PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 86(100) (2009), 5–20 DOI: 10.2298/PIM0900005D

BOUNDEDNESS OF THE BERGMAN PROJECTIONS ON L^p SPACES WITH RADIAL WEIGHTS

Milutin Dostanić

Communicated by Stevan Pilipović

Abstract. Necessary as well as sufficient conditions are given for the Bergman projections to be bounded operators on L^p spaces on the unit disc.

Let w be a continuous positive function on $[0,1)$ such that $\lim_{r\to 1-0} w(r)=0$. Denote by $d\mu(z)$ the measure $d\mu(z)=w(|z|)\,dA(z)$, where $dA(z)$ is Lebesgue's measure ($dA(z)=dx\,dy$, $z=x+i\,y$) on the unit disc D. Denote by $L^p(D,d\mu)$ (or $L^p(D)$ for short), $1\leqslant p<\infty$ the set of all complex measurable functions f for which $\|f\|_p=\bigl(\int_D |f(z)|^p d\mu(z)\bigr)^{1/p}<\infty$ and by $L^p_a(D,d\mu)$ (or $L^p_a(D)$ for short) the subspace of the space $L^p(D)$ comprising the functions that are analytic on D.

If $p=2$, $L^2_a(D)$ is a Hilbert subspace of $L^2(D)$ and it is called Bergman space. Let P denote the orthogonal projector of $L^2(D)$ on $L^2_a(D)$ (Bergman projection). Let $\{\delta_n\}_{n=0}^\infty$ be defined by $\delta_n=\bigl(2\pi\int_0^1 r^{2n+1}w(r)\,dr\bigr)^{1/2}$. Then, the sequence of functions $\{z^n/\delta_n\}_{n=0}^\infty$ is an orthonormal basis of $L^2_a(D)$ and so the corresponding Bergman reproducing kernel is given by $K(z,\xi)=\sum_{n=0}^\infty z^n\overline{\xi}^n/\delta_n^2$ ($z,\xi\in D$). Let $I_k^n=\bigl[\frac{k-1}{n},\frac{k}{n}\bigr]$ ($k=1,2,\dots,n$), $\Phi(\lambda)=\int_0^1 t^{2\lambda+1}\,w(t)\,dt$ ($\lambda\in(0,+\infty)$), $G(\lambda)=\frac{\Phi(\lambda+1)}{\Phi(\lambda)}$ and

$$H^n_{k\nu}(\lambda)=\int\limits_{I_k^n\times I_\nu^n}(xy)^\lambda(xy-G(\lambda))(x\,w(x))^{1/p}(y\,w(y))^{1/q}dx\,dy\ \ \Bigl(p>1,\ \frac1p+\frac1q=1\Bigr).$$

Operator $P: L^2(D)\longrightarrow L^2_a(D)$ acts in the following way:

$$Pf(z)=\int\limits_D K(z,\xi)\,f(\xi)\,d\mu(\xi),\quad z\in D.$$

We will use the same notation: $f\in L^p(D)$, $Pf=\int_D K(\cdot,\xi)f(\xi)\,d\mu(\xi)$, $1<p<\infty$. The same notation, P, will be used for this mapping.

2000 *Mathematics Subject Classification*: 47B38.

Partially supported by MNZZS, grant ON144010.

In this paper we consider conditions for w so that P is a bounded operator on $L^p(D)$ $(1<p<\infty)$.

The boundedness of the Bergman projection is a fact of fundamental importance. In the case of the unit disc, boundedness of the Beergman projection was studied in [**3**], [**8**] and it immediately gives the duality between the Bergman spaces. Also, the boundedness of the Bergman projection is used to establish corresponding theorems that concern duality and interpolation of analytic Besov spaces [**8**, Th. 5.3.6, Th. 5.3.8, pp. 94–97]

1. The Main Result

Let $w \in C[0,1)$, $w>0$ in $[0,1)$, $\lim_{r\to 1-0} w(r)=0$ and let the function $T:[0,1)\to\mathbb{R}$ be defined by $T(r)=\sup_{m\geqslant 0} r^m/\delta_m^2$, $r\in[0,1)$. Assume that for some $c_0\in(0,1)$ all the functions $H_{k\nu}^n$ have uniformly bounded number of zeroes on the interval $(1,+\infty)$ for all $I_k^n, I_\nu^n \subset [c_0,1]$. The main result is given by the following Theorem:

THEOREM 1.1. (a) *The necessary condition for operator P to be bounded on $L^p(D)$ $(1<p<\infty)$ is:*

$$\sup_{m\geqslant 0}\frac{1}{\delta_m^2}\left(\int_0^1 x^{mp+1}w(x)\,dx\right)^{1/p}\left(\int_0^1 x^{mq+1}w(x)\,dx\right)^{1/q}<\infty,\qquad \frac{1}{p}+\frac{1}{q}=1. \tag{1.1}$$

(b) *If condition* (1.1) *holds, then the condition*

$$\begin{cases} \sup\limits_{0\leqslant x<1} w(x)^{1/p}\int_0^1 T(xy)w(y)^{1/q}\,dy<+\infty \\ \sup\limits_{0\leqslant x<1} w(x)^{1/q}\int_0^1 T(xy)w(y)^{1/p}\,dy<+\infty \end{cases} \tag{1.2}$$

is sufficient for the Bergman projection P to be a bounded operator on $L^p(D)$ $(1<p<\infty)$.

REMARK 1.1. From condition (1.1) it follows that $\forall r\in[0,1)$ and $\forall m\geqslant 0$:

$$\frac{1}{\delta_m^2}\left(\int_{\sqrt{r}}^1 r^{(mp+1)/2}w(x)\,dx\right)^{1/p}\cdot\left(\int_{\sqrt{r}}^1 r^{(mq+1)/2}w(x)\,dx\right)^{1/q}$$
$$\leqslant \frac{1}{\delta_m^2}\left(\int_{\sqrt{r}}^1 x^{mp+1}w(x)\,dx\right)^{1/p}\cdot\left(\int_{\sqrt{r}}^1 x^{mq+1}w(x)\,dx\right)^{1/q}\leqslant M_0 \quad (<+\infty)$$

i.e., $T(r)\cdot\sqrt{r}\int_{\sqrt{r}}^1 w(x)\,dx \leqslant M_0$ and so

$$T(r)=O\left(\left(\int_{\sqrt{r}}^1 w(x)\,dx\right)^{-1}\right)\quad \text{as } r\to 1-. \tag{1.3}$$

Therefore if condition (1.1) holds, then the sufficient condition (1.2) can be replaced (keeping in mind (1.3)) by more operative condition:

$$
\left\{
\begin{aligned}
&\sup_{0\leqslant x<1} w(x)^{1/p}\int_0^1 \frac{w(y)^{1/q}}{\int_{\sqrt{xy}}^1 w(t)\,dt}\,dy<+\infty\\
&\sup_{0\leqslant x<1} w(x)^{1/q}\int_0^1 \frac{w(y)^{1/p}}{\int_{\sqrt{xy}}^1 w(t)\,dt}\,dy<+\infty
\end{aligned}
\right.
\tag{1.2$'$}
$$

Example 1.1. Let $w(r)=(1-r^2)^\alpha L\big(\frac{1}{1-r^2}\big)$, $\alpha>0$, where L is a slowly varying nondecreasing C^1 function. In a similar way as in [**1**] we can show that for the function $K(\lambda)=\int_0^1 x^\lambda w(x)\,dx$, $\lambda>0$, there holds $K(\lambda)\asymp\frac{L(\lambda)}{\lambda^{\alpha+1}}$, $\lambda\to+\infty$ i.e., there exists constants $C_1,C_2>0$ independent of λ such that $C_1\leqslant K(\lambda)\frac{\lambda^{\alpha+1}}{L(\lambda)}\leqslant C_2$ for $\lambda\geqslant\lambda_0$. From that it follows directly that the weight w satisfies the necessary condition of Theorem 1. Let us prove that w satisfies the sufficient conditions of Theorem 1. It is enough to check the conditions in (1.2$'$). We want to prove that

$$
\sup_{0\leqslant x<1} w^{1/p}(x)\int_0^1 w^{1/q}(y)\bigg(\int_{\sqrt{xy}}^1 w(t)\,dt\bigg)^{-1}dy<\infty. \tag{1.4}
$$

In a similar way we can prove that

$$
\sup_{0\leqslant x<1} w^{1/q}(x)\int_0^1 w^{1/p}(y)\bigg(\int_{\sqrt{xy}}^1 w(t)\,dt\bigg)^{-1}dy<\infty.
$$

Since

$$
\begin{aligned}
\int_r^1 w(t)\,dt &\sim \frac12\int_{r^2}^1 (1-t)^\alpha L\Big(\frac{1}{1-t}\Big)\,dt\\
&=\frac12\int_{1/(1-r^2)}^{+\infty}\frac{L(u)}{u^{\alpha+2}}du\sim\frac{1}{2(\alpha+1)}\big(1-r^2\big)^{\alpha+1}L\Big(\frac{1}{1-r^2}\Big),\quad r\to1-
\end{aligned}
$$

(here we used the asymptotic formula $\int_x^\infty\frac{L(u)}{u^{\alpha+2}}du\sim\frac{L(x)}{(\alpha+1)x^{\alpha+1}}$), we conclude that (1.4) is true if

$$
\sup_{0\leqslant x<1} w(x)^{1/p}\int_0^1\frac{w(y)^{1/q}dy}{(1-xy)^{\alpha+1}L(1/(1-xy))}<\infty \tag{1.5}
$$

is true. After the change of variables $x=e^{-u}$, $y=e^{-v}$ in (1.5) we obtain

$$
\sup_{u>0} w\left(e^{-u}\right)^{1/p}\int_0^{+\infty}\frac{w\left(e^{-v}\right)e^{-v}}{\left(1-e^{-(u+v)}\right)^{\alpha+1}L\left(1/(1-e^{-(u+v)})\right)}dv<\infty. \tag{1.6}
$$

Since L is a slowly varying function, we have

$$\lim_{u\to 0+}\frac{L\left(1/(1-e^{-2u})\right)\cdot\left(1-e^{-2u}\right)^{\alpha}}{(2u)^{\alpha}L(1/u)}=1$$
$$\lim_{u\to 0+}\frac{L\left(1/(1-e^{-u})\right)\cdot(1-e^{-u})^{\alpha+1}}{u^{\alpha+1}L(1/u)}=1$$

and so, inequality (1.6) is satisfied if

$$\sup_{0<u\leqslant 1}(u^{\alpha}L(1/u))^{1/p}\int_0^1\frac{\left(v^{\alpha}L\left(\frac{1}{v}\right)\right)^{1/q}}{(u+v)^{\alpha+1}L\left(\frac{1}{u+v}\right)}dv<\infty \tag{1.7}$$

is true. Let

$$\mathcal{A}(u)=(u^{\alpha}L(1/u))^{1/p}\int_0^1\frac{\left(v^{\alpha}L\left(\frac{1}{v}\right)\right)^{1/q}}{(u+v)^{\alpha+1}L\left(\frac{1}{u+v}\right)}dv.$$

It is clear that inequality (1.7) is true if the function $\mathcal{A}$ is bounded in a neighborhood of the point $u=0$. After changing variables $v=u\cdot t$, $t\in(0,\frac{1}{u})$ we obtain

$$\mathcal{A}(u)=\int_0^{1/u}\frac{t^{\alpha/q}}{(1+t)^{\alpha+1}}\frac{\left(L\left(\frac{1}{u}\right)\right)^{1/p}\cdot\left(L\left(\frac{1}{u}\cdot\frac{1}{t}\right)\right)^{1/q}}{L\left(\frac{1}{u}\cdot\frac{1}{t+1}\right)}dt.$$

Let $\frac{1}{u}=\lambda$. Then the boundedness of function $\mathcal{A}$ in a neighborhood of the point $u=0$ is equivalent to the boundedness of function $\lambda\mapsto\mathcal{A}\left(\frac{1}{\lambda}\right)$ in a neighborhood of $\lambda=+\infty$.

If $t=\frac{1}{s}$, we get

$$\begin{aligned}\mathcal{A}\Big(\frac{1}{\lambda}\Big)&=\int_{1/\lambda}^{+\infty}\frac{s^{\alpha/p-1}}{(1+s)^{\alpha+1}}\frac{(L(\lambda))^{1/p}\cdot(L(\lambda s))^{1/q}}{L\left(\lambda\cdot\frac{s}{s+1}\right)}ds=\int_{\frac{1}{\lambda}}^{1}(\cdot)\,ds+\int_1^{+\infty}(\cdot)\,ds\\&=H_1(\lambda)+H_2(\lambda).\end{aligned}$$

Now, we prove that H_1 and H_2 are bounded functions in a neighborhood of $\lambda=+\infty$. From $s\leqslant 1$ it follows $s\lambda\leqslant\lambda$ and we have $L(\lambda s)\leqslant L(\lambda)$ (L non-decreases) and

$$H_1(\lambda)\leqslant L(\lambda)\int_{1/\lambda}^{1}\frac{t^{\alpha/p-1}}{(1+t)^{\alpha+1}}\frac{ds}{L\left(\lambda\cdot\frac{s}{s+1}\right)}.$$

After changing variables $t = \frac{s}{s+1}$, we obtain from the previous inequality

$$\begin{aligned}
H_1(\lambda) &\leqslant L(\lambda) \int\limits_{\lambda/(\lambda+1)}^{1/2} t^{\alpha/p-1}(1-t)^{\alpha/q} \frac{dt}{L(\lambda t)} \\
&\leqslant L(\lambda) \int\limits_{\lambda/(\lambda+1)}^{1/2} t^{\alpha/p-1} \frac{dt}{L(\lambda t)} \quad (\text{change } \lambda t = \xi) \\
&= \frac{L(\lambda)}{\lambda^{\alpha/p}} \int\limits_{\lambda/(\lambda+1)}^{\frac{\lambda}{2}} \xi^{\alpha/p-1} \frac{d\xi}{L(\xi)} \\
&= \frac{L(\lambda)}{\lambda^{\alpha/p}} \int\limits_{\lambda/(\lambda+1)}^{1} \frac{\xi^{\alpha/p-1}}{L(\xi)} d\xi + \frac{L(\lambda)}{\lambda^{\alpha/p}} \int\limits_{1}^{\lambda/2} \frac{\xi^{\alpha/p-1}}{L(\xi)} d\xi = R_1(\lambda) + R_2(\lambda).
\end{aligned}$$

It is clear that $\lim_{\lambda\to+\infty} R_1(\lambda) = 0$ and it follows that R_1 is a bounded function in a neighborhood of $\lambda = +\infty$. Since L is a slowly varying function we have

$$\lim_{\lambda\to+\infty} \frac{\lambda L'(\lambda)}{L(\lambda)} = 0$$

and hence

$$\lim_{x\to+\infty} \frac{L(x)}{x^{\alpha/p}} \int\limits_{1}^{x} \frac{t^{\alpha/p-1}}{L(t)} dt = \frac{p}{\lambda}$$

From the previous equality follows that there exists $\lim_{\lambda\to+\infty} R_2(\lambda)$ and is finite. So, H_1 is a bounded function in a neighborhood of $\lambda = +\infty$.

If $1 \leqslant s < \infty$, then $\frac{1}{2} \leqslant \frac{s}{s+1} < 1$ and we have $L\big(\frac{\lambda}{2}\big) \leqslant L\big(\lambda\frac{s}{s+1}\big)$ and

$$H_2(\lambda) \leqslant \frac{(L(\lambda))^{1/p}}{L(\lambda/2)} \int\limits_{1}^{+\infty} \frac{s^{\alpha/p-1}}{(1+s)^{\alpha+1}} (L(\lambda s))^{1/q} ds.$$

Having in mind that $\lim_{\lambda\to\infty} \frac{L(\lambda)}{L(\lambda/2)} = 1$ and

$$\lim_{\lambda\to\infty} \int\limits_{1}^{+\infty} \frac{s^{\alpha/p-1}}{(1+s)^{\alpha+1}} \left(\frac{L(\lambda s)}{L(\lambda)}\right)^{1/q} ds = \int\limits_{1}^{+\infty} \frac{s^{\alpha/p-1}}{(1+s)^{\alpha+1}} ds$$

[**5**, Th. 2.6, pp. 63–64], we obtain

$$\lim_{\lambda\to\infty} \frac{(L(\lambda))^{1/p}}{L(\lambda/2)} \int\limits_{1}^{+\infty} \frac{s^{\alpha/p-1}}{(1+s)^{\alpha+1}} (L(\lambda s))^{1/q} ds = \int\limits_{1}^{+\infty} \frac{s^{\alpha/p-1}}{(1+s)^{\alpha+1}} ds < \infty.$$

So, H_2 is also a bounded function in a neighborhood of $\lambda = +\infty$. (In the case $L(\lambda) \equiv 1$ the corresponding result is derived in [**3**, p. 10].)

If, for instance $L(x) = (\ln x)^\beta$ ($\beta \geqslant 0$, $x \geqslant 1$), then the functions $H^n_{k\nu}$ $\big($which correspond to the weights $w(x) = \big(1 - x^2\big)^\alpha (\ln(1/(1-x^2)))^\beta\big)$ satisfy the condition on uniform boundedness of the number of zeroes on $(1, +\infty)$. (This follows from [**2**, Theorem 1.7, pp. 76–78].)

Example 1.2. Let $w(r) = (1-r^2)^A \exp\big(-B/(1-r^2)^\alpha\big)$, $A \in \mathbb{R}$, $B > 0$, $\alpha > 0$. In [**1**] it was demonstrated that for the function $\Phi(\lambda) = \int_0^1 r^\lambda w(r)\, dr$ the following asymptotic formula holds:

$$\Phi(\lambda) \sim C \cdot \lambda^D \exp\big(-E \cdot \lambda^{\alpha/(\alpha+1)}\big), \quad \lambda \to \infty. \tag{1.8}$$

(C, D, E are constants that depend only on A, B and α; $E > 0$).

From (1.8) it follows that the necessary condition (1.1) is not satisfied (except $p = q = 2$). This means that the corresponding Bergman projection is not bounded on $L^p(D)$ for any $p \neq 2$.

Corollary 1.1. *If function w satisfies conditions* (1.1) *and* (1.2) (*i.e.,* (1.1) *and* (1.2′)), *then the dual of the space $L^p_a(D)$, $(1 < p < \infty)$ is the space $L^q_a(D)$, under the integral pairing*

$$\langle f, g\rangle = \int\limits_D f(z)\,\overline{g(z)}\, d\mu(z), \quad f \in L^p_a(D),\ g \in L^q_a(D),\ \Big(\frac{1}{p} + \frac{1}{q} = 1\Big)$$

Note that the identification isomorphism $(L^p_a(D))^* = L^q_a(D)$ need not be isometric for $p \neq 2$.

2. Proof

In the proof of the main result we use the following two Theorems:

Theorem 2.1 (Marcinkiewicz's Theorem [**7**, pp. 346–348]). *Let $\lambda_0, \lambda_1 \dots$ be a sequence such that for a constant M holds: $\sum_{j=2^\nu}^{2^{\nu+1}-1} |\lambda_j - \lambda_{j+1}| \leqslant M$ and $|\lambda_\nu| \leqslant M$ for all $\nu = 0, 1, 2, \dots$. Then for any $p > 1$*

$$\Big\|\sum_{\nu\geqslant 0} \lambda_\nu\, c_\nu\, e^{i\nu x}\Big\|_p \leqslant M \cdot A(p) \Big\|\sum_{\nu\geqslant 0} c_\nu\, e^{i\nu x}\Big\|_p.$$

Here $A(p)$ is a constant that depends only on p and $\|\cdot\|_p$ is a norm in the space $L^p(0, 2\pi)$.

Theorem 2.2 (Schur's test [**3**, p. 9]). *Denote by $\mathbb{R}^n_p$ $(p > 1)$ the space $\mathbb{R}^n$ with the norm $\|x\|_p = \sqrt[p]{\sum_{i=1}^n |x_i t|^p}$, $x = (x_1, \dots, x_n)$ and let*

$$C_n = \begin{pmatrix} c^{(n)}_{11} & c^{(n)}_{12} & \cdots & c^{(n)}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ c^{(n)}_{n1} & c^{(n)}_{n2} & \cdots & c^{(n)}_{nn} \end{pmatrix} : \mathbb{R}^n_p \to \mathbb{R}^n_p.$$

If there exists a constant λ ($\lambda < \infty$) which does not depend on n such that

$$\max_{1\leqslant k\leqslant n}\sum_{\nu=1}^{n}\bigl|c_{k\nu}^{(n)}\bigr| \leqslant \lambda, \quad \max_{1\leqslant \nu\leqslant n}\sum_{k=1}^{n}\bigl|c_{k\nu}^{(n)}\bigr| \leqslant \lambda$$

then for any $n \in \mathbb{N}$ and any $x \in \mathbb{R}_p^n$ holds $\|C_n x\|_p \leqslant \lambda \|x\|_p$.

Lemma 2.1. *Let $\{j_{mn}\}_{n=1}^{\infty}$ be the positive zeroes of the Bessel function of the first kind J_m ($m = 0, \pm1, \pm2, \ldots$). Then, for any $m \in \mathbb{Z}$ and $\alpha, \beta \geqslant 0$, the system of functions $\bigl\{J_m(j_{mn}r)\cdot r^{\alpha}\cdot w(r)^{\beta}\bigr\}_{n=1}^{\infty}$ is complete in $L^p(0,1)$, $(1 \leqslant p < \infty)$.*

Proof. It is sufficient to prove the lemma for $m \geqslant 0$ since $(-1)^m J_{-m} = J_m$ for all $m \in \mathbb{Z}$. Suppose that $m \geqslant 0$ and that the system $\bigl\{J_m(j_{mn}r)\cdot r^{\alpha}\cdot w(r)^{\beta}\bigr\}_{n=1}^{\infty}$ is not complete in $L^p(0,1)$. This would mean that there exists function $g \in L^q(0,1)$ $\bigl(\frac{1}{p}+\frac{1}{q}=1\bigr)$ such that

$$\int_0^1 g(r)\, r^{\alpha} w(r)^{\beta} J_m(j_{mn}r)\, dr = 0 \tag{2.1}$$

for all $n = 1, 2, 3, \ldots$ and $g \neq 0$ in $L^q(0,1)$. Let us show that $g = 0$ almost everywhere on $[0,1]$ (i.e., $g = 0$ in $L^q(0,1)$). Define the function

$$G_m(\lambda) = \frac{1}{J_m(\lambda)}\int_0^1 g(r)\, r^{\alpha} w(r)^{\beta} J_m(\lambda r)\, dr, \quad \lambda \in \mathbb{C}.$$

As all the zeros of the function $\lambda \mapsto J_m(\lambda)/\lambda^m$ are real and simple, from (2.1) it follows that G_m is an entire function. Furthermore, its order of growth is not bigger than 1. From the asymptotic formula (see [**6**])

$$J_m(z) = \frac{1}{\sqrt{\pi z}}\Bigl[\cos\Bigl(z - \frac{m\pi}{2} - \frac{\pi}{4}\Bigr) + O\Bigl(\frac{1}{z}\Bigr)\Bigr], \quad |z| \to \infty,\ |\arg z| \leqslant \pi - \varepsilon$$

it follows that

$$\lim_{r\to\pm\infty} G_m\bigl(re^{\pm\, i\pi/4}\bigr) = 0. \tag{2.2}$$

From (2.2), by the Phragmen–Lindelöf and Liouville Theorem, it follows that $G_m(\lambda) \equiv 0$ i.e.,

$$\int_0^1 g(r)\, r^{\alpha} w(r)^{\beta} J_m(\lambda r)\; dr \equiv 0.$$

Now by representing function J_m by the power series and applying the Müntz–Szasz theorem, we get $g(r) = 0$ almost everywhere on $[0,1]$. □

Lemma 2.2. *Let $z = re^{i\theta}$ and $f_{mn}(z) = J_m(j_{mn}r)e^{im\theta}$ ($m \in \mathbb{Z}$, $n \in \mathbb{N}$). For any $1 < p < \infty$ the system of functions $\{f_{mn}\}_{m\in\mathbb{Z},\, n\in\mathbb{N}}$ is complete in $L^p(D)$.*

Proof. It follows by the standard method, from Lemma 1 and the completeness of the system $\bigl\{e^{imx}\bigr\}_{m\in\mathbb{Z}}$ in $L^q(0, 2\pi)$. □

Let $A_m\colon L^p(0,1)\to L^p(0,1)$, $(m\geqslant 0,\ 1<p<\infty)$ be linear operators defined by

$$A_m f(x)=\frac{2\pi}{\delta_m^2}x^{m+1/p}w(x)^{1/p}\int_0^1 y^{m+1/q}w(y)^{1/q}f(y)\,dy \qquad \Big(\frac{1}{p}+\frac{1}{q}=1\Big).$$

LEMMA 2.3. *The Bergman projection P is a bounded operator on $L^p(D)$ $(1<p<\infty)$ if and only if there exists a constant c_p (depending only on p) such that*

$$\int_0^1\int_0^{2\pi}\Big|\sum_{m\geqslant 0}e^{imx}A_m\Phi_m(y)\Big|^p dx\,dy\leqslant c_p^p\int_0^1\int_0^{2\pi}\Big|\sum_{m\geqslant 0}e^{imx}\Phi_m(y)\Big|^p dx\,dy \tag{2.3}$$

for any (finite) choice of functions $\Phi_m\in L^p(0,1)$.

PROOF. By the direct evaluation one gets

$$(Pf_{mn})(z)=\begin{cases}0, & m<0\\ \dfrac{2\pi}{\delta_m^2}z^m\int_0^1 r^{m+1}w(r)\,J_m(j_{mn})\,dr, & m\geqslant 0.\end{cases} \tag{2.4}$$

Let

$$f=\sum_{m,n}c_{mn}f_{mn} \tag{2.5}$$

be a finite sum. Since, according to Lemma 2, the system $\{f_{mn}\}_{m\in\mathbb{Z},\,n\in\mathbb{N}}$ is complete in $L^p(D)$, it follows that P is a bounded operator if and only if there exists a constant $B(p)$ (which depends only on p) such that the inequality

$$\|Pf\|_p\leqslant B(p)\,\|f\|_p \tag{2.6}$$

holds for any choice of function f of type (2.5). Expanding (2.6) (keeping in mind (2.4) and (2.5)) we get

$$\begin{aligned}\int_0^1\int_0^{2\pi}\Big|\sum_n\sum_{m\geqslant 0}2\pi c_{mn}\frac{r^m e^{im\theta}}{\delta_m^2}\,w(r)^{1/p}\,r^{1/p}\int_0^1 y^{m+1}w(y)\,J_m(j_{mn}y)\,dy\Big|^p dr\,d\theta&\\ \leqslant B(p)^p\int_0^1\int_0^{2\pi}\Big|\sum_m e^{im\theta}\sum_n c_{mn}J_m(j_{mn}r)\,r^{1/p}w(r)^{1/p}\Big|^p dr\,d\theta.&\end{aligned} \tag{2.7}$$

Letting

$$\Phi_m(r)=\sum_n c_{mn}\,r^{1/p}w(r)^{1/p}J_m(j_{mn}r), \tag{2.8}$$

we get

$$\sum_n 2\pi c_{mn}\frac{r^m}{\delta_m^2}\,w(r)^{1/p}\,r^{1/p}\int_0^1 y^{m+1}\,w(y)\,J_m(j_{mn}y)\,dy=A_m\Phi_m(r)$$

and hence, the inequality (2.7) becomes

$$\int_0^1\int_0^{2\pi}\Big|\sum_{m\geqslant 0} e^{im\theta}A_m\Phi_m(r)\Big|^p dr\,d\theta \leqslant B(p)^p\int_0^1\int_0^{2\pi}\Big|\sum_{m} e^{im\theta}\,\Phi_m(r)\Big|^p dr\,d\theta. \tag{2.9}$$

Since the system of functions $\{r^{1/p}w(r)^{1/p}J_m(j_{mn}r)\}_{n=1}^{\infty}$ is complete in $L^p(0,1)$ for every $m\in\mathbb{Z}$ (Lemma 1), then (2.6) implies (2.9) not only for functions of type (2.8) but also for arbitrary $\Phi_m\in L^p(0,1)$ (because all the operators A_m are bounded and the sums in (2.9) are finite).

Conversely, if (2.9) holds for arbitrary (finite) choice of $\Phi_m\in L^p(0,1)$, then by choosing Φ_m as in (2.8) we get that (2.6) holds for functions f of type (2.5). Since the system $\{f_{mn}\}_{m\in\mathbb{Z},\,n\in\mathbb{N}}$ is complete in $L^p(D)$, (by Lemma 2) (2.6) holds for every $f\in L^p(D)$ and, hence, P is a bounded operator. Therefore, P is bounded on $L^p(D)$ if and only if (2.9) holds for any (finite) choice of $\Phi_m\in L^p(0,1)$. Let us now show that (2.9) is equivalent with the following inequality:

$$\int_0^1\int_0^{2\pi}\Big|\sum_{m\geqslant 0} e^{im\theta}A_m\Phi_m(r)\Big|^p dr\,d\theta \leqslant c_p^p\int_0^1\int_0^{2\pi}\Big|\sum_{m\geqslant 0} e^{im\theta}\Phi_m(r)\Big|^p dr\,d\theta. \tag{2.10}$$

for any (finite) choice of $\Phi_m\in L^p(0,1)$ (where c_p depends only on p). By letting $\Phi_m=0$ for $m<0$ in (2.9), we obtain (2.10) with $c_p=B(p)$.

Conversely, suppose that (2.10) holds. By the Riesz Theorem [**7**], the following inequality holds

$$\int_0^{2\pi}\Big|\sum_{m\geqslant 0} e^{i\,m\theta}\Phi_m(r)\Big|^p d\theta \leqslant K_1(p)^p\int_0^{2\pi}\Big|\sum_{m} e^{i\,m\theta}\,\Phi_m(r)\Big|^p d\theta$$

(one may put $K_1(p)=1/\sin(\pi/p)$ based on Hollenbeck–Verbitsky result [**4**]), then by integrating (over r) the previous inequality we obtain

$$\int_0^1\int_0^{2\pi}\Big|\sum_{m\geqslant 0} e^{im\theta}\Phi_m(r)\Big|^p dr\,d\theta \leqslant K_1(p)^p\int_0^1\int_0^{2\pi}\Big|\sum_{m} e^{im\theta}\Phi_m(r)\Big|^p dr\,d\theta. \tag{2.11}$$

So, from (2.10) and (2.11) it follows that (2.9) holds with $B(p)=c_p\cdot K_1(p)$ Thus, Lemma 3 is proven. ☐

Lemma 2.4. *The sequence $(\theta_{mk}^{n\nu})_{m=0}^{\infty}$ ($n\in\mathbb{N}$, $m\geqslant 0$, $\nu,k\in\{1,2,\dots,n\}$) has a uniformly bounded number of the intervals of monotonicity.*

Proof. Since

$$\theta_{m+1,k}^{n\nu}-\theta_{m,k}^{n\nu}=n\left(\int_0^1 x^{2m+3}w(x)\,dx\ \int_0^1 x^{2m+1}w(x)\,dx\right)^{-1}\cdot\Phi(m)\cdot H_{k\nu}^{n}(m)$$

we have, by the hypothesis on zeroes of the functions $H_{k\nu}^n$, that the sequence $(\theta_{mk}^{n\nu})_{m=0}^{\infty}$ has uniformly bounded number of the intervals of monotonicity, for all

n, k, ν such that $I_k^n, I_\nu^n \subset [c_0, 1]$. If at least one of the intervals I_k^n, I_ν^n is contained in $[0, c_0]$, then

$$H_{k\nu}^n(\lambda) \leqslant (c_0 - G(\lambda)) \int\limits_{I_k^n \times I_\nu^n} (xy)^\lambda (x\, w(x))^{1/p} (y\, w(y))^{1/q} dx\, dy.$$

Having in mind that the function $\lambda \mapsto G(\lambda)$ is increasing and that $\lim_{\lambda \to +\infty} G(\lambda) = 1$, we have $H_{k\nu}^n(\lambda) < 0$ for all $\lambda \geqslant \lambda_0 = \lambda_0(c_0)$. Hence, the sequence $(\theta_{mk}^{n\nu})_{m=0}^\infty$ decreases for $m \geqslant [\lambda_0] + 1$ and so it has uniformly bounded number of the intervals of monotonicity $\square$

Lemma 2.5. *Let*

$$\theta_{mk}^{n\nu} = \frac{2\pi}{\delta_m^2} \cdot n \cdot \int\limits_{(k-1)/n}^{k/n} x^{m+1/p} w(x)^{1/p} dx \cdot \int\limits_{(\nu-1)/n}^{\nu/n} y^{m+1/q} w(y)^{1/q} dy$$

$(n \in \mathbb{N},\ m \geqslant 0,\ \nu, k \in \{1, 2, \dots, n\})$. *Then*

$$\Big\| \sum_{m \geqslant 0} \theta_{mk}^{n\nu}\, a_m\, e^{imx} \Big\|_p \leqslant c_{k\nu}^{(n)} \Big\| \sum_{m \geqslant 0} a_m\, e^{imx} \Big\|_p \quad (p > 1),$$

where

$$c_{k\nu}^{(n)} = 4N_0 \pi n \cdot A(p) \cdot \int\limits_{(k-1)/n}^{k/n} dx \int\limits_{(\nu-1)/n}^{\nu/n} Tt(xy)\, w(x)^{1/p} w(y)^{1/q}\, dy$$

and $A(p)$ is the constant from Theorem 2 *(N_0 does not depend on n, k, ν).*

Proof. It is sufficient to show that the sequence $\{\theta_{mk}^{n\nu}\}_{m=0}^\infty$ satisfies the conditions of Theorem 2. By Lemma 4, there is a positive number N_0 such that sequence $\{\theta_{mk}^{n\nu}\}_{m=0}^\infty$, has the number of the intervals of monotonicity not greater than N_0 for $n \in \mathbb{N}$ and every $k, \nu \in \{1, 2, \dots, n\}$.

Let $s \in \mathbb{N}$. Then the following holds

$$\begin{aligned}
\sum_{j=2^s}^{2^{s+1}-1} \left| \theta_{jk}^{n\nu} - \theta_{j+1,k}^{n\nu} \right| &\leqslant 2N_0 \sup_{m \geqslant 0} |\theta_{mk}^{n\nu}| \\
&= \sup_{m \geqslant 0} 4N_0 \pi n \int\limits_{(k-1)/n}^{k/n} dx \int\limits_{(\nu-1)/n}^{\nu/n} \frac{(xy)^m}{\delta_m^2}\, w(x)^{1/p} w(y)^{1/q} x^{1/p} y^{1/q} dy \\
&\leqslant 4N_0 \pi n \int\limits_{(k-1)/n}^{k/n} dx \int\limits_{(\nu-1)/n}^{\nu/n} T(xy)\, w(x)^{1/p} w(y)^{1/q} dy.
\end{aligned}$$

The proof of the Lemma is completed by applying Theorem 2 with

$$M = 4N_0\pi n \int\limits_{(k-1)/n}^{k/n} dx \int\limits_{(\nu-1)/n}^{\nu/n} T(xy)\, w(x)^{1/p} w(y)^{1/q} dy. \qquad \square$$

Lemma 2.6. *Let* $\{a_{mk}^{(n)}\}$ *be complex numbers* $(0 \leqslant m \leqslant N,\ 1 \leqslant k \leqslant n)$. *If the condition* (1.2) *is satisfied, then there exists a constant* B_p (*which depends only on* p) *such that*

$$\sum_{k=1}^{n} \int\limits_{0}^{2\pi} \left| \sum_{\nu=1}^{n} \left(\sum_{m=0}^{N} \theta_{mk}^{n\nu}\, a_{m\nu}^{(n)}\, e^{imx} \right) \right|^p dx \leqslant B_p^p \sum_{\nu=1}^{n} \int\limits_{0}^{2\pi} \left| \sum_{m=0}^{N} a_{m\nu}^{(n)}\, e^{imx} \right|^p dx. \tag{2.12}$$

Proof. According to Lemma 5, the following holds

$$\left\| \sum_{m=0}^{N} \theta_{mk}^{n\nu}\, a_{m\nu}^{(n)}\, e^{imx} \right\|_p \leqslant c_{k\nu}^{(n)} \left\| \sum_{m=0}^{N} a_{m\nu}^{(n)}\, e^{imx} \right\|_p, \tag{2.13}$$

$(n \in \mathbb{N},\ \nu, k \in \{1, 2, \ldots, n\})$ and so we obtain

$$\begin{aligned} \sum_{k=1}^{n} \int\limits_{0}^{2\pi} \left| \sum_{\nu=1}^{n} \left(\sum_{m=0}^{N} \theta_{mk}^{n\nu}\, a_{m\nu}^{(n)}\, e^{imx} \right) \right|^p dx &= \sum_{k=1}^{n} \left\| \sum_{\nu=1}^{n} \left(\sum_{m=0}^{N} \theta_{mk}^{n\nu}\, a_{m\nu}^{(n)}\, e^{imx} \right) \right\|_p^p \\ &\qquad\qquad \text{according to Minkowski inequality} \\ \leqslant \sum_{k=1}^{n} \left(\sum_{\nu=1}^{n} \left\| \sum_{m=0}^{N} \theta_{mk}^{n\nu}\, a_{m\nu}^{(n)}\, e^{imx} \right\|_p \right)^p &\leqslant \sum_{k=1}^{n} \left(\sum_{\nu=1}^{n} c_{k\nu}^{(n)} \left\| \sum_{m=0}^{N} a_{m\nu}^{(n)}\, e^{imx} \right\|_p \right)^p \\ &\qquad\qquad \text{according to (2.13).} \end{aligned} \tag{2.14}$$

According to Theorem 3, putting $x_\nu = \left\| \sum_{m=0}^{N} a_{m\nu}^{(n)}\, e^{imx} \right\|_p$ $(\nu = 1, 2, \ldots, n)$, one gets

$$\sum_{k=1}^{n} \left(\sum_{\nu=1}^{n} c_{k\nu}^{(n)} \left\| \sum_{m=0}^{N} a_{m\nu}^{(n)}\, e^{imx} \right\|_p \right)^p \leqslant \lambda^p \sum_{\nu=1}^{n} \left\| \sum_{m=0}^{N} a_{m\nu}^{(n)}\, e^{imx} \right\|_p^p, \tag{2.15}$$

where

$$\lambda = \sup_{n \geqslant 1} \max \left\{ \max_{1 \leqslant k \leqslant n} \sum_{\nu=1}^{n} c_{k\nu}^{(n)},\ \max_{1 \leqslant \nu \leqslant n} \sum_{k=1}^{n} c_{k\nu}^{(n)} \right\}.$$

Let us estimate λ. Since

$$\sum_{k=1}^{n} c_{k\nu}^{(n)} = 4N_0\pi A(p)\, n \int\limits_{(\nu-1)/n}^{\nu/n} dy \int\limits_{0}^{1} T(xy)\, w(x)^{1/p} w(y)^{1/q}\, dx,$$

by the Mean Value Theorem (for integrals), we get

$$\int\limits_{(\nu-1)/n}^{\nu/n} dy \int\limits_0^1 T(xy)\, w(x)^{1/p} w(y)^{1/q} dx = \frac{1}{n} \int\limits_0^1 T(x\theta_n)\, w(x)^{1/p} w(\theta_n)^{1/q} dx,$$

where $\frac{\nu-1}{n} \leqslant \theta_n \leqslant \frac{\nu}{n}$, and hence, it follows

$$\begin{aligned}\sum_{k=1}^{n} c_{k\nu}^{(n)} &= 4N_0\pi A(p)\, w(\theta_n)^{1/q} \int\limits_0^1 T(x\theta_n)\, w(x)^{1/p} dx \\ &\leqslant 4N_0\pi A(p) \sup_{0\leqslant y<1} w(y)^{1/q} \int\limits_0^1 T(xy)\, w(x)^{1/p} dx.\end{aligned}$$

Similarly, we obtain

$$\sum_{\nu=1}^{n} c_{k\nu}^{(n)} \leqslant 4N_0\pi A(p) \sup_{0\leqslant x<1} w(x)^{1/p} \int\limits_0^1 T(xy)\, w(y)^{1/q} dy.$$

Therefore,

$$\lambda \leqslant B_p = 4N_0\pi A(p)\, \max \begin{Bmatrix} \sup\limits_{x<1} w(x)^{1/p} \int\limits_0^1 T(xy)\, w(y)^{1/q} dy \\ \sup\limits_{y<1} w(y)^{1/q} \int\limits_0^1 T(xy)\, w(x)^{1/p} dx \end{Bmatrix}$$

From (2.14) and (2.15) it follows that (2.12) holds with the constant

$$B_p = 4N_0\pi A(p)\, \max \begin{Bmatrix} \sup\limits_{x<1} w(x)^{1/p} \int\limits_0^1 T(xy)\, w(y)^{1/q} dy \\ \sup\limits_{y<1} w(y)^{1/q} \int\limits_0^1 T(xy)\, w(x)^{1/p} dx \end{Bmatrix} \qquad \square$$

3. Proof of Theorem 1

Necessity. If $P : L^p(D) \to L^p(D)$ $(1 < p < \infty)$ is a bounded operator, according to Lemma 3 inequality (2.3) holds for any (finite) choice of $\Phi_m \in L^p(0,1)$. Letting in (2.3) all the functions Φ_m, except one, to be zero, one gets

$$\int\limits_0^1 \int\limits_0^{2\pi} \left|e^{ikx} A_k \Phi_k(y)\right|^p dx\, dy \leqslant c_p^p \int\limits_0^1 \int\limits_0^{2\pi} \left|e^{ikx} \Phi_k(y)\right|^p dx\, dy$$

i.e., $\|A_k\Phi_k\|_p \leqslant c_p \|\Phi_k\|_p$ for each $\Phi_k \in L^p(0,1)$ $(k = 0, 1, \ldots)$. From this we obtain

$$\|A_k\|_p \leqslant c_p \tag{3.1}$$

for each $k = 0, 1, \ldots$ (Here $\|S\|_p$ denotes the norm of operator S on space $L^p(0,1)$). Since

$$\|A_k\|_p = \frac{2\pi}{\delta_k^2}\left(\int_0^1 x^{kp+1} w(x)\,dx\right)^{1/p} \cdot \left(\int_0^1 x^{kq+1} w(x)\,dx\right)^{1/q}$$

from (3.1), the necessary condition of Theorem 1 follows (i.e., inequality (1.1)).

Sufficiency. Let conditions (1.1) and (1.2) of Theorem 1 hold. To establish the boundedness of operator P it is sufficient to prove inequality (2.3) in the case when functions Φ_m are continuous on $[0,1]$. (If (2.3) holds when Φ_m are continuous, then it will also hold for $\Phi_m, \in L^p(0,1)$ because operators A_m are bounded and the space $C[0,1]$ is dense in $L^p(0,1)$.).

Consider functions $G_m \in C[0,1]$, $m = 0, 1, \ldots N$ and let

$$\alpha = \max_{0\leqslant m\leqslant N} \max_{0\leqslant x\leqslant 1} |G_m(x)|.$$

Let $a_{mk}^{(n)} \stackrel{\text{def}}{=} G_m(\frac{k}{n})$ and

$$\Phi_m^{(n)}(x) = \begin{cases} a_{m1}^{(n)}, & 0 \leqslant x < \frac{1}{n} \\ a_{m2}^{(n)}, & \frac{1}{n} \leqslant x < \frac{2}{n} \\ \vdots & \\ a_{mk}^{(n)}, & \frac{k-1}{n} \leqslant x < \frac{k}{n} \\ \vdots & \\ a_{mn}^{(n)}, & \frac{n-1}{n} \leqslant x \leqslant \frac{n}{n} \end{cases}$$

It is clear that the sequence of functions $\left\{\Phi_m^{(n)}(x)\right\}_{n=1}^{\infty}$ converges uniformly on $[0,1]$ towards $G_m(x)$. Note that $\max_{0\leqslant x\leqslant 1}\left|\Phi_m^{(n)}(x)\right| \leqslant \alpha$ for $n \geqslant 1$ and $m \in \{0, 1, \ldots, N\}$. Let

$$V_{mk}^{(n)} = \frac{2\pi}{\delta_m^2}\, n \int_{(k-1)/n}^{k/n} x^{m+1/p} w(x)^{1/p}\,dx \int_0^1 y^{m+1/q} w(y)^{1/q}\,\Phi_m^{(n)}(y)\,dy.$$

Then the inequality (2.12) (Lemma 6) can be written as

$$\sum_{k=1}^{n} \int_0^{2\pi} \left|\sum_{m=0}^{N} e^{imx}\, V_{mk}^{(n)}\right|^p dx \leqslant B_p^p \sum_{k=1}^{n} \int_0^{2\pi} \left|\left(\sum_{m=0}^{N} a_{mk}^{(n)}\, e^{imx}\right)\right|^p dx. \tag{3.2}$$

Since

$$\int_0^1 \int_0^{2\pi} \left|\sum_{m=0}^{N} e^{imx}\, A_m \Phi_m^{(n)}(y)\right|^p dx\,dy = \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} dy \int_0^{2\pi} \left|\sum_{m=0}^{N} e^{imx}\, A_m \Phi_m^{(n)}(y)\right|^p dx$$

and since the continuity of function $y \mapsto \int_0^{2\pi} \left|\sum_{m=0}^{N} e^{imx} A_m \Phi_m^{(n)}(y)\right|^p dx$ (on $[0,1]$) implies that there exist points ξ_{nk} such that $\frac{k-1}{n} \leqslant \xi_{nk} \leqslant \frac{k}{n}$ $(k = 1, 2, \dots, n)$ and

$$\int_{(k-1)/n}^{k/n} dy \int_0^{2\pi} \left|\sum_{m=0}^{N} e^{imx} A_m \Phi_m^{(n)}(y)\right|^p dx = \frac{1}{n} \int_0^{2\pi} \left|\sum_{m=0}^{N} e^{imx} A_m \Phi_m^{(n)}(\xi_{nk})\right|^p dx,$$

we obtain

$$\int_0^1 \int_0^{2\pi} \left|\sum_{m=0}^{N} e^{imx} A_m \Phi_m^{(n)}(y)\right|^p dx\, dy = \sum_{k=1}^{n} \frac{1}{n} \int_0^{2\pi} \left|\sum_{m=0}^{N} e^{imx} A_m \Phi_m^{(n)}(\xi_{nk})\right|^p dx. \tag{3.3}$$

Functions $x \mapsto x^{m+1/p} w(x)^{1/p}/\delta_m^2$ are continuous $[0,1]$ (and even uniformly continuous; we define w at $x = 1$ as $w(1) = 0$ for each $m = 0, 1, \dots, N$, and so, for a given $\varepsilon > 0$ there exists a natural number n_0 such that for $n \geqslant n_0$
(3.4)

$$\left|\frac{\xi_{nk}^{m+\frac{1}{p}} w(\xi_{nk})^{1/p}}{\delta_m^2} - \frac{n}{\delta_m^2} \int_{(k-1)/n}^{k/n} x^{m+1/p} w(x)^{1/p} dx\right| < \varepsilon \left(2\pi\alpha \cdot \int_0^1 y^{1/q} w(y)^{1/q} dy\right)^{-1}$$

for each $m = 0, 1, \dots, N$ and each $k \in \{1, 2, \dots, n\}$.

Since

$$A_m \Phi_m^{(n)}(\xi_{nk}) = \frac{2\pi}{\delta_m^2} \cdot \xi_{nk}^{m+1/p} \cdot w(\xi_{nk})^{1/p} \int_0^1 y^{m+1/q} w(y)^{1/q} \Phi_m^{(n)}(y)\, dy$$

from (3.4) we get

$$\left|A_m \Phi_m^{(n)}(\xi_{nk}) - V_{mk}^{(n)}\right| < \varepsilon \tag{3.5}$$

for $n \geqslant n_0$, $m \in \{0, 1, \dots, N\}$ and $k \in \{1, 2, \dots, n\}$.

Let $n \geqslant n_0$. From

$$\begin{aligned} &\sum_{k=1}^{n} \frac{1}{n} \int_0^{2\pi} \left|\sum_{m=0}^{N} e^{imx} A_m \Phi_m^{(n)}(\xi_{nk})\right|^p dx \\ &\qquad = \sum_{k=1}^{n} \frac{1}{n} \int_0^{2\pi} \left|\sum_{m=0}^{N} e^{imx} V_{mk}^{(n)} + \sum_{m=0}^{N} e^{imx} \Big(A_m \Phi_m^{(n)}(\xi_{nk}) - V_{mk}^{(n)}\Big)\right|^p dx, \end{aligned}$$

using the inequality $(|a|+|b|)^p \leqslant 2^{p-1}(|a|^p+|b|^p)$, $p \geqslant 1$, we get

$$\sum_{k=1}^{n}\frac{1}{n}\int_0^{2\pi}\Big|\sum_{m=0}^{N} e^{imx}A_m\Phi_m^{(n)}(\xi_{nk})\Big|^p dx$$
$$\leqslant 2^{p-1}\left[\sum_{k=1}^{n}\frac{1}{n}\left(\int_0^{2\pi}\Big|\sum_{m=0}^{N} e^{imx}V_{mk}^{(n)}\Big|^p dx+\int_0^{2\pi}\Big|\sum_{m=0}^{N} e^{imx}\left(A_m\Phi_m^{(n)}(\xi_{nk})-V_{mk}^{(n)}\right)\Big|^p dx\right)\right]$$
$$\leqslant 2^{p-1}\sum_{k=1}^{n}\frac{1}{n}\int_0^{2\pi}\Big|\sum_{m=0}^{N} e^{imx}V_{mk}^{(n)}\Big|^p dx+(N+1)^p\varepsilon^p\cdot 2\pi\cdot 2^{p-1}$$

(using the inequality (3.5) in the last line).

Therefore, for $n \geqslant n_0$ we have:

$$\sum_{k=1}^{n}\frac{1}{n}\int_0^{2\pi}\Big|\sum_{m=0}^{N} e^{imx}A_m\Phi_m^{(n)}(\xi_{nk})\Big|^p dx \leqslant 2^p\pi\,\varepsilon^p\,(N+1)^p+2^{p-1}\sum_{k=1}^{n}\frac{1}{n}\int_0^{2\pi}\Big|\sum_{m=0}^{N} e^{imx}V_{mk}^{(n)}\Big|^p dx \tag{3.6}$$

From (3.2) and (3.6) we obtain (for $n \geqslant n_0$)

$$\sum_{k=1}^{n}\frac{1}{n}\int_0^{2\pi}\Big|\sum_{m=0}^{N} e^{imx}A_m\Phi_m^{(n)}(\xi_{nk})\Big|^p dx$$
$$\leqslant 2^p\pi\,\varepsilon^p\,(N+1)^p+2^{p-1}B_p^p\sum_{k=1}^{n}\frac{1}{n}\int_0^{2\pi}\Big|\sum_{m=0}^{N} e^{imx}a_{mk}^{(n)}\Big|^p dx$$
$$= 2^p\pi\,\varepsilon^p\,(N+1)^p+2^{p-1}B_p^p\int_0^{1}\int_0^{2\pi}\Big|\sum_{m=0}^{N} e^{imx}\Phi_m^{(n)}(y)\Big|^p dx\,dy.$$

If $n \geqslant n_0$, the previous inequality and (3.3) give

$$\int_0^{1}\int_0^{2\pi}\Big|\sum_{m=0}^{N} e^{imx}A_m\Phi_m^{(n)}(y)\Big|^p dx\,dy \leqslant 2^p\pi\,\varepsilon^p\,(N+1)^p+2^{p-1}B_p^p\int_0^{1}\int_0^{2\pi}\Big|\sum_{m=0}^{N} e^{imx}\Phi_m^{(n)}(y)\Big|^p dx\,dy. \tag{3.7}$$

Since the sequence $\{\Phi_m^{(n)}\}_{n=1}^{\infty}$ converges uniformly towards G_m on $[0,1]$, then, due to the boundedness of the operators A_m, we get $A_m\Phi_m^{(n)}(y)\to A_mG_m(y)$, $n\to\infty$, and so, taking the limit as $n \to \infty$ in (3.7) we obtain

$$\int_0^1 \int_0^{2\pi} \left| \sum_{m=0}^{N} e^{imx} A_m G_m(y) \right|^p dx\, dy \\ \leqslant 2^p \pi \varepsilon^p (N+1)^p + 2^{p-1} B_p^p \int_0^1 \int_0^{2\pi} \left| \sum_{m=0}^{N}, e^{imx} G_m(y) \right|^p dx\, dy.$$

Keeping in mind that $\varepsilon > 0$ is arbitrary, then, as $\varepsilon \to 0^+$ we obtain

$$\int_0^1 \int_0^{2\pi} \left| \sum_{m=0}^{N} e^{imx} A_m G_m(y) \right|^p dx\, dy \leqslant 2^{p-1} B_p^p \int_0^1 \int_0^{2\pi} \left| \sum_{m=0}^{N} e^{imx} G_m(y) \right|^p dx\, dy.$$

Consequently inequality (2.3) holds (with $c_p = 2^{1-1/p} \cdot B_p$) if the functions Φ_m are continuous which proves Theorem 1. □

Remark 3.1. The necessary condition for the boundedness of the Bergman projection P on $L^p(D)$ can be expressed as $\sup_{m \geqslant 0} \|\Phi_m\|_p \cdot \|\Phi_m\|_q < \infty$ where $\Phi_m(z) = z^m/\delta_m$, and can be obtained if we apply operator P to the functions $f_m(z) = |z|^{m/(p-1)}\, e^{im\theta}$, $(z = re^{i\theta})$, $m = 0, 1, 2, \ldots$.

Remark 3.2. It would be interesting to find a weight w that satisfies the necessary but does not satisfy the sufficient condition of Theorem 1. Such a weight should tend to 0 (when $r \to 1-$) faster then $r \mapsto \left(1 - r^2\right)^A$ but slower then $\left(1 - r^2\right)^A \exp\left(-B\left(1 - r^2\right)^{-\alpha}\right)$, $A, B, \alpha > 0$. That is connected with non-trivial analysis of the asymptotic behavior (when $\lambda \to +\infty$) of the function $\lambda \mapsto \int_0^1 r^\lambda w(r)\, dr$ which will be the subject of further investigations.

References

[1] M. R. Dostanić, *Unboundedeness of the Bergman projections on L^p spaces with exponential weights*, Proc. Edinbourgh Math. Soc. 47 (2004), 111–117.

[2] M. V. Fedoryuk, *Asymptotics Integrals and Series*, Nauka, Moscow, 1987.

[3] H. Hadenmalm, B. Korenblum, K. Zhu, *Theory of Bergman spaces*, Springer-Verlag, Berlin, 2000.

[4] B. Hollenbeck, I. E. Verbitsky, *Best constants for the Riesz projection*, J. Func. Anal. 175 (2000), 370–392.

[5] E. Seneta, *Regularly Varying Functions*, Nauka, Moscow 1985.

[6] G. N. Watson *A treatise of the theory of Bessel functions*, 2-nd edition, Cambridge University Press, Cambridge, 1966.

[7] A. Zygmund, *Trigonometric Series*, Mir, Moscow, 1965.

[8] K. Zhy, *Operator Theory in Function Spaces*, Marcel Dekker, New York and Basel, 1990.

Matematički fakultet
Studentski trg 16
Beograd
Serbia
domi@matf.bg.ac.rs

(Received 08 08 2008)
(Revised 16 04 2009)