1}{\pi pq}\min\{q-\gamma-\psi(q), p-\gamma-\psi(p)\} < \frac{1}{\pi}\langle v_p, v_q\rangle \leqslant \sqrt{\log p \log q/pq}.$$

In addition we have

$$pq\langle v_p, v_q\rangle = G(p,q) = \sum_{r=1}^{pq-1}\Big(\psi\Big(\frac{r+1}{pq}\Big)-\psi\Big(\frac{r}{pq}\Big)\Big)\Big\{\frac{r}{p}\Big\}\Big\{\frac{r}{q}\Big\}$$

and for $x>0$ and $0<y<1$, from Sulaiman [**27**, Theorem 2.2] we have the inequality $\psi(x+y)-\psi(x)\geqslant\psi(y)$. Taking $x=\frac{r}{pq}$ and $y=\frac{1}{pq}$, from the above inequalities we obtain

$$G(p,q)\geqslant \psi\Big(\frac{1}{pq}\Big)\sum_{r=1}^{pq-1}\Big\{\frac{r}{p}\Big\}\Big\{\frac{r}{q}\Big\}\geqslant\frac{(p-1)(q-1)}{4}\psi\Big(\frac{1}{pq}\Big).$$

Furthermore

$$\begin{aligned}\frac{1}{pq}\max\Big\{\frac{(p-1)(q-1)}{4}\psi\Big(\frac{1}{pq}\Big), \min\{q-\gamma-\psi(q), p-\gamma-\psi(p)\}\Big\}&\\ <\langle v_p, v_q\rangle \leqslant \sqrt{\log p\log q/pq}.&\end{aligned}$$

Then we have

$$\begin{aligned}\frac{2}{\pi}\max\Big\{\frac{(p-1)(q-1)}{4}\psi\Big(\frac{1}{pq}\Big), \min\{q-\gamma-\psi(q), p-\gamma-\psi(p)\}\Big\}&\\ <\frac{2}{\pi}G(p,q)\leqslant\frac{2}{\pi}\sqrt{pq\log p\log q}.&\end{aligned}$$

From (1.8) we obtain

$$\frac{2}{\pi}G(p,q)+\frac{2}{\pi}=\frac{1}{\pi}\log p^{q-1}q^{p-1}-(V(p,q)+V(q,p))-pg\Big(\frac{1}{p}\Big)-qg\Big(\frac{1}{q}\Big).$$

This implies relation (6.2). $\square$

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