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ON THE LOCATION OF THE ZEROS OF CERTAIN POLYNOMIALS

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Abstract. We extend Aziz and Mohammad's result that the zeros, of a polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$, $ta_j \geqslant a_{j-1} > 0$, $j=2,3,\dots,n$ for certain $t\ (>0)$, with moduli greater than $t(n-1)/n$ are simple, to polynomials with complex coefficients. Then we improve their result that the polynomial $P(z)$, of degree n, with complex coefficients, does not vanish in the disc

$$|z-ae^{i\alpha}| < a/(2n);\ a>0,\ \max_{|z|=a}|P(z)| = |P(ae^{i\alpha})|,$$

for $r<a<2, r$ being the greatest positive root of the equation

$$x^n - 2x^{n-1} + 1 = 0,$$

and finally obtained an upper bound, for moduli of all zeros of a polynomial, (better, in many cases, than those obtainable from many other known results).

1. Introduction and statement of results

While thinking about zeros of a polynomial, Aziz and Mohammad [**1**] proved the following result, thereby suggesting that certain zeros of a polynomial with non-negative increasing coefficients may be simple.

Theorem A. *Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that $a_n \geqslant a_{n-1} \geqslant \dots \geqslant a_1 \geqslant a_0 > 0$. Then all the zeros of $P(z)$, of modulus $\geqslant n/(n+1)$, are simple.*

In [**2**] Aziz and Mohammad obtained, a generalization as well as a refinement, of Theorem A.

Theorem B. *Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n>1$, such that for some $t>0$ one has $ta_j \geqslant a_{j-1} > 0$, $j=2,3,\dots,n$, a_0 may be a real or a complex number. Then all the zeros of $P(z)$, of modulus greater than $t(n-1)/n$, are simple.*

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In this paper, we extend Theorem B to polynomials with complex coefficients. More precisely we prove

THEOREM 1.1. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n(>1)$ such that for certain $t > 0$*

$$t|a_j| \geqslant |a_{j-1}|, \quad j = 2, 3, \ldots, n \tag{1.1}$$

and for certain real α and β: $|\operatorname{Arg} a_j - \beta| \leqslant \alpha \leqslant \pi/2$, $j = 1, 2, 3, \ldots, n$, then $p(z)$ can not have a zero of order ($\geqslant 2$), with modulus greater than

$$\frac{t(n-1)}{n}\left\{\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{n|a_n|}\sum_{j=0}^{n-2}\left(\frac{n}{t(n-1)}\right)^{n-1-j}(j+1)|a_{j+1}|\right\}.$$

In other words all the zeros of $p(z)$, with moduli greater than

$$\frac{t(n-1)}{n}\left\{\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{n|a_n|}\sum_{j=0}^{n-2}\left(\frac{n}{t(n-1)}\right)^{n-1-j}(j+1)|a_{j+1}|\right\},$$

are simple.

THEOREM 1.2. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n(>1)$, with $\operatorname{Re} a_j = \alpha_j$, $\operatorname{Im} a_j = \beta_j$, $j = 0, 1, \ldots, n$. If for certain $t > 0$*

$$t\alpha_j \geqslant \alpha_{j-1} \geqslant 0, \quad j = 2, 3, \ldots, n \quad \text{and} \quad \alpha_n > 0, \tag{1.2}$$

then $p(z)$ can not have a zero of order ($\geqslant 2$), with modulus greater than

$$\frac{t(n-1)}{n}\left\{1 + \frac{2}{n\alpha_n}\sum_{j=0}^{n-1}\left(\frac{n}{t(n-1)}\right)^{n-j}(j+1)|\beta_{j+1}|\right\}. \tag{1.3}$$

In other words all the zeros of $p(z)$, with moduli greater than (1.3) *are simple.*

Next we obtain the following result for polynomials with complex coefficients, similar to Theorem B, but not an extension of Theorem B to polynomials with complex coefficients.

THEOREM 1.3. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n(>1)$ such that for certain $t > 0$: $t|a_j| \geqslant |a_{j-1}|$, $j = 2, 3, \ldots, n$. If k is the greatest positive root of the trinomial equation $x^n - 2x^{n-1} + 1 = 0$, then $p(z)$ can not have a zero of order ($\geqslant 2$), with modulus greater than $kt(n-1)/n$. In other words all the zeros of $p(z)$, with moduli greater than $kt(n-1)/n$, are simple.*

Further in [**2**], Aziz and Mohammad also obtained a zero free region for polynomials with complex coefficients.

THEOREM C. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial with complex coefficients. Then for no real $a > 0$, $P(z)$ vanishes in the disk $|z - ae^{i\alpha}| < a/(2n)$, where*

$$\operatorname*{Max}_{|z|=a}|P(z)| = |P(ae^{i\alpha})|.$$

We obtain a refinement of Theorem C, for $r < a < 2$, r being the greatest positive root of the equation $x^n - 2x^{n-1} + 1 = 0$.

Theorem 1.4. *Let $p(z)$ be a polynomial of degree n $(\geqslant 2)$ and K, the greatest positive root of the equation $x^{n+1} - 2x^n + 1 = 0$. If*

$$\max_{|z|=a} |p(z)| = |p(ae^{i\alpha_a})|, \quad a > 0, \tag{1.4}$$

$$R_a = \begin{cases} \frac{a^n(a-1)}{n(a^n-1)}, & 0 < a < K,\ a \neq 1, \\ 1/n^2, & a = 1, \\ 1/n, & a \geqslant K, \end{cases} \tag{1.5}$$

then for no $a > 0$, $p(z)$ vanishes in the disc $|z - ae^{i\alpha_a}| < R_a$.

Finally we obtain an upper bound for moduli of all zeros of a polynomial (better in many cases than those obtainable from many other known results).

Theorem 1.5. *Let $p(z) = a_0 z^n + \sum_{k=m}^{n} a_k z^{n-k}$ be a polynomial, with*

$$r = \max_{m \leqslant k \leqslant n-1} \left| \frac{a_{k+1}}{a_k} \right| \tag{1.6}$$

and let ξ be unique positive root of the equation

$$x^m - r x^{m-1} - \frac{|a_m|}{|a_0|} = 0. \tag{1.7}$$

Then all the zeros of $p(z)$ lie in $|z| < \xi$.

By taking $m = 1$ in Theorem 1.5 we get

Corollary 1.1. *Let $p(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n$ be a polynomial, with*

$$R = \max \left(\left| \frac{a_2}{a_1} \right|, \left| \frac{a_3}{a_2} \right|, \ldots, \left| \frac{a_n}{a_{n-1}} \right| \right).$$

Then all the zeros of $p(z)$ lie in $|z| < R + |a_1/a_0|$.

Remark 1.1. In many cases Corollary 1.1 gives better upper bounds than those obtainable by other known results. For the polynomial

$$p_1(z) = a_0 z^5 + a_1 z^4 + a_2 z^3 + a_3 z^2 + a_4 z + a_5,$$

with $|a_0| = 1$, $|a_1| = 2$, $|a_2| = 3$, $|a_3| = 4$, $|a_4| = 5$, $|a_5| = 6$, all the zeros lie in

(i) $|z| < 7$, by Cauchy [**3**], [**10**, Thoerem (27,2)], [**11, 12**],
(ii) $|z| \leqslant 4$, by Kojima [**8, 9**],
(iii) $|z| \leqslant R$, $R > 3.6$, by Govil and Rahman [**6**, Theorem 1],
(iv) $|z| \leqslant R$, $6.9 < R < 7$, by Dehmer [**4**, Theorem 3.2],
(v) $|z| \leqslant R$, $R > 3.8$, by Jain [**7**, Corollary 1],
(vi) $|z| \leqslant R$, $3.9 < R < 4$, by Dehmer and Mowshowitz [**5**, Theorem 2],
(vii) $|z| \leqslant 4$, by Dehmer and Mowshowitz [**5**, Theorem 4],
(viii) $|z| < 3.5$, by Corollary 1.1.

2. Lemmas

For the proofs of the theorems we require the following lemmas.

LEMMA 2.1. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial with complex coefficients such that for certain real* β *and* α *one has* $|\operatorname{Arg} a_j - \beta| \leqslant \alpha \leqslant \pi/2$, $j = 0, 1, \dots, n$ *and* $|a_n| \geqslant |a_{n-1}| \geqslant \dots \geqslant |a_0|$, *then* $p(z)$ *has all its zeros on or inside the circle*

$$|z| = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{j=0}^{n-1}|a_j|.$$

This lemma is due to Govil and Rahman [**6**].

LEMMA 2.2. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial such that for certain* $a > 0$

$$|a_n| \geqslant a|a_{n-1}| \geqslant a^2|a_{n-2}| \geqslant \dots \geqslant a^{n-1}|a_1| \geqslant a^n|a_0| \tag{2.1}$$

and for certain real β *and* α

$$|\operatorname{Arg} a_j - \beta| \leqslant \alpha \leqslant \pi/2, \quad j = 0, 1, \dots, n \tag{2.2}$$

then $p(z)$ *has all its zeros in*

$$|z| \leqslant \frac{1}{a}\left(\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{j=0}^{n-1}a^{n-j}|a_j|\right).$$

PROOF OF LEMMA 2.2. Using (2.1) and (2.2) we can say that polynomial $p(z/a)$ satisfies the hypotheses of Lemma 2.1 and therefore has all its zeros in

$$|z| \leqslant \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{j=0}^{n-1}a^{n-j}|a_j|.$$

Accordingly the polynomial $p(z) \equiv p\left(\frac{az}{a}\right)$ will have all its zeros in

$$|z| \leqslant \frac{1}{a}\left\{\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{j=0}^{n-1}a^{n-j}|a_j|\right\}. \qquad \square$$

LEMMA 2.3. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial. If* $\operatorname{Re} a_j = \alpha_j$, $\operatorname{Im} a_j = \beta_j$, $j = 0, 1, \dots, n$ *and* $\alpha_n \geqslant \alpha_{n-1} \geqslant \dots \geqslant \alpha_1 \geqslant \alpha_0 \geqslant 0$, $\alpha_n > 0$, *then* $p(z)$ *has all its zeros in*

$$|z| \leqslant 1 + \frac{2}{\alpha_n}\sum_{j=0}^{n}|\beta_j|.$$

This lemma is due to Govil and Rahman [**6**].

LEMMA 2.4. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n (> 1), *with*

$$\operatorname{Re} a_j = \alpha_j, \qquad \operatorname{Im} a_j = \beta_j, \quad j = 0, 1, \dots, n. \tag{2.3}$$

If for certain $a > 0$

$$\alpha_n \geqslant a\alpha_{n-1} \geqslant a^2\alpha_{n-2} \geqslant \dots \geqslant a^{n-1}\alpha_1 \geqslant a^n\alpha_0 \geqslant 0, \quad \alpha_n > 0 \tag{2.4}$$

then $p(z)$ *has all its zeros in*

$$|z| \leqslant \frac{1}{a}\left\{1+\frac{2}{\alpha_n}\sum_{j=0}^{n} a^{n-j}|\beta_j|\right\}.$$

Proof of Lemma 2.4. Using (2.4) and (2.3) we can say that the polynomial $p(z/a)$ satisfies the hypotheses of Lemma 2.3 and therefore has all its zeros in

$$|z| \leqslant 1+\frac{2}{\alpha_n}\sum_{j=0}^{n} a^{n-j}|\beta_j|.$$

Accordingly the polynomial $p(z) \equiv p(az/a)$ will have all its zeros in

$$|z| \leqslant \frac{1}{a}\left\{1+\frac{2}{\alpha_n}\sum_{j=0}^{n} a^{n-j}|\beta_j|\right\}. \qquad \square$$

Lemma 2.5. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial with complex coefficients such that for certain* $a > 0$

$$|a_n| \geqslant a|a_{n-1}| \geqslant a^2|a_{n-2}| \geqslant \cdots \geqslant a^{n-1}|a_1| \geqslant a^n|a_0|.$$

Then $p(z)$ *has all its zeros in* $|z| \leqslant K_1/a$, *where* K_1 *is the greatest positive root of the trinomial equation* $K^{n+1} - 2K^n + 1 = 0$.

This lemma is due to Govil and Rahman [**6**].

Lemma 2.6. *Let* $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, *be a polynomial with complex coefficients. Then for every positive real number* r, *all the zeros of* $P(z)$ *lie in the disc*

$$|z| \leqslant \max\left\{r, \sum_{j=0}^{n-1} |a_j/a_n| \frac{1}{r^{n-j-1}}\right\}.$$

This lemma is due to Aziz and Mohammad [**2**].

Lemma 2.7. *Let* $p(z)$ *be a polynomial of degree* n. *Then*

$$\max_{|z|=1} |p'(z)| \leqslant n \max_{|z|=1} |p(z)|.$$

This lemma is due to Bernstein [**13**].

On applying Lemma 2.7 to the polynomial $p(rz)$, we get

Lemma 2.8. *Let* $p(z)$ *be a polynomial of degree* n $(\geqslant 1)$. *Then for every positive* r *we have* $\max_{|z|=r} |p'(z)| \leqslant (n/r) \max_{|z|=r} |p(z)|$.

By repeated application of Lemma 2.8, we get

Lemma 2.9. *Let* $p(z)$ *be a polynomial of degree* $n(\geqslant 1)$. *Then for every positive* r *we have*

$$\max_{|z|=r} |p^k(z)| \leqslant \frac{n(n-1)\dots(n-k+1)}{r^k} \max_{|z|=r} |p(z)|, \quad k = 1, 2, \dots, n.$$

3. Proofs of the theorems

PROOF OF THEOREM 1.1. We have

$$p'(z)=\sum_{j=0}^{n-1}(j+1)a_{j+1}z^j=\sum_{j=0}^{n-1}b_jz^j,\ \text{say .}\tag{3.1}$$

Now

$$t|a_{j+1}|\geqslant |a_j|,\quad j=1,2,\dots,n-1,\ (\text{by (1.1)}).\tag{3.2}$$

Further $\frac{n-1}{n}\geqslant\frac{j}{j+1}$, $j=1,2,\dots,n-1$, which, by (3.2), makes it possible to write $t\frac{n-1}{n}(j+1)|a_{j+1}|\geqslant j|a_j|$, $j=1,2,\dots,n-1$, i.e., $|b_j|\geqslant\frac{n}{t(n-1)}|b_{j-1}|$, $j=1,2,\dots,n-1$, (by (3.1)). We can now apply Lemma 2.2 to the polynomial $p'(z)$, of degree $(n-1)$ and say that all the zeros of $p'(z)$ lie in

$$|z|\leqslant\frac{t(n-1)}{n}\Bigg\{\cos\alpha+\sin\alpha+\frac{2\sin\alpha}{n|a_n|}\sum_{j=0}^{n-2}\Big(\frac{n}{t(n-1)}\Big)^{n-1-j}(j+1)|a_{j+1}|\Bigg\}.$$

Therefore all the zeros of $p(z)$, with moduli greater than

$$\frac{t(n-1)}{n}\Bigg\{\cos\alpha+\sin\alpha+\frac{2\sin\alpha}{n|a_n|}\sum_{j=0}^{n-2}\Big(\frac{n}{t(n-1)}\Big)^{n-1-j}(j+1)|a_{j+1}|\Bigg\},$$

are simple. $\square$

PROOF OF THEOREM 1.2. We have

$$p'(z)=\sum_{j=0}^{n-1}(j+1)a_{j+1}z^j=\sum_{j=0}^{n-1}b_jz^j,\ \text{say.}\tag{3.3}$$

Now

$$t\alpha_{j+1}\geqslant\alpha_j\geqslant 0,\quad j=1,2,\dots,n-1,\ (\text{by (1.2)}).\tag{3.4}$$

Further $\frac{n-1}{n}\geqslant\frac{j}{j+1}$, $j=1,2,\dots,n-1$, which, by (3.4), makes it possible us write $\frac{t(n-1)}{n}(j+1)\alpha_{j+1}\geqslant j\alpha_j\geqslant 0$, $j=1,2,\dots,n-1$, i.e.,

$$\operatorname{Re}b_j\geqslant\frac{n}{t(n-1)}\operatorname{Re}b_{j-1}\geqslant 0,\quad j=1,2,\dots,n-1,\ (\text{by (3.3)}).\tag{3.5}$$

As $\operatorname{Re}b_{n-1}>0$, by (3.3) and (1.2), we can say by using (3.5) that the polynomial $p'(z)$ satisfies the hypotheses of Lemma 2.4 and therefore has all its zeros in

$$|z|\leqslant\frac{t(n-1)}{n}\Bigg\{1+\frac{2}{n\alpha_n}\sum_{j=0}^{n-1}\Big(\frac{n}{t(n-1)}\Big)^{n-j}(j+1)|\beta_{j+1}|\Bigg\}.$$

Accordingly all the zeros of $p(z)$, with moduli greater than

$$\frac{t(n-1)}{n}\Bigg\{1+\frac{2}{n\alpha_n}\sum_{j=0}^{n-1}\Big(\frac{n}{t(n-1)}\Big)^{n-j}(j+1)|\beta_{j+1}|\Bigg\}$$

are simple. $\square$

Proof of Theorem 1.3. It is similar to the proof of Theorem 1.1 with one change: Lemma 2.5 instead of Lemma 2.2. □

Proof of Theorem 1.4. With $w = ae^{i\alpha_a}$, let us consider the polynomial

$$\begin{aligned}F(z) &= p(wz/n + w)\\ &= p(w) + z(w/n)p'(w) + \frac{z^2}{2!}(w/n)^2 p''(w) + \cdots + \frac{z^n}{n!}(w/n)^n p^{(n)}(w).\end{aligned} \tag{3.6}$$

Then the polynomial

$$G(z) = z^n F(1/z) = \sum_{k=0}^{n} (w/n)^k \frac{p^{(k)}(w)}{k!} z^{n-k}, \tag{3.7}$$

will have all its zeros, by using Lemma 2.6 with $r = a$, in the disc

$$|z| \leqslant \max\left\{a, \sum_{k=0}^{n-1} |w/n|^{n-k} \frac{|p^{(n-k)}(w)|}{(n-k)!|p(w)|a^{n-k-1}}\right\},$$

and as

$$|p(w)| \geqslant \frac{a^k}{n(n-1)\dots(n-k+1)} |p^{(k)}(w)|, \geqslant \frac{|w|^k |p^{(k)}(w)|}{n^k k!}, \quad k = 1, 2, \dots, n,$$

by (1.4) and Lemma 2.9, we can say that $G(z)$ has all its zeros in the disc

$$|z| \leqslant \max\left\{a, \sum_{k=0}^{n-1} \frac{1}{a^{n-k-1}}\right\} = \frac{a}{nR_a}, \text{ (by (1.5))}.$$

Therefore by (3.7), we can say that $F(z)$ has all its zeros in $|z| \geqslant \frac{nR_a}{a}$, i.e., $F(z)$ does not vanish in $|z| < \frac{nR_a}{a}$ and accordingly, by using (3.6), we can say that $p(z)$ does not vanish in the disc $|z - w| < R_a$. □

Proof of Theorem 1.5. Using (1.6) we get

$$|a_k| \leqslant r^{k-m} |a_m|, \quad m \leqslant k \leqslant n. \tag{3.8}$$

Now for $|z| > r$, we have

$$\begin{aligned}|p(z)| &\geqslant |a_0||z|^n \left\{1 - \frac{1}{|a_0|} \sum_{k=m}^{n} \frac{|a_k|}{|z|^k}\right\},\\ &\geqslant |a_0||z|^n \left\{1 - \frac{|a_m|r^{-m}}{|a_0|} \sum_{k=m}^{n} \left(\frac{r}{|z|}\right)^k\right\}, \text{ (by (3.8))},\\ &> |a_0||z|^n \left\{1 - \frac{|a_m|r^{-m}}{|a_0|} \sum_{k=m}^{\infty} \left(\frac{r}{|z|}\right)^k\right\},\\ &= |a_0||z|^n \left\{\frac{|z|^m - r|z|^{m-1} - (|a_m|/|a_0|)}{(|z|^m - r|z|^{m-1})}\right\}.\end{aligned} \tag{3.9}$$

Further if $g(x) = x^m - rx^{m-1} - \frac{|a_m|}{|a_0|}$, then

$$g(r) = -\frac{|a_m|}{|a_0|} < 0, \tag{3.10}$$

and as ξ is unique positive root of the equation (1.7), we can say that

$$x^m - rx^{m-1} - \frac{|a_m|}{|a_0|} \geqslant 0 \quad \text{for} \quad x \geqslant \xi, \tag{3.11}$$

with

$$\xi > r, \ \text{(by (3.10))}. \tag{3.12}$$

Now by (3.9), (3.11) and (3.12) we can say that $|p(z)| > 0$ for $|z| \geqslant \xi$. □

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