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# ON SOME CLASS OF INTEGRAL OPERATORS RELATED TO THE BERGMAN PROJECTION

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Abstract. We consider the integral operator

$$C_\alpha f(z) = \int_D \frac{f(\xi)}{(1-z\bar{\xi})^\alpha}\, dA(\xi), \quad z \in D,$$

where $0 < \alpha < 2$ and $D$ is the unit disc in the complex plane. and investigate boundedness of it on the space $L^p(D, d\lambda)$, $1 < p < \infty$, where $d\lambda$ is the Möbius invariant measure in $D$. We also consider the spectral properties of $C_\alpha$ when it acts on the Hilbert space $L^2(D, d\lambda)$, i.e., in the case $p = 2$, when $C_\alpha$ maps $L^2(D, d\lambda)$ into the Dirichlet space.

## 1. Introduction and notation

Throughout the paper let $D = \{z : |z| < 1\}$ be the open unit disc in complex plane $\mathbb{C}$ and let $dA(z) = \frac{1}{\pi}\, dx\, dy$, $z = x + iy$ stands for the normalized area measure in $\mathbb{C}$. For $1 < p < \infty$ we consider the Besov space $B_p$ of $D$, $1 < p < \infty$, which is defined to be the space of all analytic functions $f$ in $D$ such that

$$\|f\|_{B_p} = \Big( \int_D |f'(z)|^p (1-|z|^2)^p d\lambda(z) \Big)^{1/p} < \infty,$$

where $d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}$ is the Möbius invariant measure on $D$. It is known that $\|\cdot\|_{B_p}$ is complete seminorm on $B_p$. It should be pointed that $B_p$ is a Banach space with norm $\|f\| = |f(0)| + \|f\|_{B_p}$. For $p = 2$ the space $B_2$ is the classical Dirichlet space, and appropriate semi-inner product is given by the formula

$$\langle f, g \rangle = \int_D f'(z)\, \overline{g'(z)}\, dA(z), \quad f, g \in B_2. \tag{1.1}$$

The weighted Bergman projection $P_s$, $-1 < s < \infty$ represents a central operator which appears in the research concerning the analytic function spaces. It is given by

$$P_s f(z) = (s+1) \int_D \frac{(1-|\omega|^2)^s}{(1-z\bar{\omega})^{2+s}} f(\omega)\, dA(\omega), \quad z \in D.$$

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Particulary, the ordinary Bergman projection $P = P_0$ arises as the orthogonal projection from $L^2(D; dA)$ onto an analytic function subspace. It connects $B_p$ and $L^p(D, d\lambda)$. This relation is expressed in the next theorem.

THEOREM 1.1. *Suppose $f \in H(D)$ and $1 \leqslant p \leqslant \infty$. Then*

$$f \in B_p \Leftrightarrow f \in P(L^p(D, d\lambda)).$$

The inclusion operator $V$ from $B_p$ into $L^p(D, d\lambda)$ is given by

$$Vf(z) = 3(1 - |z|^2)^2 \int_D \frac{f(\xi)\, dA(\xi)}{(1 - z\bar{\xi})^4}, \quad z \in D.$$

More precisely we have the following lemma (see [**8**]).

LEMMA 1.1. *The operator $V$ is an embedding from $B_p$ into $L^p(D, d\lambda)$ for all $1 < p < \infty$ if on $B_p = P(L^p(D, d\lambda))$ is given the quotient norm.*

In this paper we consider the class of the operators

$$C_\alpha f(z) = \int_D \frac{f(\xi)}{(1 - z\bar{\xi})^\alpha}\, d\xi, \quad z \in D,$$

where $0 < \alpha < 2$. For $\alpha = 2$ we have the Bergman projection. The norm of the Bergman projection from $L^p(D, d\lambda)$ onto $B_p$ was estimated in [**7**]. In Theorem 1.3 we prove that $C_\alpha$ is a bounded mapping from $L^p(D, d\lambda)$ into $B_p$ for all $0 < \alpha < 2$ and $1 < p < \infty$. We investigate in the next section some of its spectral properties in the context of the Lebesgue space $L^2(D, d\lambda)$ and the Besov space $B_2$.

By boundedness of an operator $T : L^p(D, d\lambda) \to B_p$ we mean that there exists a constant $C > 0$ such that $\|Tf\|_{B_p} \leqslant C\|f\|_{L^p(D,d\lambda)}$.

In this section we observe boundedness of $C_\alpha$ defined on $L^p(D, d\lambda)$. We firstly state a technical lemma and a proposition (the Schur test).

LEMMA 1.2. *Suppose $z \in D$, $c$ is real, $t > -1$, and*

$$I_{c,t}(z) = \int_D \frac{(1 - |\omega|^2)^t}{|1 - z\bar{\omega}|^{2+t+c}}\, dA(\omega).$$

*Then we have*

(a) *If $c < 0$, then $I_{c,t}(z)$ is bounded in $z$.*
(b) *If $c > 0$, then $I_{c,t}(z) \sim \frac{1}{(1-|z|^2)^c}$, $|z| \to 1^-$.*
(c) *If $c = 0$, then $I_{0,t}(z) \sim \log \frac{1}{1-|z|^2}$, $|z| \to 1^-$.*

PROPOSITION 1.1. *Suppose $K$ is a nonnegative measurable function on $X \times X$, where $(X, \mu)$ is a measure space. Let $T$ be an integral operator induced by $K$, that is $Tf(x) = \int_X K(x, y)\, f(y)\, d\mu(y)$, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If there exist constants $C_1, C_2 > 0$ and a positive measurable function $h$ on $X$ such that*

$$\int_X K(x, y)\, h(y)^q d\mu(y) \leqslant C_1 h(x)^q \quad \text{for } \mu\text{-almost every } x \in X,$$

$$\int_X K(x, y)\, h(x)^p d\mu(x) \leqslant C_2 h(y)^p \quad \text{for } \mu\text{-almost every } y \in X,$$

*then $T$ is bounded on $L^p(X, d\mu)$ with the norm less than or equal to $C_1^{1/q} C_2^{1/p}$.*

In the proof of Theorem 1.3 we will use the Gauss hypergeometric functions and its basic properties. Following [**1**] we recall some facts for the sake of easy reference.

The Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n \quad \text{for } |z|<1, $$

and by continuation elsewhere. Here $(a)_n = a(a+1)\cdots(a+n-1)$ denotes the shifted factorial, where $a$ is any complex number.

The identity

$$ {}_2F_1(a,b;c;z) = (1-z^2)^{c-a-b}\, {}_2F_1(c-a,c-b;c;z) \tag{1.2} $$

is known as Euler identity. The following properties of hypergeometric function are also going to be of interest

$$ \frac{\partial}{\partial x} {}_2F_1(a,b;c;z) = \frac{ab}{c}\, {}_2F_1(a+1,b+1;c+1;z), \tag{1.3} $$

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{for } \operatorname{Re}(c-a-b)>0. $$

It is known that ${}_2F_1(a,b;c;z)$ diverges in general for $z=1$ if $\operatorname{Re}(c-a-b) \leqslant 0$. The next theorem, due to Gauss, describes the asymptotic behaviour of the hypergeometric functions as $z \to 1^-$.

THEOREM 1.2. *If* $\operatorname{Re}(c-a-b)<0$, *then*

$$ \lim_{x\to 1^-} \frac{{}_2F_1(a,b;c;z)}{(1-z)^{c-a-b}} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}. $$

*For* $c=a+b$ *we have*

$$ \lim_{x\to 1^-} \frac{{}_2F_1(a,b;c;z)}{\log\left(\frac{1}{1-z}\right)} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}. $$

THEOREM 1.3. *For* $0<\alpha<2$, $C_\alpha$ *is a bounded mapping from* $L^p(D,d\lambda)$ *into* $B_p$ $(1<p<\infty)$. *The norm may be estimated by* $\|C_\alpha\|_{L^p(D,d\lambda)\to B_p} < \alpha C_1^{1/q} C_2^{1/p}$. *Here*

$$ C_1 = \begin{cases} \frac{p\Gamma(2+\frac{1}{p})\Gamma(\alpha-1-\frac{1}{p})}{(p+1)\Gamma^2(\frac{\alpha+1}{2})}, & \alpha > 1+\frac{1}{p}; \\ \frac{pq\Gamma(1+\frac{1}{p})}{e\pi(p+1)}, & \alpha = 1+\frac{1}{p}; \\ \frac{p\Gamma(2+\frac{1}{p})\Gamma(1+\frac{1}{p}-\alpha)}{(p+1)\Gamma^2(\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2})}, & \alpha < 1+\frac{1}{p}, \end{cases} \quad \textit{and} \quad C_2 = \begin{cases} \frac{q\Gamma(1+\frac{1}{q})\Gamma(\alpha-\frac{1}{q})}{\Gamma^2(\frac{\alpha+1}{2})}, & \alpha > \frac{1}{q}; \\ \frac{pq\Gamma(1+\frac{1}{q})}{\pi(p+1)e}, & \alpha = \frac{1}{q}; \\ \frac{q\Gamma(1+\frac{1}{q})\Gamma(\frac{1}{q}-\alpha)}{\Gamma^2(\frac{1}{2}+\frac{1}{q}-\frac{\alpha}{2})}, & \alpha < \frac{1}{q}, \end{cases} $$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

PROOF. First of all, it is easy to see that $Cf$ is an analytic function for every $f \in L^p(D,d\lambda)$. Further, for $f \in L^p(D,d\lambda)$ we have

$$ (Cf)'(z) = \alpha \int_D \frac{\bar{\xi}}{(1-z\bar{\xi})^{\alpha+1}} f(\xi)\, dA(\xi), $$

$$\|Cf\|_{B_p}^p = \alpha^p \int_D \left|(1-|z|^2)^{1-2/p} \int_D \frac{\bar{\xi}}{(1-z\bar{\xi})^{\alpha+1}}\, f(\xi)\, dA(\xi)\right|^p dA(z).$$

Thus

$$\begin{aligned}\|Cf\|_{B_p}^p &= \alpha^p \int_D \left|(1-|z|^2)^{1-2/p} \int_D \frac{\bar{\xi}}{(1-z\bar{\xi})^{\alpha+1}}\, f(\xi)\, dA(\xi)\right|^p dA(z)\\ &= \alpha^p \int_D \left|(1-|z|^2) \int_D \frac{\bar{\xi}(1-|\xi|^2)^2}{(1-z\bar{\xi})^{\alpha+1}}\, f(\xi)\, d\lambda(\xi)\right|^p d\lambda(z)\end{aligned}$$

Therefore, we should consider the operator

$$Tf(z) = (1-|z|^2) \int_D \frac{f(\xi)(1-|\xi|^2)^2}{|1-z\bar{\xi}|^{\alpha+1}}\, d\lambda(\xi)$$

on $L^p(D, d\lambda)$. We will prove that it is bounded there and we will estimate its norm. Using the obvious relation $\|C_\alpha\|_{L^p(D,d\lambda)\to B_p} \leqslant \alpha \|T\|_{L^p(D,d\lambda)\to L^p(D,d\lambda)}$ we can estimate the norm of $C_\alpha$.

We use Proposition 1.1 and test function $h(z) = (1-|z|^2)^{1/pq}$ for the kernel $K(z,\xi) = \frac{(1-|z|^2)(1-|\xi|^2)^2}{|1-z\xi|^{\alpha+1}}$. We have to prove existence of the constants $C_1, C_2$ such that

$$\begin{aligned}\int_D K(z,\xi)\, h^q(\xi)\, d\lambda(\xi) &\leqslant C_1 h^q(z), \quad z \in D,\\ \int_D K(z,\xi)\, h^p(z)\, d\lambda(z) &\leqslant C_2 h^p(\xi), \quad \xi \in D,\end{aligned}$$

which is equivalent with

$$\begin{aligned}(1-|z|^2)^{1/q} \int_D \frac{(1-|\xi|^2)^{1/p}}{|1-z\bar{\xi}|^{\alpha+1}}\, dA(\xi) &\leqslant C_1, \quad z \in D,\\ (1-|\xi|^2)^{1+1/p} \int_D \frac{(1-|z|^2)^{1/q-1}}{|1-z\bar{\xi}|^{\alpha+1}}\, dA(z) &\leqslant C_2, \quad \xi \in D.\end{aligned} \tag{1.4}$$

From Lemma 1.2 we can easily check that both functions on the left-hand side in (1.4) are bounded and consequently relations (1.4) are true for some constants $C_1$ and $C_2$. In the sequel we will determine the upper bounds for the constants $C_1$ and $C_2$.

By using the uniform convergence and orthogonality we have

$$\begin{aligned}(1-|z|^2)^{1/q} \int_D \frac{(1-|\xi|^2)^{1/p}}{|1-z\bar{\xi}|^{\alpha+1}} dA(\xi) &= (1-|z|^2)^{1/q} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{p}+1)\Gamma^2(\frac{\alpha+1}{2}+n)}{\Gamma(n+\frac{1}{p}+2)\Gamma^2(\frac{\alpha+1}{2})n!} |z|^{2n}\\ &= \frac{p}{1+p}(1-|z|^2)^{1/q}\, {}_2F_1\Big(\frac{\alpha+1}{2}, \frac{\alpha+1}{2}; 2+\frac{1}{p}, |z|^2\Big).\end{aligned}$$

In a similar way we obtain that

$$(1-|\xi|^2)^{1+1/p} \int_D \frac{(1-|z|^2)^{1/q-1}}{|1-z\bar{\xi}|^{\alpha+1}} dA(z)$$

$$= q(1-|\xi|^2)^{1+1/p}{}_2F_1\Big(\frac{\alpha+1}{2},\frac{\alpha+1}{2};1+\frac{1}{q},|\xi|^2\Big).$$

Let us denote

$$C_1 = \frac{p}{1+p}\sup_{|z|<1}(1-|z|^2)^{1/q}{}_2F_1\Big(\frac{\alpha+1}{2},\frac{\alpha+1}{2};2+\frac{1}{p},|z|^2\Big),$$
$$C_2 = q\sup_{|\xi|<1}(1-|\xi|^2)^{1+1/p}{}_2F_1\Big(\frac{\alpha+1}{2},\frac{\alpha+1}{2};1+\frac{1}{q},|\xi|^2\Big).$$

Then the Schur test implies $\|C_\alpha\|_{L^p(d\lambda)\to B_p} \leqslant \alpha C_1^{1/q}C_2^{1/p}$. By using the Euler transformation (1.2) for the hypergeometric functions, we obtain

$$C_1 = \frac{p}{1+p}\sup_{|z|<1}(1-|z|^2)^{2-\alpha}{}_2F_1\Big(\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2},\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2};2+\frac{1}{p},|z|^2\Big),$$
$$C_2 = q\sup_{|\xi|<1}(1-|\xi|^2)^{2-\alpha}{}_2F_1\Big(\frac{1}{2}+\frac{1}{q}-\frac{\alpha}{2},\frac{1}{2}+\frac{1}{q}-\frac{\alpha}{2};1+\frac{1}{q},|\xi|^2\Big).$$

Both functions

$${}_2F_1\Big(\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2},\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2};2+\frac{1}{p},|z|^2\Big),$$
$${}_2F_1\Big(\frac{1}{2}+\frac{1}{q}-\frac{\alpha}{2},\frac{1}{2}+\frac{1}{q}-\frac{\alpha}{2};1+\frac{1}{q},|\xi|^2\Big)$$

are increasing in $|z|$ and $|\xi|$, respectively (see (1.3)).

We distinguish the following five cases:

1) If $\alpha > 1+\frac{1}{p}$, then

$$C_1 < \frac{p}{p+1}{}_2F_1\Big(\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2},\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2};2+\frac{1}{p},1\Big) = \frac{p\Gamma(2+\frac{1}{p})\Gamma(\alpha-1-\frac{1}{p})}{(p+1)\Gamma^2(\frac{\alpha+1}{2})},$$
$$C_2 < q\,{}_2F_1\Big(\frac{1}{2}+\frac{1}{q}-\frac{\alpha}{2},\frac{1}{2}+\frac{1}{q}-\frac{\alpha}{2};1+\frac{1}{q},1\Big) = \frac{q\Gamma(1+\frac{1}{q})\Gamma(\alpha-\frac{1}{q})}{\Gamma^2(\frac{\alpha+1}{2})}.$$

2) If $\alpha < \frac{1}{q}$, then according to Theorem 1.2 we have

$$C_1 \leqslant \frac{p}{p+1}\limsup_{|z|\to1^-}(1-|z|^2)^{2-\alpha}\frac{{}_2F_1\big(\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2},\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2};2+\frac{1}{p},1\big)}{(1-|z|^2)^{\alpha-1-\frac{1}{p}}}$$
$$< \frac{p\Gamma(2+\frac{1}{p})\Gamma(1+\frac{1}{p}-\alpha)}{(p+1)\Gamma^2(\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2})},$$
$$C_2 \leqslant q\limsup_{|\xi|\to1^-}(1-|\xi|^2)^{2-\alpha}\frac{{}_2F_1\big(\frac{1}{2}+\frac{1}{q}-\frac{\alpha}{2},\frac{1}{2}+\frac{1}{q}-\frac{\alpha}{2};1+\frac{1}{q},1\big)}{(1-|\xi|^2)^{\alpha-1/q}}$$
$$< \frac{q\Gamma(1+\frac{1}{q})\Gamma(\frac{1}{q}-\alpha)}{\Gamma^2(\frac{1}{2}+\frac{1}{q}-\frac{\alpha}{2})}.$$

3) If $\alpha = 1 + \frac{1}{p}$, then

$$C_1 \leqslant \frac{p}{p+1} \limsup_{|z|\to 1^-} (1-|z|^2)^{1/q} \log \frac{1}{1-|z|^2} \frac{{}_2F_1\big(\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2}, \frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2}; 2+\frac{1}{p}, 1\big)}{\log \frac{1}{1-|z|^2}}$$
$$< \frac{pq}{e(p+1)} \frac{\Gamma(3+\frac{2}{p}-\alpha)}{\Gamma^2(\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2})} = \frac{pq\Gamma(1+\frac{1}{p})}{e\pi(p+1)}, \qquad C_2 < \frac{q\Gamma(1+\frac{1}{q})\Gamma(\frac{2}{p})}{\Gamma^2(\frac{\alpha+1}{2})},$$

since the maximal value of the function $\phi(x) = (1-x)^{1/q} \log \frac{1}{1-x}$, $x \in (0,1)$ is $\frac{q}{e}$.

4) If $\alpha = \frac{1}{q}$, then

$$C_1 < \frac{p\Gamma(2+\frac{1}{p})\Gamma(\frac{2}{p})}{(p+1)\Gamma^2(\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2})}, \quad C_2 < \frac{pq\Gamma(1+\frac{1}{q})}{\pi(p+1)e}.$$

5) If $\frac{1}{q} < \alpha < 1 + \frac{1}{p}$, then

$$C_1 < \frac{p\Gamma(2+\frac{1}{p})\Gamma(1+\frac{1}{p}-\alpha)}{(p+1)\Gamma^2(\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2})}, \quad C_2 < \frac{q\Gamma(1+\frac{1}{q})\Gamma(\alpha-\frac{1}{q})}{\Gamma^2(\frac{\alpha+1}{2})}. \qquad \square$$

## 2. Hilbert case and spectral properties

Folowing [**6**] let us recall some basic facts from spectral-operator theory. Let us firstly recall that for the bounded measurable function $A(z,\xi)$ the operator

$$Af(z) = \int_D \frac{A(x,\xi)f(\xi)}{|z-\xi|^\alpha}\, dA(\xi), \quad z \in D$$

is compact on $L^2(D, dA)$, where $0 < \alpha < 2$.

For a compact operator $T$ defined on a separable Hilbert space $H$, let $s_n(T)$, $n \geqslant 1$ denote the eigenvalues of the operator $(T^*T)^{1/2}$ arranged in nondecreasing order [**4**]. In general, if $T$ is a compact operator on a separable Hilbert space $H$, then there exist orthonormal sets $\{e_n\}$ and $\{\sigma_n\}$ in $H$ such that

$$Tx = \sum_n \lambda_n \langle x, e_n\rangle \sigma_n, \quad x \in H,$$

where $\lambda_n$ is $n$-th singular value of $T$.

For $0 < p < \infty$, we define the Schattene $p$-class of $H$ denoted by $S_p(H)$, or simply $S_p$, to be the space of all compact operators $T$ on $H$ with singular value sequence $\{\lambda_n\}$ belonging to $l^p$ ($p$-summable sequence space). The Schattene class $S_p$ is a Banach space for the range $1 \leqslant p < \infty$, and appropriate norm of the operator $T \in S_p$ is given by $\|T\|_p = \big(\sum_n |\lambda_n|^p\big)^{1/p}$.

THEOREM 2.1. *Let $A$ be a compact operator. Then $A$ has the norm convergent expansion*

$$A = \sum_{n=1}^{N} \mu_n(A) \langle \phi_n, \cdot\rangle \psi_n \tag{2.1}$$

*(where $N$ is a finite non-negative integer or infinity), each $\mu_n(A) > 0$, $\mu_1(A) \geqslant \mu_2(A) \geqslant \cdots$, and $(\phi_n)$ and $(\psi_n)$ are (not necessarily complete) orthonomal sets. Moreover, $\mu_n(A)$ are uniquely determined and $\phi$'s and $\psi$'s are essentially uniquely determined.*

Here $\mu_n(A)$ are singular values of $A$ and formula (2.1) is called canonical expansion for $A$.

Now we state a known result related to minimax properties of eigenvalues for compact nonnegative operators [**4**].

THEOREM 2.2. *Let $A$ $(\neq 0)$ be a nonnegative compact operator and let $\varphi_j$ $(j = 1, 2, \ldots)$ be an orthonomal system of its eigenvalues which is complete in the range of $A$, so that $A\varphi_j = \lambda_j(A)\varphi_j$, $j = 1, 2, \ldots$ where $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots$. Then its eigenvalues have the following minimax properties*

$$\lambda_1(A) = \max_{\varphi \in H} \frac{\langle A\varphi, \varphi\rangle}{\langle \varphi, \varphi\rangle} \tag{2.2}$$

*where the maximum in* (2.2) *is attained only for those eigenvalues of $A$ that correspond to $\lambda_1(A)$.*

$$\lambda_{j+1}(A) = \min_{L \in N_j} \max_{\varphi \in L^T} \frac{\langle A\varphi, \varphi\rangle}{\langle \varphi, \varphi\rangle}, \quad j = 1, 2, \ldots \tag{2.3}$$

*where $N_j$ is the set of all $j$-dimensional lineals of $H$, and the minimum in* (2.3) *is attained when $L$ coincides with the linear hull $L_j$ of the eigenvectors $\varphi_1, \varphi_2, \ldots, \varphi_j, \ldots$ so that*

$$\lambda_{j+1}(A) = \max_{\varphi \in L_j^T} \frac{\langle A\varphi, \varphi\rangle}{\langle \varphi, \varphi\rangle}.$$

We note that

$$s_1(A) = \|A\|. \tag{2.4}$$

Dostanić [**3**] investigated the singular values of the operator $S : L^2(D) \to L^2(D)$ defined by

$$Sf(z) = \frac{1}{\pi} \int_D \frac{\bar{\xi}}{1 - z\bar{\xi}} m(\xi)\, f(\xi)\, dA(\xi),$$

where $m \in C(\bar{D})$. He obtained that $s_n(S) \sim \frac{1}{2n\pi} \int_0^{2\pi} |m(e^{i\theta})|\, d\theta$.

The next theorem is our second main result and is related to finding singular numbers of the operator $VC_\alpha$.

THEOREM 2.3. *The operator $VC_\alpha : L^2(D, d\lambda) \to L^2(D, d\lambda)$ is compact for $0 < \alpha < 2$. The following asymptotic formula holds*

$$s_n(C_\alpha|_{L^2(D,d\lambda)}) = s_n(VC_\alpha) = \frac{\Gamma(n + \alpha)}{12\Gamma(\alpha)\,\Gamma(n + 2)} \sim \frac{1}{n^{2-\alpha}}, \quad n \to \infty.$$

In the proof we will need the following inequalities for $\Gamma$ function (see [**2**]).

PROPOSITION 2.1. *Let $m, p$ and $k$ be real numbers with $m, p > 0$ and $p > k > -m$. If $k(p - m - k) \geqslant 0$ $(\leqslant 0)$, then $\Gamma(p)\Gamma(m) \geqslant$ $(\leqslant)\Gamma(p - k)\,\Gamma(m + k)$.*

PROOF. From Theorem 1.3 and properties of the operator $V$ we have that $VC_\alpha$ maps $L^2(D, d\lambda)$ into itself. Let $H(\cdot,\cdot)$ be appropriate kernel of $VC_\alpha$, i.e.,

$$VC_\alpha f(z) = \int_D H(z,\xi)\, f(\xi)\, dA(\xi).$$

The the kernel $H(\cdot,\cdot)$ is given by

$$H(z,\xi) = (1-|z|^2)^2 \int_D \frac{dA(t)}{(1-z\bar{t})^4(1-\bar{\xi}t)^\alpha}.$$

On the other hand

$$\begin{aligned} H(z,\xi) &= (1-|z|^2)^2 \int_D \frac{dA(t)}{(1-z\bar{t})^4(1-\bar{\xi}t)^\alpha} \\ &= (1-|z|^2)^2 \int_D \sum_{n=0}^{\infty} \frac{\Gamma(n+4)}{\Gamma(4)n!}(z\bar{t})^n \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)k!}(\bar{\xi}t)^k dA(t) \\ &= (1-|z|^2)^2 \sum_{n=0}^{\infty} \frac{\Gamma(n+4)\Gamma(n+\alpha)}{12\Gamma(\alpha)n!(n+1)!} z^n \bar{\xi}^n. \end{aligned}$$

So,

$$\begin{aligned} VC_\alpha f(z) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+4)\Gamma(n+\alpha)}{12\Gamma(\alpha)n!(n+1)!}(1-|z|^2)^2 z^n \int_D f(\xi)\bar{\xi}^n dA(\xi) \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(n+4)\Gamma(n+\alpha)}{12\Gamma(\alpha)n!(n+1)!}(1-|z|^2)^2 z^n \int_D f(\xi)(1-|\xi|^2)^2 \bar{\xi}^n d\lambda(\xi). \end{aligned}$$

Let us note that $e_n(z) = \sqrt{\frac{1}{2}(n+3)(n+2)(n+1)}(1-|z|^2)^2 z^n$, $n = 0,1,2,\ldots$ represents orthonomal set in $L^2(D, d\lambda)$, which implies that

$$VC_\alpha f(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{6\Gamma(\alpha)\Gamma(n+2)} e_n(z)\langle f, e_n\rangle, \quad z \in D.$$

Since, Stirling's formula implies $s_n(VC_\alpha) \sim \frac{1}{n^{2-\alpha}}$, as $n \to \infty$, and by using the fact

$$\|VC_\alpha f\|_2^2 = \sum_{n=0}^{\infty} \Big(\frac{\Gamma(n+\alpha)}{6\Gamma(\alpha)\Gamma(n+2)}\Big)^2 |\langle f, e_n\rangle|^2, \quad f \in L^2(D, d\lambda),$$

we conclude that $VC_\alpha$ is compact for $0 < \alpha < 2$. The sequence $\big(\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)(n+1)!}\big)$ is decreasing in $n$, and this is a consequence of Proposition 2.1 with $p = \alpha + n + 1$, $m = n + 2$, $k = 1$. Then by Theorem 2.1 we get $s_n(VC_\alpha) = \frac{\Gamma(n+\alpha)}{6\Gamma(\alpha)\Gamma(n+2)}$. □

The next corollary is a direct consequence of the previous theorem.

COROLLARY 2.1. *For $0 < \alpha < 2$, $VC_\alpha \in S_p$ holds, where $p > \frac{1}{2-\alpha}$ and*

$$\|VC_\alpha\|_p = \frac{1}{12\Gamma(\alpha)}\Big(\sum_{n=0}^{\infty}\Big(\frac{\Gamma(n+\alpha)}{\Gamma(n+2)}\Big)^p\Big)^{1/p}.$$

According to (2.4) and Theorem 2.3 we easily obtain the following result.

THEOREM 2.4. *If $VC_\alpha : L^2(D, d\lambda) \to L^2(D, d\lambda)$, $0 < \alpha < 2$, then*

$$\|VC_\alpha\|_{L^2(d\lambda)\to L^2(d\lambda)} = \frac{\Gamma(1+\alpha)}{12\Gamma(\alpha)}.$$

In the next theorem we will consider the operator $C_\alpha$ defined on the Dirichlet space $B_2$.

THEOREM 2.5. *The operator $C_\alpha : B_2 \to B_2$ is compact for $0 < \alpha < 2$ and*

$$s_n(C|_{B_2}) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)(n+1)!} \sim \frac{1}{n^{2-\alpha}}, \quad n \to \infty.$$

PROOF. Let us note that the sequence $e_n(z) = \frac{z^n}{\sqrt{n}}$, $n \geqslant 1$ is orthonomal in $B_2$ according to the invariant integral pairing defined in (1.1). Then, for the function $f \in B_2, f(z) = \sum_{n=0}^{\infty} a_n z^n$, where we can add the condition $a_0 = 0$, we have

$$\langle f, e_n\rangle = \sqrt{n}\int_D \sum_{k=1}^{\infty} k a_k z^{k-1}\bar{z}^{n-1} dA(z) = \sqrt{n}a_n,$$

$$C_\alpha f(z) = \int_D \frac{f(\xi)}{(1-z\bar{\xi})^\alpha} dA(\xi) = \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)a_n}{\Gamma(\alpha)(n+1)!} z^n = \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)e_n(z)}{\Gamma(\alpha)(n+1)!}\langle f, e_n\rangle.$$

By Stirling's formula we obtain $\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+2)} \sim \frac{1}{n^{2-\alpha}}$, $n \to \infty$. On the other hand,

$$\|C_\alpha f\|^2_{B_2} = \sum_{n=0}^{\infty} \left(\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\,\Gamma(n+2)}\right)^2 |\langle f, e_n\rangle|^2$$

So, we conclude that $C_\alpha$ is a compact operator on $B_2$ and Theorem 2.1 implies $s_n(C_\alpha|_{B_2}) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)(n+1)!}$. □

COROLLARY 2.2. *For the operator $C_\alpha : B_2 \to B_2$, $(0 < \alpha < 2)$ holds $C_\alpha \in S_p$, where $p > \frac{1}{2-\alpha}$, and*

$$\|C_\alpha\|_p = \frac{1}{\Gamma(\alpha)}\left(\sum_{n=0}^{\infty}\left(\frac{\Gamma(n+\alpha)}{\Gamma(n+2)}\right)^p\right)^{1/p}.$$

A direct consequence of Theorem 2.5 is that $\|C_\alpha\|_{B_2\to B_2} = s_1(C_\alpha|_{B_2})$, i.e.,

$$\|C_\alpha\|_{B_2\to B_2} = \frac{\Gamma(\alpha+1)}{2\Gamma(\alpha)}.$$

However, we present here a direct way for finding the norm of $C_\alpha$ on $B_2$ without using singular numbers.

THEOREM 2.6. *The operator $C_\alpha : B_2 \to B_2$, $0 < \alpha < 2$ is bounded and*

$$\|C_\alpha\|_{B_2\to B_2} = \frac{\Gamma(\alpha+1)}{2\Gamma(\alpha)}.$$

PROOF. Every function in $B_2$ can be approximated in norm by a sequence of polynomials. It is enough to find the norm of $C_\alpha$ on the set of polynomials $p_m(z) = \sum_{k=0}^{m} a_k z^k$, where $m$ is a nonnegative integer. From the proof of Theorem 2.5 we get

$$C_\alpha f(z) = \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)e_n(z)}{\Gamma(\alpha)(n+1)!}\langle f, e_n\rangle.$$

Thus,

$$C_\alpha p_m(z) = \sum_{k=1}^{m} \frac{\Gamma(\alpha+n)a_n z^n}{\Gamma(\alpha)(n+1)!},$$

$$\|C_\alpha p_m\|_{B_2}^2 = \sum_{n=1}^{m} \frac{n|\Gamma(\alpha+n)|^2|a_n|^2}{|\Gamma(\alpha)(n+1)!|^2}, \quad \|p_m\|_{B_2}^2 = \sum_{n=1}^{m} n|a_n|^2.$$

We want to find the minimal constant $A$ such that

$$\sum_{n=1}^{m} \frac{n|\Gamma(\alpha+n)|^2|a_n|^2}{|\Gamma(\alpha)(n+1)!|^2} \leqslant A^2 \sum_{n=1}^{m} n|a_n|^2, \tag{2.5}$$

for every polynomial $p_m$. In the above inequality we can treat the sequences $(n|a_n|^2)$ and $\left(\frac{|\Gamma(\alpha+n)|^2}{|\Gamma(\alpha)(n+1)!|^2}\right)$ as elements of $l^1$ and $l^\infty$, respectively, so (2.5) can be rewritten as

$$\left\langle (n|a_n|^2), \left(\frac{|\Gamma(\alpha+n)|^2}{|\Gamma(\alpha)(n+1)!|^2}\right)\right\rangle \leqslant A^2 \|(n|a_n|^2)\|_{l^1},$$

where $\langle (\xi_n), (\eta_n)\rangle = \sum_{n=1}^{\infty} \xi_n\eta_n$. By using the duality argument $(l^1)^* = l^\infty$ we obtain

$$A^2 = \sup_{n\geqslant 1} \frac{\Gamma^2(\alpha+n)}{\Gamma^2(\alpha)((n+1)!)^2} = \frac{\Gamma^2(\alpha+1)}{4\Gamma(\alpha)^2}, \tag{2.6}$$

i.e.,

$$\|C_\alpha\|_{B_2\to B_2} = \frac{\Gamma(\alpha+1)}{2\Gamma(\alpha)}.$$

In (2.6) we used again the fact that the sequence $\left(\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)(n+1)!}\right)$ is decreasing in $n$. □

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