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DISJUNCTION OF BOOLEAN EQUATIONS

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Abstract. We give a formula for all the solutions of a disjunction of Boolean equations.

1. Introduction

Since Boole, systems of equations over a Boolean algebra have been extensively studied. Some results on solving systems of Boolean equations were summarized in Rudeanu's books [**5, 6**]. The problem of solving the generalized systems of Boolean equations (systems which are built using conjunctions and disjunctions of Boolean equations and Boolean inequations) still stays open for further discussion. Banković has given all the solutions related to: Boolean inequations [**1**], systems of a Boolean equation and a Boolean inequation [**2**], and systems of two Boolean inequations [**3**]. Marovac in [**7**] has considered systems of k Boolean inequation and described all their solutions. In this paper, we deal with the problem of solving the disjunction of k Boolean equations.

Let $(B, \cap, \cup, ', 0, 1)$ be a Boolean algebra and n be a natural number.

Definition 1.1. Let $x \in B$. Then

$$x^1 = x, \qquad x^0 = x'.$$

If $X = (x_1, \dots, x_n) \in B^n$ and $A = (a_1, \dots, a_n) \in \{0,1\}^n$ then

$$X^A = x_1^{a_1} \cap \cdots \cap x_n^{a_n}.$$

In the sequel, $\cap$ will be omitted.

Definition 1.2. [**5**, Definition 1.13] The Boolean functions of n variables (BFn) over the Boolean algebra $(B, \cup, \cdot, ', 0, 1)$ are determined by the following rules:

(0) For every $a \in B$, constant function $f_a : B^n \to B$ defined by

$$f_a(x_1, \dots, x_n) = a \quad (\forall x_1, \dots, x_n \in B)$$

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is a BFn.

(1) For every $i = 1, 2, \dots, n$, the projection function $\varepsilon_i : B^n \to B$ defined by

$$\varepsilon_i(x_1, \dots, x_n) = x_i \quad (\forall x_1, \dots, x_n \in B)$$

is a BFn.

(2) If $f, g : B^n \to B$ are BFn, then the functions $f \cup g$, fg, $f' : B^n \to B$ defined by

$$(f \cup g)(x_1, \dots, x_n) = f(x_1, \dots, x_n) \cup g(x_1, \dots, x_n)\, (\forall x_1, \dots, x_n \in B),$$
$$(fg)(x_1, \dots, x_n) = f(x_1, \dots, x_n) \cdot g(x_1, \dots, x_n)\, (\forall x_1, \dots, x_n \in B),$$
$$f'(x_1, \dots, x_n) = (f(x_1, \dots, x_n))' \;\; (\forall x_1, \dots, x_n \in B)$$

are BFn.

(3) Any BFn is obtained by applying the rules (0), (1) and (2) a finite number of times.

Theorem 1.1. [**5**, Corollary 1.1] *The function $f : B^n \to B$ is Boolean if and only if it can be written in the canonical disjunctive form*

$$f(X) = \bigcup_A f(A) X^A.$$

2. Generalized systems of Boolean equations

When we generally refer to a system of equations of any kind, we refer to equations that are linked by logical conjunction. The idea of generalized system of equations is to link the equations by any logical operation, not merely by conjunction.

Definition 2.1. [**6**, Definition 5.1] The generalized systems of Boolean equations (GSBE's for short) over a Boolean algebra are defined recursively as follows:

(i) every Boolean equation $f(X) = 0$ is a GSBE;

(ii) the negation, logical conjunction and logical disjunction of any GSBE's is a GSBE;

(iii) every GSBE is obtained by applying rules (i) and (ii) finitely many times.

The problem of solving GSBE's reduces to a particular case of it.

Definition 2.2. [**6**, Definition 5.3] An elementary GSBE is either a Boolean equation $f(X) = 0$ or a system of the form

$$f_1(X) \neq 0 \wedge \cdots \wedge f_k(X) \neq 0$$

or of the form

$$g(X) = 0 \wedge f_1(X) \neq 0 \wedge \cdots \wedge f_k(X) \neq 0.$$

If $k = 1$ we shall call them an atomic GSBE. An atomic GSBE of the form $f(X) \neq 0$ will be called a Boolean inequation.

Proposition 2.1. [**6**, Proposion 5.1] *Every GSBE is equivalent to a logical disjunction of elementary GSBE's, possibly a single elementary GSBE.*

THEOREM 2.1. [**6**, Theorem 5.1] *The set of solutions of any GSBE is the union of the sets of solutions of several elementary GSBE's.*

3. Boolean equations

To solve a Boolean equation $f(X) = 0$ means to determine all $X \in B^n$ such that $f(X) = 0$ holds, i.e., to determine the set $S = \{X \in B^n \mid f(X) = 0\}$.

THEOREM 3.1. [**5**, Theorem 2.3] *Let* $f : B^n \to B$ *be a Boolean function. The equation* $f(X) = 0$ *has a solution if and only if* $\prod_A f(A) = 0$.

Let $T = (t_1, \ldots, t_n)$. The following definition comes from Prešić [**4**]. Prešić's definition refers to an arbitrary equation. It can be also found in [**6**, Definition 1.2]. Here it is applied to Boolean equations.

DEFINITION 3.1. Let $f, F_1, \ldots, F_n : B^n \to B$ be Boolean functions and $F = (F_1, \ldots, F_n)$. The formula $X = F(T)$, or in scalar form $x_i = F_i(t_1, \ldots, t_n)$ $(i = 1, \ldots, n)$, expresses the general solution of the Boolean equation $f(X) = 0$ if and only if, for every $X \in B^n$

$$f(X) = 0 \Leftrightarrow (\exists T) X = F(T).$$

LEMMA 3.1. [**5**, Lemma 2.2] *Suppose that the equation* $ax \cup bx' = 0$ *has a solution* $(ab = 0)$. *Then*

$$ax \cup bx' = 0 \Leftrightarrow (\exists t)(x = a't \cup bt') \tag{3.1}$$
$$ax \cup bx' = 0 \Leftrightarrow b \leqslant x \leqslant a'$$

for all $x \in B$.

4. Disjunction of k Boolean equations

Here we describe all the solutions of a system of Boolean equations connected by logical disjunction

$$f_1(X) = 0 \vee f_2(X) = 0 \vee \cdots \vee f_k(X) = 0. \tag{4.1}$$

The solution set of this system is the union of the solutions of the Boolean equations in it. If an equation has no solution, it will be eliminated from the system. In the sequel, we shall consider the system of Boolean equations of the form (4.1), where each equation has a solution.

Let $k = 2$. Then, system (4.1) can be written as $f(X) = 0 \vee g(X) = 0$.

THEOREM 4.1. *Let* $f, g : B^n \to B$ *be Boolean functions. Then*

$$f(X) = 0 \vee g(X) = 0 \Leftrightarrow (\exists T)(\exists s)\big((s = 0 \vee s = 1) \wedge (X = s'\Phi(T) \cup s\Psi(T)\big)$$

where $\Phi(T)$ *and* $\Psi(T)$ *express the general solutions of equations* $f(X) = 0$ *and* $g(X) = 0$, *respectively.*

Proof. Let $(\exists T)(\exists s)\big((s=0 \vee s=1) \wedge (X = s'\Phi(T) \cup s\Psi(T)\big)$. If $s=1$, then the formula $X = s'\Phi(T)\cup s\Psi(T)$ gives $X=\Psi(T)$. Since $\Psi(T)$ expresses the general solution of the equation $g(X)=0$, then $g(X)=0$. Therefore, $f(X)=0 \vee g(X)=0$. Similarly, if $s=0$ then $X=\Phi(T)$. Since $\Phi(T)$ expresses the general solution of the equation $f(X)=0$, then $f(X)=0$. Therefore, $f(X)=0 \vee g(X)=0$. Let $f(X)=0 \vee g(X)=0$. If $f(X)=0$, then there is T such that $X=\Phi(T)$, because $X=\Phi(T)$ determines the general solution of $f(X)=0$. Therefore $X = s'\Phi(T)\cup s\Psi(T)$ for $s=0$. Similarly, if $g(X)=0$, then there is T such that $X=\Psi(T)$, because $X=\Psi(T)$ determines the general solution of $g(X)=0$. Therefore $X = s'\Phi(T)\cup s\Psi(T)$ for $s=1$. □

Example 4.1. Let $a,b,c,d \in B$. Solve the system

$$ax \cup bx' = 0 \vee cx \cup dx' = 0.$$

According to (3.1) we can take $\Phi(t) = a't \cup bt'$ and $\Psi(t) = c't \cup dt'$. Using Theorem 4.1 we get

$$\text{(4.2)} \quad ax \cup bx' = 0 \vee cx \cup dx' = 0 \\ \Leftrightarrow (\exists t)(\exists s)\big((s=0 \vee s=1) \wedge (x = s'(a't \cup bt') \cup s(c't \cup dt')\big).$$

Example 4.2. Let $B = \{0,1,m,l,k,m',l',k'\}$. Solve the system

$$m'x' = 0 \vee m'x = 0.$$

In accordance with Example 4.1 we have $a=0$, $b=m'$, $c=m'$, $d=0$. Then

$$m'x' = 0 \vee m'x = 0 \Leftrightarrow (\exists t)(\exists s)\big((s=0 \vee s=1) \wedge (x = s't \cup m'st' \cup mst)\big).$$

Since $t \in \{0,1,m,l,k,m',l',k'\}$ and $s \in \{0,1\}$ we get $x \in \{m',1,0,m\}$.

Theorem 4.2. *Let $f_1,\dots,f_k : B^n \to B$ be Boolean functions. Then*

$$f_1(X)=0 \vee \cdots \vee f_k(X)=0 \Leftrightarrow \\ (\exists T)(\exists s_1)\cdots(\exists s_{k-1})\big((s_i=0 \vee s_i=1) \wedge \cdots \wedge (s_{k-1}=0 \vee s_{k-1}=1) \wedge \\ X = s_1'\Phi_1(T) \cup \cdots \cup s_1\cdots s_{k-2}s_{k-1}'\Phi_{k-1}(T) \cup s_1\cdots s_{k-1}\Phi_k(T)\big)$$

where for every $i \in \{1,\dots,k\}$, $\Phi_i(T)$ expresses the general solution of the equation $f_i(X)=0$.

Proof. Let

$$\text{(4.3)} \quad (\exists T)(\exists s_1)\cdots(\exists s_{k-1})((s_1=0 \vee s_1=1) \wedge ... \wedge (s_{k-1}=0 \vee s_{k-1}=1) \wedge \\ X = s_1'\Phi_1(T) \cup \cdots \cup s_1\cdots s_{k-2}s_{k-1}'\Phi_{k-1}(T) \cup s_1\cdots s_{k-1}\Phi_k(T)).$$

Let s_i be the number of the sequence $s_1,\dots,s_{k-1}$ such that $s_i=0$ and $i=1$ or $s_1=1,\dots,s_{i-1}=1$. Then formula (4.3) gives $X=\Phi_i(T)$. Thus $f_i(X)=0$. Therefore, $f_1(X)=0 \vee \cdots \vee f_k(X)=0$. If $s_i=1$ for each $i \in \{1,\dots,k-1\}$, formula(4.3) gives $X=\Phi_k(T)$. Thus $f_k(X)=0$. Therefore, $f_1(X)=0 \vee \cdots \vee f_k(X)=0$. Let $f_1(X)=0 \vee \cdots \vee f_k(X)=0$. Then $f_j(X)=0$ for some $j \in \{1,\dots,k\}$. Then there is T such that $X=\Phi_j(T)$. Thus, X can be presented by formula (4.3), where:

a) if $j < k$ then $s_1 = 1, \dots, s_{j-1} = 1$ and $s_j = 0$,
b) if $j = k$ then $s_i = 1$ for each $i \in \{1 \dots k-1\}$. $\square$

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