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REGULARLY VARYING FUNCTIONS

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In memoriam Tatjana Ostrogorski.

Abstract. We consider some elementary functions of the components of a regularly varying random vector such as linear combinations, products, minima, maxima, order statistics, powers. We give conditions under which these functions are again regularly varying, possibly with a different index.

1. Introduction

Regular variation is one of the basic concepts which appears in a natural way in different contexts of applied probability theory. Feller's [**21**] monograph has certainly contributed to the propagation of regular variation in the context of limit theory for sums of iid random variables. Resnick [**50, 51, 52**] popularized the notion of multivariate regular variation for multivariate extreme value theory. Bingham et al. [**3**] is an encyclopedia where one finds many analytical results related to one-dimensional regular variation. Kesten [**28**] and Goldie [**22**] studied regular variation of the stationary solution to a stochastic recurrence equation. These results find natural applications in financial time series analysis, see Basrak et al. [**2**] and Mikosch [**39**]. Recently, regular variation has become one of the key notions for modelling the behavior of large telecommunications networks, see e.g. Leland et al. [**35**], Heath et al. [**23**], Mikosch et al. [**40**].

It is the aim of this paper to review some known results on basic functions acting on regularly varying random variables and random vectors such as sums, products, linear combinations, maxima and minima, and powers. These results are often useful in applications related to time series analysis, risk management, insurance and telecommunications. Most of the results belong to the folklore but they are often wide spread over the literature and not always easily accessible. We will give references whenever we are aware of a proved result and give proofs if this is not the case.

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We focus on functions of finitely many regularly varying random variables. With a few exceptions (the tail of the marginal distribution of a linear process, functionals with a random index) we will *not* consider results where an increasing or an infinite number of such random variables or vectors is involved. We exclude distributional limit results e.g. for partial sums and maxima of iid and strictly stationary sequences, tail probabilities of subadditive functionals acting on a regularly varying random walk (e.g. ruin probabilities) and heavy-tailed large deviation results, tails of solutions to stochastic recurrence equations.

We start by introducing the notion of a multivariate regularly varying vector in Section 2. Then we will consider sum-type functionals of regularly varying vectors in Section 3. Functionals of product-type are considered in Section 4. In Section 5 we finally study order statistics and powers.

2. Regularly varying random vectors

In what follows, we will often need the notion of a regularly varying random vector and its properties; we refer to Resnick [**50**] and [**51**, Section 5.4.2]. This notion was further developed by Tatjana Ostrogorski in a series of papers, see [**42, 43, 44, 45, 46, 47**].

Definition 2.1. An $\mathbb{R}^d$-valued random vector $\mathbf{X}$ and its distribution are said to be *regularly varying with limiting non-null Radon measure* μ on the Borel σ-field $\mathcal{B}(\overline{\mathbb{R}}_0^d)$ of $\overline{\mathbb{R}}_0^d = \overline{\mathbb{R}}^d \setminus \{\mathbf{0}\}$ if

$$\frac{P(x^{-1}\mathbf{X} \in \cdot)}{P(|\mathbf{X}| > x)} \xrightarrow{v} \mu\,, \quad x \to \infty\,. \tag{2.1}$$

Here $|\cdot|$ is any norm in $\mathbb{R}^d$ and $\xrightarrow{v}$ refers to vague convergence on $\mathcal{B}(\overline{\mathbb{R}}_0^d)$.

Since μ necessarily has the property $\mu(t\,A) = t^{-\alpha}\mu(A)$, $t > 0$, for some $\alpha > 0$ and all Borel sets A in $\overline{\mathbb{R}}_0^d$, we say that $\mathbf{X}$ is *regularly varying with index* α *and limiting measure* μ, for short $\mathbf{X} \in \mathrm{RV}(\alpha, \mu)$. If the limit measure μ is irrelevant we also write $\mathbf{X} \in \mathrm{RV}(\alpha)$. Relation (2.1) is often used in different equivalent disguises. It is equivalent to the *sequential definition of regular variation*: there exist $c_n \to \infty$ such that $n\,P(c_n^{-1}\,\mathbf{X} \in \cdot) \xrightarrow{v} \mu$. One can always choose (c_n) increasing and such that $n\,P(|\mathbf{X}| > c_n) \sim 1$. Another aspect of regular variation can be seen if one switches in (2.1) to a polar coordinate representation. Writing $\widetilde{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$ for any $\mathbf{x} \neq \mathbf{0}$ and $\mathbb{S}^{d-1} = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$ for the unit sphere in $\mathbb{R}^d$, relation (2.1) is equivalent to

$$\frac{P(|\mathbf{X}| > x\,t\,, \widetilde{\mathbf{X}} \in \cdot)}{P(|\mathbf{X}| > x)} \xrightarrow{w} t^{-\alpha}\,P(\boldsymbol{\Theta} \in \cdot) \quad \text{for all } t > 0, \tag{2.2}$$

where $\boldsymbol{\Theta}$ is a random vector assuming values in $\mathbb{S}^{d-1}$ and $\xrightarrow{w}$ refers to weak convergence on the Borel σ-field of $\mathbb{S}^{d-1}$.

Plugging the set $\mathbb{S}^{d-1}$ into (2.2), it is straightforward that the norm $|\mathbf{X}|$ is regularly varying with index α.

The special case $d = 1$ refers to a regularly varying random variable X with index $\alpha \geqslant 0$:

$$P(X > x) \sim p\, x^{-\alpha} L(x) \quad \text{and} \quad P(X \leqslant -x) \sim q\, x^{-\alpha} L(x), \ \ p + q = 1, \tag{2.3}$$

where L is a *slowly varying function*, i.e., $L(cx)/L(x) \to 1$ as $x \to \infty$ for every $c > 0$. Condition (2.3) is also referred to as a *tail balance condition*. The cases $p = 0$ or $q = 0$ are not excluded. Here and in what follows we write $f(x) \sim g(x)$ as $x \to \infty$ if $f(x)/g(x) \to 1$ or, if $g(x) = 0$, we interpret this asymptotic relation as $f(x) = o(1)$.

3. Sum-type functions

3.1. Partial sums of random variables. Consider regularly varying random variables $X_1, X_2, \ldots$, possibly with different indices. We write

$$S_n = X_1 + \cdots + X_n, \quad n \geqslant 1,$$

for the partial sums. In what follows, we write $\overline{G} = 1 - G$ for the right tail of a distribution function G on $\mathbb{R}$.

LEMMA 3.1. *Assume $|X_1|$ is regularly varying with index $\alpha \geqslant 0$ and distribution function F. Assume $X_1, \ldots, X_n$ are random variables satisfying*

$$\lim_{x\to\infty} \frac{P(X_i > x)}{\overline{F}(x)} = c_i^+ \quad \textit{and} \quad \lim_{x\to\infty} \frac{P(X_i \leqslant -x)}{\overline{F}(x)} = c_i^-, \quad i = 1, \ldots, n, \tag{3.1}$$

for some non-negative numbers $c_i^{\pm}$ and

$$\begin{aligned} \lim_{x\to\infty} \frac{P(X_i > x, X_j > x)}{\overline{F}(x)} &= \lim_{x\to\infty} \frac{P(X_i \leqslant -x, X_j > x)}{\overline{F}(x)} \\ &= \lim_{x\to\infty} \frac{P(X_i \leqslant -x, X_j \leqslant -x)}{\overline{F}(x)} = 0, \quad i \neq j. \end{aligned} \tag{3.2}$$

Then

$$\lim_{x\to\infty} \frac{P(S_n > x)}{\overline{F}(x)} = c_1^+ + \cdots + c_n^+ \quad \textit{and} \quad \lim_{x\to\infty} \frac{P(S_n \leqslant -x)}{\overline{F}(x)} = c_1^- + \cdots + c_n^-.$$

In particular, if the X_i's are independent non-negative regularly varying random variables then

$$P(S_n > x) \sim P(X_1 > x) + \cdots + P(X_n > x). \tag{3.3}$$

The proof of (3.3) can be found in Feller [**21**, p. 278], cf. Embrechts et al. [**18**, Lemma 1.3.1]. The general case of possibly dependent non-negative X_i's was proved in Davis and Resnick [**14**, Lemma 2.1]; the extension to general X_i's follows along the lines of the proof in [**14**]. Generalizations to the multivariate case are given in Section 3.6 below.

The conditions in Lemma 3.1 are sharp in the sense that they cannot be substantially improved. A condition like (3.1) with not all $c_i^{\pm}$'s vanishing is needed in order to ensure that at least one summand X_i is regularly varying. Condition (3.2) is a so-called *asymptotic independence condition*. It cannot be avoided as the trivial example $X_2 = -X_1$ for a regularly varying X_1 shows. Then (3.1) holds but (3.2) does not and $S_2 = 0$ a.s.

A partial converse follows from Embrechts et al. [**17**].

LEMMA 3.2. *Assume $S_n = X_1 + \cdots + X_n$ is regularly varying with index $\alpha \geqslant 0$ and X_i are iid non-negative. Then the X_i's are regularly varying with index α and*

$$P(S_n > x) \sim n\, P(X_1 > x)\,, \quad n \geqslant 1\,. \tag{3.4}$$

Relation (3.4) can be taken as the definition of a subexponential distribution. The class of those distributions is larger than the class of regularly varying distributions, see Embrechts et al. [**18**, Sections 1.3, 1.4 and Appendix A3]. Lemma 3.2 remains valid for subexponential distributions in the sense that subexponentiality of S_n implies subexponentiality of X_1. This property is referred to as *convolution root closure* of subexponential distributions.

PROOF. Since S_n is regularly varying it is subexponential. Then the regular variation of X_i follows from the convolution root closure of subexponential distributions, see Proposition A3.18 in Embrechts et al. [**18**]. Relation (3.4) is a consequence of (3.3). $\square$

An alternative proof is presented in the proof of Proposition 4.8 in Faÿ et al. [**20**]. It strongly depends on the regular variation of the X_i's: Karamata's Tauberian theorem (see Feller [**21**, XIII, Section 5]) is used.

In general, one cannot conclude from regular variation of $X+Y$ for independent X and Y that X and Y are regularly varying. For example, if $X+Y$ has a Cauchy distribution, in particular $X + Y \in \mathrm{RV}(1)$, then X can be chosen Poisson, see Theorem 6.3.1 on p. 71 in Lukacs [**37**]. It follows from Lemma 3.12 below that $Y \in \mathrm{RV}(1)$.

3.2. Weighted sums of iid regularly varying random variables. We assume that (Z_i) is an iid sequence of regularly varying random variables with index $\alpha \geqslant 0$ and tail balance condition (2.3) (applied to $X = Z_i$). Then it follows from Lemma 3.1 that for any real constants ψ_i

$$P(\psi_1\, Z_1 + \cdots + \psi_m\, Z_m > x) \sim P(\psi_1\, Z_1 > x) + \cdots + P(\psi_m\, Z_1 > x)\,.$$

Then evaluating $P(\psi_i\, Z_1 > x) = P(\psi_i^+ Z_i^+ > x) + P(\psi_i^- Z_i^- > x)$, where $x^\pm = 0 \vee (\pm x)$ we conclude the following result which can be found in various books, e.g. Embrechts et al. [**18**, Lemma A3.26].

LEMMA 3.3. *Let (Z_i) be an iid sequence of regularly varying random variables satisfying the tail balance condition* (2.3). *Then for any real constants ψ_i and $m \geqslant 1$,*

$$P(\psi_1\, Z_1 + \cdots + \psi_m\, Z_m > x) \sim P(|Z_1| > x) \sum_{i=1}^m \left[p\, (\psi_i^+)^\alpha + q\, (\psi_i^-)^\alpha\right]. \tag{3.5}$$

The converse of Lemma 3.3 is in general incorrect, i.e., regular variation of $\psi_1\, Z_1 + \cdots + \psi_m\, Z_m$ with index $\alpha > 0$ for an iid sequence (Z_i) does in general not imply regular variation of Z_1, an exception being the case $m = 2$ with $\psi_i > 0$, $Z_i \geqslant 0$ a.s., $i = 1, 2$, cf. Jacobsen et al. [**27**].

3.3. Infinite series of weighted iid regularly varying random variables. The question about the tail behavior of an infinite series

$$X = \sum_{i=0}^{\infty} \psi_j Z_j \tag{3.6}$$

for an iid sequence (Z_i) of regularly varying random variables with index $\alpha > 0$ and real weights occurs for example in the context of extreme value theory for linear processes, including ARMA and FARIMA processes, see Davis and Resnick [**11, 12, 13**], Klüppelberg and Mikosch [**29, 30, 31**], cf. Brockwell and Davis [**5**, Section 13.3], Resnick [**51**, Section 4.5], Embrechts et al. [**18**, Section 5.5 and Chapter 7].

The problem about the regular variation of X is only reasonably posed if the infinite series (3.6) converges a.s. Necessary and sufficient conditions are given by Kolmogorov's 3-series theorem, cf. Petrov [**48, 49**]. For example, if $\alpha > 2$ (then $\mathrm{var}(Z_i) < \infty$), the conditions $E(Z_1) = 0$ and $\sum_i \psi_i^2 < \infty$ are necessary and sufficient for the a.s. convergence of X.

The following conditions from Mikosch and Samorodnitsky [**41**] are best possible in the sense that the conditions on (ψ_i) coincide with or are close to the conditions in the 3-series theorem. Similar results, partly under stronger conditions, can be found in Lemma 4.24 of Resnick [**51**] for $\alpha \leqslant 1$ (attributed to Cline [**7, 8**]), Theorem 2.2 in Kokoszka and Taqqu [**32**] for $\alpha \in (1, 2)$.

LEMMA 3.4. *Let (Z_i) be an iid sequence of regularly varying random variables with index $\alpha > 0$ which satisfy the tail balance condition* (2.3). *Let (ψ_i) be a sequence of real weights. Assume that one of the following conditions holds:*

(1) $\alpha > 2$, $E(Z_1) = 0$ *and* $\sum_{i=0}^{\infty} \psi_i^2 < \infty$.
(2) $\alpha \in (1, 2]$, $E(Z_1) = 0$ *and* $\sum_{i=0}^{\infty} |\psi_i|^{\alpha-\varepsilon} < \infty$ *for some* $\varepsilon > 0$.
(3) $\alpha \in (0, 1]$ *and* $\sum_{i=0}^{\infty} |\psi_i|^{\alpha-\varepsilon} < \infty$ *for some* $\varepsilon > 0$.

Then

$$P(X > x) \sim P(|Z_1| > x) \sum_{i=0}^{\infty} \left[p\,(\psi_i^+)^\alpha + q\,(\psi_i^-)^\alpha \right] . \tag{3.7}$$

The conditions on (ψ_j) in the case $\alpha \in (0, 2]$ can be slightly relaxed if one knows more about the slowly varying L. In this case the following result from Mikosch and Samorodnitsky [**41**] holds.

LEMMA 3.5. *Let (Z_i) be an iid sequence of regularly varying random variables with index $\alpha \in (0, 2]$ which satisfy the tail balance condition* (2.3). *Assume that $\sum_{i=1}^{\infty} |\psi_i|^\alpha < \infty$, that the infinite series* (3.6) *converges a.s. and that one of the following conditions holds:*

(1) *There exist constants $c, x_0 > 0$ such that $L(x_2) \leqslant c\,L(x_1)$ for all $x_0 < x_1 < x_2$.*

(2) *There exist constants* $c, x_0 > 0$ *such that* $L(x_1\,x_2) \leqslant c\,L(x_1)\,L(x_2)$ *for all* $x_1, x_2 \geqslant x_0 > 0$

Then (3.7) *holds.*

Condition (2) holds for Pareto-like tails $P(Z_1 > x) \sim c\,x^{-\alpha}$, in particular for α-stable random variables Z_i and for student distributed Z_i's with α degrees of freedom. It is also satisfied for $L(x) = (\log_k x)^\beta$, any real β, where $\log_k$ is the kth time iterated logarithm.

Classical time series analysis deals with the strictly stationary linear processes

$$X_n = \sum_{i=0}^{\infty} \psi_i\, Z_{n-i}\,, \quad n \in \mathbb{Z}\,,$$

where (Z_i) is an iid white noise sequence, cf. Brockwell and Davis [**5**]. In the case of regularly varying Z_i's with $\alpha > 2$, $\mathrm{var}(Z_1)$ and $\mathrm{var}(X_1)$ are finite and therefore it makes sense to define the autocovariance function $\gamma_X(h) = \mathrm{cov}(X_0, X_h) = \mathrm{var}(Z_1) \sum_i \psi_i \psi_{i+|h|}$, $h \in \mathbb{Z}$. The condition $\sum_i \psi_i^2 < \infty$ (which is necessary for the a.s. convergence of X_n) does not only capture *short range dependent* sequences (such as ARMA processes for which $\gamma_X(h)$ decays exponentially fast to zero) but also *long range dependent* sequences (X_n) in the sense that $\sum_h |\gamma_X(h)| = \infty$. Thus Lemma 3.4 also covers long range dependent sequences. The latter class includes fractional ARIMA processes; cf. Brockwell and Davis [**5**, Section 13.2], and Samorodnitsky and Taqqu [**56**].

Notice that (3.7) is the direct analog of (3.5) for the truncated series. The proof of (3.7) is based on (3.5) and the fact that the remainder term $\sum_{i=m+1}^{\infty} \psi_i\, Z_i$ is negligible compared to $P(|Z_1| > x)$ when first letting $x \to \infty$ and then $m \to \infty$. More generally, the following result holds:

LEMMA 3.6. *Let* A *be a random variable and let* Z *be positive regularly varying random variable with index* $\alpha \geqslant 0$. *Assume that for every* $m \geqslant 0$ *there exist finite positive constants* $c_m > 0$, *random variables* A_m *and* B_m *such that the representation* $A \stackrel{d}{=} A_m + B_m$ *holds and the following three conditions are satisfied:*

$$P(A_m > x) \sim c_m\, P(Z > x)\,, \quad \text{as } x \to \infty\,,$$

$$c_m \to c_0\,, \quad \text{as } m \to \infty\,,$$

$$\lim_{m\to\infty} \limsup_{x\to\infty} \frac{P(B_m > x)}{P(Z > x)} = 0 \quad \text{and } A_m, B_m \text{ are independent for every } m \geqslant 1 \text{ or}$$

$$\lim_{m\to\infty} \limsup_{x\to\infty} \frac{P(|B_m| > x)}{P(Z > x)} = 0\,.$$

Then $P(A > x) \sim c_0\, P(Z > x)$.

PROOF. For every $m \geqslant 1$ and $\varepsilon \in (0,1)$.

$$P(A > x) \leqslant P(A_m > x(1-\varepsilon)) + P(B_m > \varepsilon\, x)\,.$$

Hence

$$\begin{aligned}
\limsup_{x\to\infty} \frac{P(A>x)}{P(Z>x)} &\leqslant \limsup_{x\to\infty} \frac{P(A_m > x(1-\varepsilon))}{P(Z>x)} + \limsup_{x\to\infty} \frac{P(B_m>\varepsilon\, x)}{P(Z>x)} \\
&= c_m\,(1-\varepsilon)^{-\alpha} + \varepsilon^{-\alpha}\,\limsup_{x\to\infty}\frac{P(B_m>\varepsilon\, x)}{P(Z>\varepsilon\, x)} \\
&\to c_0\,(1-\varepsilon)^{-\alpha} \quad \text{as } m\to\infty \\
&\to c_0 \quad \text{as } \varepsilon\downarrow 0.
\end{aligned}$$

Similarly, for independent A_m and B_m,

$$\begin{aligned}
\liminf_{x\to\infty} \frac{P(A>x)}{P(Z>x)} &\geqslant \liminf_{x\to\infty} \frac{P(A_m > x\,(1+\varepsilon)\,, B_m \geqslant -\varepsilon\, x)}{P(Z>x)} \\
&= \liminf_{x\to\infty} \frac{P(A_m > x\,(1+\varepsilon))}{P(Z>x)}\, P(B_m \geqslant -\varepsilon\, x) \\
&= c_m\,(1+\varepsilon)^{-\alpha} \to c_0\,, \quad \text{as } m\to\infty,\ \varepsilon\downarrow 0.
\end{aligned}$$

If A_m and B_m are not necessarily independent a similar bound yields

$$\begin{aligned}
\liminf_{x\to\infty} \frac{P(A>x)}{P(Z>x)} &\geqslant \liminf_{x\to\infty} \frac{P(A_m > x\,(1+\varepsilon)\,, |B_m| \leqslant \varepsilon\, x)}{P(Z>x)} \\
&\geqslant \liminf_{x\to\infty} \frac{P(A_m > x\,(1+\varepsilon))}{P(Z>x)} - \limsup_{x\to\infty}\frac{P(|B_m|>\varepsilon\, x)}{P(Z>x)} \\
&= c_m\,(1+\varepsilon)^{-\alpha} \to c_0\,, \quad \text{as } m\to\infty,\ \varepsilon\downarrow 0.
\end{aligned}$$

Combining the upper and lower bounds, we arrive at the desired result. □

We also mention that Resnick and Willekens [**53**] study the tails of the infinite series $\sum_i \mathbf{A}_i\,\mathbf{Z}_i$, where $(\mathbf{A}_i)$ is an iid sequence of random matrices, independent of the iid sequence $(\mathbf{Z}_i)$ of regularly varying vectors $\mathbf{Z}_i$.

3.4. Random sums. We consider an iid sequence (X_i) of non-negative random variables, independent of the integer-valued non-negative random variable K. Depending on the distributional tails of K and X_1, one gets rather different tail behavior for the random sum $S_K = \sum_{i=1}^K X_i$. The following results are taken from Faÿ et al. [**20**].

LEMMA 3.7. (1) *Assume X_1 is regularly varying with index $\alpha>0$, $EK<\infty$ and $P(K>x) = o(P(X_1>x))$. Then, as $x\to\infty$,*

$$P(S_K > x) \sim EK\, P(X_1 > x)\,. \tag{3.8}$$

(2) *Assume K is regularly varying with index $\beta \geqslant 0$. If $\beta = 1$, assume that $EK<\infty$. Moreover, let (X_i) be an iid sequence such that $E(X_1)<\infty$ and $P(X_1>x) = o(P(K>x))$. Then, as $x\to\infty$,*

$$P(S_K > x) \sim P(K > (E(X_1))^{-1}\, x) \sim (E(X_1))^{\beta}\, P(K>x)\,. \tag{3.9}$$

(3) *Assume S_K is regularly varying with index $\alpha>0$ and $E(K^{1\vee(\alpha+\delta)})<\infty$ for some positive δ. Then X_1 is regularly varying with index α and $P(S_K>x)\sim EK\, P(X_1>x)$.*

(4) *Assume S_K is regularly varying with index $\alpha > 0$. Suppose that $E(X_1) < \infty$ and $P(X_1 > x) = o(P(S_K > x))$ as $x \to \infty$. In the case $\alpha = 1$ and $E(S_K) = \infty$, assume that $x\,P(X_1 > x) = o(P(S_K > x))$ as $x \to \infty$. Then K is regularly varying with index α and*

$$P(S_K > x) \sim (E(X_1))^\alpha\, P(K > x)\,.$$

(5) *Assume $P(K > x) \sim c\,P(X_1 > x)$ for some $c > 0$, that X_1 is regularly varying with index $\alpha \geqslant 1$ and $E(X_1) < \infty$. Then*

$$P(S_K > x) \sim (EK + c\,(E(X_1))^\alpha)\, P(X_1 > x)\,.$$

Relations (3) and (4) are the partial converses of the corresponding relations (1) and (2). The law of large numbers stands behind the form of relation (3.9), whereas relation (3.8) is expected from the results in Section 3.1.

Relations of type (3.8) appear in a natural way in risk and queuing theory when the summands X_i are subexponential and K has a moment generating function in some neighborhood of the origin, see for example the proof of the Cramér-Lundberg ruin bound in Section 1.4 of Embrechts et al. [**18**].

For $\alpha \in (0, 2)$ some of the results in Lemma 3.7 can already be found in Resnick [**50**] and even in the earlier papers by Stam [**57**], Embrechts and Omey [**19**]. The restriction to $\alpha < 2$ is due to the fact that some of the proofs depend on the equivalence between regular variation and membership in the domain of attraction of infinite variance stable distributions. Resnick [**50**] also extends some of his results to the case when K is a stopping time.

In the following example the assumptions of Lemma 3.7 are not necessarily satisfied. Assume (X_i) is a sequence of iid positive α-stable random variables for some $\alpha < 1$. Then $S_K \overset{d}{=} K^{1/\alpha} X_1$ and $P(X_1 > x) \sim c\,x^{-\alpha}$ for some $c > 0$; cf. Feller [**21**] or Samorodnitsky and Taqqu [**56**]. If $EK < \infty$ then Breiman's result (see Lemma 4.2 below) yields $P(S_K > x) \sim EK\,P(X > x)$ in agreement with (3.8). If $EK = \infty$ we have to consider different possibilities. If K is regularly varying with index 1, then $K^{1/\alpha} \in \mathrm{RV}(\alpha)$. Then we are in the situation of Lemma 4.2 below and S_K is regularly varying with index α. If we assume that $K \in \mathrm{RV}(\beta)$ for some $\beta < 1$, then $K^{1/\alpha} \in \mathrm{RV}(\beta\alpha)$ and the results of Lemma 4.2 ensure that $P(S_K > x) \sim E(X^{\alpha\beta})P(K^{1/\alpha} > x)$.

The latter result can be extended by using a Tauberian argument.

LEMMA 3.8. *Assume that $K, X_1 > 0$ are regularly varying with indices $\beta \in [0, 1)$ and $\alpha \in [0, 1)$, respectively. Then*

$$P(S_K > x) \sim P(K > [P(X > x)]^{-1}) \sim P(M_K > x)\,,$$

where $M_n = \max_{i=1,\dots,n} X_i$.

PROOF. By Karamata's Tauberian theorem (see Feller [**21**, XIII, Section 5]) $1 - E(\mathrm{e}^{-sK}) \sim s^\beta L_K(1/s)$ as $s \downarrow 0$ provided that $P(K > x) = x^{-\beta} L_K(x)$ for some slowly varying function L. In the same way, $1 - E(\mathrm{e}^{-tX_1}) \sim t^\alpha L_X(1/t)$ as $t \downarrow 0$.

Then

$$\begin{aligned}1-E(\mathrm{e}^{-t\,S_K}) &= 1-E\left(\exp\left\{K\,\log\left(E\left(\mathrm{e}^{-t\,X_1}\right)\right)\right\}\right)\\ &\sim \left[-\log\left(E\left(\mathrm{e}^{-t\,X_1}\right)\right)\right]^{\beta}\,L_K(1/[-\log\left(E\left(\mathrm{e}^{-t\,X_1}\right)\right)])\\ &\sim \left[1-E\left(\mathrm{e}^{-t\,X_1}\right)\right]^{\beta} L_K\left(1/\left[1-E\left(\mathrm{e}^{-t\,X_1}\right)\right]\right)\\ &\sim [t^\alpha L_X(1/t)]^\beta L_K([t^\alpha L_X(1/t)]^{-1})\\ &= t^{\alpha\beta}L(1/t)\,,\end{aligned}$$

where $L(x)=L_X^\beta(x)\,L_K(x^\alpha/L_X(x))$ is a slowly varying function. Again by Karamata's Tauberian theorem, $P(S_K>x)\sim x^{-\alpha\,\beta}\,L(x)$. Notice that the right-hand side is equivalent to the tail $P(K>[P(X_1>x)]^{-1})\sim P(M_K>x)$. The latter equivalence follows from (5.1) below. □

3.5. Linear combinations of a regularly varying random vector. Assume $\mathbf{X}\in \mathrm{RV}(\alpha,\mu)$ and let $\mathbf{c}\in\mathbb{R}^d$, $\mathbf{c}\neq 0$, be a constant. The set $A_{\mathbf{c}}=\{\mathbf{x}:\mathbf{c}'\mathbf{x}>1\}$ is bounded away from zero and $\mu(\partial A_{\mathbf{c}})=0$. Indeed, it follows from the scaling property of μ that $\mu(\{\mathbf{x}:\mathbf{c}'\mathbf{x}=y\})=y^{-\alpha}\,\mu(\{\mathbf{x}:\mathbf{c}'\mathbf{x}=1\})$, $y>0$. If $\mu(\{\mathbf{x}:\mathbf{c}'\mathbf{x}=1\})>0$ this would contradict the finiteness of $\mu(A_{\mathbf{c}})$.

Therefore, from (2.1),

$$\frac{P(x^{-1}\mathbf{X}\in A_{\mathbf{c}})}{P(|\mathbf{X}|>x)}=\frac{P(\mathbf{c}'\mathbf{X}>x)}{P(|\mathbf{X}|>x)}\to\mu(A_{\mathbf{c}})\,.$$

We conclude the following, see also Resnick [**52**], Section 7.3.

LEMMA 3.9. *For $\mathbf{c}\in\mathbb{R}$, $\mathbf{c}\neq\mathbf{0}$, $\mathbf{c}'\mathbf{X}$ is regularly varying with index α if $\mu(A_{\mathbf{c}})\neq 0$. In general,*

$$\frac{P(\mathbf{c}'\mathbf{X}>x)}{P(|\mathbf{X}|>x)}\to\mu(\{\mathbf{x}:\mathbf{c}'\mathbf{x}>1\})\,,$$

where the right-hand side possibly vanishes. In particular, if $\mu(\{\mathbf{x}:\mathbf{c}_i'\mathbf{x}>1\})>0$ for the basis vector $\mathbf{c}_i=(0,\dots,0,1,0,\dots,0)'$ with 1 in the ith component then $(X_i)^+$ is regularly varying with index α.

A natural question arises: given that

$$\frac{P(\mathbf{c}'\mathbf{X}>x)}{L(x)x^{-\alpha}}=C(\mathbf{c})\quad\text{for all } \mathbf{c}\neq\mathbf{0} \text{ and } C(\mathbf{c})\neq 0 \text{ for at least one } \mathbf{c}\tag{3.10}$$

holds for some function C, is then $\mathbf{X}$ regularly varying in the sense of Definition 2.1? This would yield a Cramér–Wold device analog for regularly varying random vectors.

The answer to this question is not obvious. Here are some partial answers. The first three statements can be found in Basrak et al. [**1**], the last statements are due to Hult and Lindskog [**26**]. Statement (5) was already mentioned (without proof) in Kesten [**28**].

LEMMA 3.10. (1) (3.10) *implies that $\mathbf{X}$ is regularly varying with a unique spectral measure if α is not an integer.*

(2) (3.10) *when restricted to* $\mathbf{c} \in [0,\infty)^d \setminus \{\mathbf{0}\}$ *implies that* $\mathbf{X}$ *is regularly varying with a unique spectral measure if* $\mathbf{X}$ *has non-negative components and* α *is positive and not an integer,*

(3) (3.10) *implies that* $\mathbf{X}$ *is regularly varying with a unique spectral measure if* $\mathbf{X}$ *has non-negative components and* α *is an odd integer.*

(4) (1) *and* (2) *cannot be extended to integer* α *without additional assumptions on the distribution of* $\mathbf{X}$*. There exist regularly varying* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *both satisfying* (3.10) *with the same function* C *but having different spectral measures.*

(5) *For integer* $\alpha > 0$*, there exist non-regularly varying* $\mathbf{X}$ *satisfying* (3.10).

3.6. Multivariate extensions. In this section we assume that $\mathbf{X}_1$ and $\mathbf{X}_2$ are random vectors with values in $\mathbb{R}^d$. The following result due to Hult and Lindskog [**24**], see also Resnick [**52**, Section 7.3], yields an extension of Lemma 3.1 to regularly varying vectors.

LEMMA 3.11. *Assume that* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are independent regularly varying such that* $n\,P(c_n^{-1}\mathbf{X}_i \in) \xrightarrow{v} \mu_i$*,* $i=1,2$*, for some sequence* $c_n \to \infty$ *and Radon measures* μ_i*,* $i=1,2$*. Then* $\mathbf{X}_1+\mathbf{X}_2$ *is regularly varying and* $n\,P(c_n^{-1}(\mathbf{X}_1+\mathbf{X}_2) \in \cdot) \xrightarrow{v} \mu_1+\mu_2$.

The following lemma is often useful.

LEMMA 3.12. *Assume* $\mathbf{X}_1 \in \mathrm{RV}(\alpha,\mu)$ *and* $P(|\mathbf{X}_2| > x) = o(P(|\mathbf{X}_1| > x))$ *as* $x \to \infty$*. Then* $\mathbf{X}_1+\mathbf{X}_2 \in \mathrm{RV}(\alpha,\mu)$.

PROOF. It suffices to show that

$$P(x^{-1}(\mathbf{X}_1+\mathbf{X}_2) \in A) \sim P(x^{-1}\mathbf{X}_1 \in A)\,, \tag{3.11}$$

where A is any rectangle in $\mathbb{R}^d$ bounded away from zero. The latter class of sets generates vague convergence in $\mathcal{B}(\overline{\mathbb{R}}_0^d)$ and satisfies $\mu(\partial A)=0$. Assume that $A = [\mathbf{a},\mathbf{b}] = \{\mathbf{x} : \mathbf{a} \leqslant \mathbf{x} \leqslant \mathbf{b}\}$ for two vectors $\mathbf{a} < \mathbf{b}$, where $<$, $\leqslant$ are defined in the natural componentwise way. Write $\mathbf{a}^{\pm\varepsilon} = (a_1 \pm \varepsilon, \cdots, a_d \pm \varepsilon)$ and define $\mathbf{b}^{\pm\varepsilon}$ correspondingly. Define the rectangles $A^{-\varepsilon} = [\mathbf{a}^{-\varepsilon}, \mathbf{b}^{\varepsilon}]$ and $A^{\varepsilon} = [\mathbf{a}^{\varepsilon}, \mathbf{b}^{-\varepsilon}]$ in the same way as A. For small ε these sets are not empty, bounded away from zero and $A^{\varepsilon} \subset A \subset A^{-\varepsilon}$.

For small $\varepsilon > 0$,

$$\begin{aligned}
&P(x^{-1}(\mathbf{X}_1+\mathbf{X}_2) \in A)\\
&\quad = P(x^{-1}(\mathbf{X}_1+\mathbf{X}_2) \in A\,, x^{-1}|\mathbf{X}_2| > \varepsilon) + P(x^{-1}(\mathbf{X}_1+\mathbf{X}_2) \in A\,, x^{-1}|\mathbf{X}_2| \leqslant \varepsilon)\\
&\quad \leqslant P(|\mathbf{X}_2| > x\,\varepsilon) + P(x^{-1}\mathbf{X}_1 \in A^{-\varepsilon})\,.
\end{aligned}$$

Then

$$\begin{aligned}
\limsup_{x\to\infty} \frac{P(x^{-1}(\mathbf{X}_1+\mathbf{X}_2) \in A)}{P(|\mathbf{X}_1| > x)} &\leqslant \limsup_{x\to\infty} \frac{P(|\mathbf{X}_2| > x\,\varepsilon)}{P(|\mathbf{X}_1| > x)} + \limsup_{x\to\infty} \frac{P(x^{-1}\mathbf{X}_1 \in A^{-\varepsilon})}{P(|\mathbf{X}_1| > x)}\\
&= \mu(A^{-\varepsilon}) \downarrow \mu(A) \quad \text{as } \varepsilon \downarrow 0.
\end{aligned}$$

In the last step we used that A is a μ-continuity set. Similarly,

$$\begin{aligned} P(x^{-1}(\mathbf{X}_1+\mathbf{X}_2)\in A) &\geqslant P(x^{-1}\mathbf{X}_1\in A^{\varepsilon}\,,x^{-1}|\mathbf{X}_2|\leqslant\varepsilon) \\ &\geqslant P(x^{-1}\mathbf{X}_1\in A^{\varepsilon})-P(|\mathbf{X}_2|>\varepsilon\, x)\,. \end{aligned}$$

Then

$$\liminf_{x\to\infty}\frac{P(x^{-1}(\mathbf{X}_1+\mathbf{X}_2)\in A)}{P(|\mathbf{X}_1|>x)}\geqslant\liminf_{x\to\infty}\frac{P(x^{-1}\mathbf{X}_1\in A^{\varepsilon})}{P(|\mathbf{X}_1|>x)}=\mu(A^{\varepsilon})\uparrow\mu(A)\quad\text{as }\varepsilon\downarrow 0.$$

In the last step we again used that A is a μ-continuity set.

Now collecting the upper and lower bounds, we arrive at the desired relation (3.11). □

4. Product-type functions

Products are more complicated objects than sums. Their asymptotic tail behavior crucially depends on which tail of the factors in the product dominates. If the factors have similar tail behavior the results become more complicated.

Assume for the moment $d=2$. The set $A=\{\mathbf{x}:x_1\,x_2>1\}$ is bounded away from zero and therefore regular variation of $\mathbf{X}$ implies that the limit

$$\frac{P(X_1\,X_2>x^2)}{P(|\mathbf{X}|>x)}=\frac{P(x^{-1}(X_1,X_2)\in A)}{P(|\mathbf{X}|>x)}\to\mu(A)$$

exists. However, the quantity $\mu(A)$ can be rather non-informative, for example, in the two extreme cases: $\mathbf{X}=(X,X)$ for a non-negative regularly varying X with index α and $\mathbf{X}=(X_1,X_2)$, where X_1 and X_2 are independent copies of X. In the former case, with the max-norm $|\cdot|$, $\mu(A)=1$, and in the latter case $\mu(A)=0$ since μ is concentrated on the axes.

Thus, the knowledge about regular variation of $\mathbf{X}$ is useful when $\mu(A)>0$, i.e., when the components of $\mathbf{X}$ are not (asymptotically) independent. However, if $\mu(A)=0$ the regular variation of $\mathbf{X}$ is too crude in order to determine the tails of the distribution of the products of the components.

4.1. One-dimensional results. In the following result we collect some of the well known results about the tail behavior of the product of two independent non-negative random variables.

LEMMA 4.1. *Assume that X_1 and X_2 are independent non-negative random variables and that X_1 is regularly varying with index $\alpha>0$.*

(1) *If either X_2 is regularly varying with index $\alpha>0$ or $P(X_2>x)=o(P(X_1>x))$ then X_1X_2 is regularly varying with index α.*

(2) *If X_1,X_2 are iid such that $E(X_1^{\alpha})=\infty$ then $P(X_1X_2>x)/P(X_1>x)\to\infty$.*

(3) *If X_1,X_2 are iid such that $E(X_1^{\alpha})<\infty$, then the only possible limit of $P(X_1\,X_2>x)/P(X_1>x)$ as $x\to\infty$ is given by $2E(X_1^{\alpha})$ which is attained under the condition*

$$\lim_{M\to\infty}\limsup_{x\to\infty}\frac{P(X_1X_2>x\,,M<X_1X_2\leqslant x/M)}{P(X_1>x)}=0\,.$$

(4) *Assume* $P(X_1 > x) \sim c^\alpha x^{-\alpha}$ *for some* $c > 0$. *Then for iid copies* $X_1, \dots, X_n$ *of* X_1, $n \geqslant 1$,

$$P(X_1 \cdots X_n > x) \sim \frac{\alpha^{n-1} c^{n\alpha}}{(n-1)!} \, x^{-\alpha} \log^{n-1} x \,.$$

PROOF. (1) was proved in Embrechts and Goldie [**16**, p. 245].

(2) The following decomposition holds for any $M > 0$:

$$\frac{P(X_1 X_2 > x)}{P(X_1 > x)} = \tag{4.1}$$
$$\int_{(0,M]} \frac{P(X_2 > x/y)}{P(X_1 > x)} \, dP(X_1 \leqslant y) + \int_{(M,\infty)} \frac{P(X_2 > x/y)}{P(X_1 > x)} \, dP(X_1 \leqslant y) = I_1 + I_2 \,.$$

By the uniform convergence theorem, $P(X_1 > x/y)/P(X_1 > x) \to y^{-\alpha}$ uniformly for $y \in (0, M]$. Hence

$$\begin{aligned} I_1 &\to \int_0^M y^\alpha \, dP(X_1 \leqslant y) \,, \quad x \to \infty \,, \\ &\to E(X_1^\alpha) \,, \quad M \to \infty \,. \end{aligned}$$

Hence, if $E(X_1^\alpha) = \infty$, (2) applies.

(3) It follows from Chover et al. [**6**] that the only possible limits of $P(X_1 X_2 > x) / P(X_1 > x)$ are $2E(X_1^\alpha)$. The proof follows now from Davis and Resnick [**12**, Proposition 3.1].

(4) We start with the case when $P(Y_i/c > x) = x^{-\alpha}$, for $x \geqslant 1$ and an iid sequence (Y_i). Then $\sum_{i=1}^n \log(Y_i/c)$ is $\Gamma(\alpha, n)$ distributed:

$$P\left(\sum_{i=1}^n \log(Y_i/c) > x \right) = \frac{\alpha^n}{(n-1)!} \int_0^x y^{n-1} \, \mathrm{e}^{-\alpha y} \, dy \,, \quad x > 0 \,.$$

Then, by Karamata's theorem,

$$\begin{aligned} P\left(\prod_{i=1}^n (Y_i/c) > x/c^n \right) &= \frac{\alpha^n}{(n-1)!} \int_0^{\log(x/c^n)} y^{n-1} \, \mathrm{e}^{-\alpha y} \, dy \\ &= \frac{\alpha^n}{(n-1)!} \int_1^{x/c^n} (\log z)^{n-1} \, z^{-\alpha-1} \, dz \\ &\sim \frac{\alpha^{n-1}}{(n-1)!} (\log(x/c^n))^{n-1} \, (x/c^n)^{-\alpha} \\ &\sim \frac{\alpha^{n-1} \, c^{n\alpha}}{(n-1)!} (\log x)^{n-1} \, x^{-\alpha} \,. \end{aligned} \tag{4.2}$$

Next consider an iid sequence (X_i) with $P(X_1 > x) \sim c^\alpha x^{-\alpha}$, independent of (Y_i), and assume without loss of generality that $c = 1$. Denote the distribution function of $\prod_{i=2}^n Y_i$ by G and let $h(x) \to \infty$ be any increasing function such that $x/h(x) \to \infty$. Then

$$\begin{aligned}
P\left(X_1 \prod_{i=2}^{n} Y_i > x\right) &= \int_0^\infty P(X_1 > x/y)\, dG(y) \\
&= \int_0^{h(x)} \frac{P(X_1 > x/y)}{P(Y_1 > x/y)}\, P(Y_1 > x/y)\, dG(y) \\
&\quad + \int_{h(x)}^\infty P(X_1 > x/y)\, dG(y) \\
&= I_1(x) + I_2(x)\,.
\end{aligned}$$

For any $\varepsilon > 0$, sufficiently large x and $y \in (0, h(x))$,

$$1 - \varepsilon \leqslant \frac{P(X_1 > x/y)}{P(Y_1 > x/y)} \leqslant 1 + \varepsilon\,.$$

Hence

$$I_1(x) \sim \int_0^{h(x)} P(Y_1 > x/y)\, dG(y)\,.$$

Now choose, for example, $h(x) = x/\log\log x$. Then

$$I_2(x) \leqslant \overline{G}(x/\log\log x) = O((x/(\log\log x))^{-\alpha} \log^{n-2} x) = o(x^{-\alpha} \log^{n-1} x)\,.$$

A similar argument yields

$$\int_{h(x)}^\infty P(Y_1 > x/y)\, dG(y) = o(x^{-\alpha}\, \log^{n-1} x)\,.$$

In view of (4.2) we conclude that

$$P\biggl(X_1 \prod_{i=2}^{n} Y_i > x\biggr) \sim I_1(x) \sim P\biggl(\prod_{i=1}^{n} Y_i > x\biggr)$$

A similar argument shows that we may replace in the left probability any Y_i, $i = 2, \ldots, n$, by X_i. This proves (4). □

Under the assumption $\limsup_{x\to\infty} x^\alpha P(X_i > x) < \infty$ upper bounds similar to (4) were obtained by Rosiński and Woyczyński [**54**]. The tail behavior of products of independent random variables is then also reflected in the tail behavior of polynomial forms of iid random variables with regularly varying tails and in multiple stochastic integrals driven by α-stable Lévy motion; see Kwapień and Woyczyński [**34**].

In the following results for the product $X_1 X_2$ of non-negative independent random variables X_1 and X_2 we assume that the tail of one of the factors dominates the tail of the other one.

Lemma 4.2. *Assume X_1 and X_2 are non-negative independent random variables and that X_1 is regularly varying with index $\alpha > 0$.*

(1) *If there exists an $\varepsilon > 0$ such that $E(X_2^{\alpha+\varepsilon}) < \infty$, then*

$$P(X_1\, X_2 > x) \sim E(X_2^\alpha)\, P(X_1 > x)\,. \tag{4.3}$$

(2) *Under the assumptions of part* (1),

$$\sup_{x\geqslant y}\left|\frac{P(X_1\,X_2>x)}{P(X_1>x)}-E(X_2^\alpha)\right|\to 0\,,\quad \text{as } y\to\infty.$$

(3) *If* $P(X_1>x)\sim c\,x^{-\alpha}$ *and* $E(X_2^\alpha)<\infty$ *then* (4.3) *holds.*

(4) *If* $P(X_2>x)=o(P(X_1\,X_2>x))$ *then* $X_1\,X_2$ *is regularly varying with index* α.

Proof. Part (1) is usually attributed to Breiman [**4**] although he did not prove the result for general α. However, the proof remains the same for all $\alpha>0$, and it also applies to the proof of (3): a glance at relation (4.1) shows that one has to prove $\lim_{M\to\infty}\limsup_{x\to\infty} I_2=0$ by applying a domination argument. An alternative proof of (1) is given in Cline and Samorodnitsky [**9**, Theorem 3.5(v)]. Part (3) is hardly available as an explicit result; but is has been implicitly used in various disguises e.g. in the books by Samorodnitsky and Taqqu [**56**] and in Ledoux and Talagrand [**36**]. Part (2) is Lemma 2.2 in Konstantinides and Mikosch [**33**]. Part (4) is due to Embrechts and Goldie [**16**], see also Theorem 3.5(iii) in Cline and Samorodnitsky [**9**]. □

Denisov and Zwart [**15**] give best possible conditions on the distributions of X_1 and X_2 such that Breiman's result (4.3) holds.

The lemma has applications in financial time series analysis. Indeed, financial time series are often assumed to be of the form $X_n=\sigma_n\,Z_n$, where the volatility σ_n is a measurable function of past Z_i's, (Z_i) is an iid sequence and (X_n) is strictly stationary. For example, this is the case for a strictly stationary GARCH(p,q) process, see e.g. Mikosch [**39**]. In many cases of interest, Z_n is light-tailed, e.g. standard normal, but σ_n is regularly varying with some positive index α. Breiman's result implies $P(X_1>x)\sim E(Z_1^\alpha)\,P(\sigma_1>x)$. Another case of interest is a stochastic volatility model, where the strictly stationary volatility sequence (σ_n) is independent of the iid noise sequence (Z_n). A convenient example is given when $\log\sigma_n$ constitutes a Gaussian stationary process. Then σ_n is log-normal. If Z_n is regularly varying with index α then Breiman's result yields $P(X_1>x)\sim E(\sigma_1^\alpha)\,P(Z_1>x)$.

The following results contain partial converses to Breiman's result, i.e., if we know that X_1X_2 is regularly varying what can be said about regular variation of X_1 or X_2?

Lemma 4.3. *Assume that* X_1 *and* X_2 *are independent non-negative random variables and that* X_1X_2 *is regularly varying with positive index* α.

(1) *Assume that* X_2^p *for some* $p>0$ *has a Lebesgue density of the form* $f(x)=c_0\,x^\beta\,\mathrm{e}^{-c\,x^\tau}$, $x>0$, *for some constants* $\tau,c,c_0>0$, $\beta\in\mathbb{R}$, *such that* $x^\beta P(X_1>x^{-1})$ *is ultimately monotone in* x. *Then* X_1 *is regularly varying with index* α *and Breiman's result* (4.3) *holds.*

(2) *Assume* $P(X_1>x)=x^{-\alpha}$, $x\geqslant 1$. *Then* $X_2\in\mathrm{RV}(\beta)$ *for some* $\beta<\alpha$ *if and only if* $X_1X_2\in\mathrm{RV}(\beta)$.

(3) *There exist* X_1, X_2 *such that* $E(X_1^\alpha)<\infty$, X_1 *and* X_2 *are not regularly varying and* $P(X_1>x)=o(P(X_1X_2>x))$.

PROOF. (1) The idea is similar to the proof in Basrak et al. [**2**, Lemma 2.2], who assumed that X_2 is the absolute value of a normal random variable. Notice that if $X_1X_2 \in \mathrm{RV}(\alpha)$ then $(X_1X_2)^p \in \mathrm{RV}(\alpha/p)$ for $p > 0$. Therefore assume without loss of generality that $p = 1$ and we also assume for simplicity that $c = 1$. Since X_1X_2 is regularly varying there exists a slowly varying function L such that

$$\begin{aligned} L(x)\, x^{-\alpha} &= P(X_1\, X_2 > x) = \int_0^\infty P(X_1 > x/y)\, f(y)\, dy \\ &= c_0\, x^{1+\beta} \int_0^\infty P(X_1 > z^{-1})\, z^\beta \mathrm{e}^{\,-(z\,x)^\tau}\, dz \\ &= c_0\, \tau^{-1}\, x^{1+\beta} \int_0^\infty P(X_1 > v^{-1/\tau})\, v^{(\beta+1)/\tau-1}\, \mathrm{e}^{\,-v\, x^\tau}\, dv \\ &= x^{1+\beta} \int_0^\infty \mathrm{e}^{\,-r\, x^\tau}\, dU(r)\,, \end{aligned}$$

where

$$U(r) = \frac{c_0}{\tau} \int_0^r P(X_1 > v^{-1/\tau}) v^{(\beta+1)/\tau-1}\, dv = c_0 \int_0^{r^{1/\tau}} P(X_1 > z^{-1})\, z^\beta\, dz\,.$$

Thus

$$L(x^{1/\tau})\, x^{-(\alpha+\beta+1)/\tau} = \int_0^\infty \mathrm{e}^{\,-r\,x}\, dU(r)\,.$$

An application of Karamata's Tauberian theorem (see Feller [**21**, XIII, Section 5]) yields that

$$U(x) \sim \frac{L(x^{-1/\tau})\, x^{(\alpha+\beta+1)/\tau}}{\Gamma((\alpha+\beta+1)/\tau+1)}\,, \quad x \to \infty\,.$$

By assumption, $P(X_1 > z^{-1})\, z^\beta$ is ultimately monotone. Then the monotone density theorem (see Bingham et al. [**3**]) implies that

$$P(X_1 > x) \;\sim\; \frac{\tau}{c_0\, \Gamma((\alpha+\beta+1)/\tau)}\, \frac{L(x)}{x^\alpha}\,.$$

(2) This part is proved in Maulik and Resnick [**38**].

(3) An example of this kind, attributed to Daren Cline, is given in Maulik and Resnick [**38**]. □

Results for products of independent positive random variables can also be obtained by taking logarithms and then applying the corresponding results for regularly varying summands. The following example is in this line of thought.

LEMMA 4.4. *Let X_i be positive iid and such that $(\log X_1)_+ \in \mathrm{RV}(\alpha)$ for some $\alpha \geqslant 0$ and $P(X_1 \leqslant x^{-1}) = o(P(X_1 > x))$. Then for $n \geqslant 1$,*

$$P(X_1 \cdots X_n > x) \sim n\, P(X_1 > x)\,.$$

PROOF. We have for $x > 0$,

$$\begin{aligned} P(X_1 \cdots X_n > x) &= P(\log X_1 + \cdots + \log X_n > \log x) \\ &\sim n\, P(\log X_1 > \log x) = n\, P(X_1 > x)\,. \end{aligned}$$

This follows e.g. by an application of Lemma 3.3. Indeed, $(\log X_1)_+$ is regularly varying and by assumption, for $x > 0$,

$$P((\log X_1)_- > x) = P(X_1 < \mathrm{e}^{-x}) = o(P(X_1 > \mathrm{e}^{x})) = o(P((\log X_1)_+ > x)). \quad \square$$

Results for random products are rather rare. The following example is due to Samorodnitsky (personal communication). Extensions can be found in Cohen and Mikosch [**10**].

LEMMA 4.5. *Let (X_i) be an iid sequence with $P(X_1 > x) = c\,x^{-1}$ for some $x \geqslant c \geqslant 1$, K be Poisson(λ) distributed and independent of (X_i). Write $P_K = \prod_{i=1}^K X_i$. Then $P(P_K = 0) = \mathrm{e}^{-\lambda}$ and P_K has density f_P on (c,∞) satisfying as $x \to \infty$,*

$$f_P(x) \sim \frac{\mathrm{e}^{-\lambda} c^{-\lambda\, c} (\lambda\, c)^{1/4}}{2\sqrt{\pi}}\, x^{-2}\, (\log x)^{-3/4}\, \mathrm{e}^{\,2\,(\lambda\, c)^{1/2}\,(\log x)^{1/2}}\,,$$

and hence

$$P(P_K > x) \sim \frac{\mathrm{e}^{-\lambda} c^{-\lambda\, c} (\lambda\, c)^{1/4}}{2\sqrt{\pi}}\, x^{-1}\, (\log x)^{-3/4}\, \mathrm{e}^{\,2\,(\lambda\, c)^{1/2}\,(\log x)^{1/2}}\,.$$

Various results of this section can be extended to subexponential and even long-tailed distributions, see Cline and Samorodnitsky [**9**]. Resnick [**52**, Section 7.3.2] also treats the case of products with dependent regularly varying factors. Hult and Lindskog [**25**] extended Breiman's result in a functional sense to stochastic integrals $(\int_0^t \xi_{s-} d\eta_s)_{0\leqslant t\leqslant 1}$, where η is a Lévy process with regularly varying Lévy measure and ξ is a predictable integrand.

4.2. Multivariate extensions. Breiman's result (4.3) has a multivariate analog. It was proved in the context of regular variation for the finite-dimensional distributions of GARCH processes where multivariate products appear in a natural way; see Basrak et al. [**2**].

LEMMA 4.6. *Let $\mathbf{A}$ be an $m \times d$ random matrix such that $E(\|\mathbf{A}\|^{\alpha+\varepsilon}) < \infty$ for some matrix norm $\|\cdot\|$ and $\varepsilon > 0$. If $\mathbf{X} \in \mathrm{RV}(\alpha,\mu)$ assumes values in $\mathbb{R}^d$ and is independent of $\mathbf{A}$, then $\mathbf{AX}$ is regularly varying with index α and*

$$\frac{P(\mathbf{AX} \in \cdot)}{P(|\mathbf{X}| > x)} \xrightarrow{v} E\left(\mu\{\mathbf{x} : \mathbf{A}\,\mathbf{x} \in \cdot\}\right)\,.$$

5. Other functions

5.1. Powers. Let $\mathbf{X} \geqslant \mathbf{0}$ be a regularly varying random vector with index $\alpha > 0$. It is straightforward from the definition of multivariate regular variation that for $p > 0$, $\mathbf{X}^p = (X_1^p,\dots,X_d^p)$ is regularly varying with index α/p. This can be seen from the polar coordinate representation of regular variation with $|\cdot|$ the max-norm, see (2.2):

$$\frac{P(|\mathbf{X}^p| > t\,x\,, \widetilde{\mathbf{X}^p} \in \cdot)}{P(|\mathbf{X}^p| > x)} = \frac{P(|\mathbf{X}| > (t\,x)^{1/p}\,, (\widetilde{\mathbf{X}})^p \in \cdot)}{P(|\mathbf{X}| > x^{1/p})} \to t^{-\alpha/p}\, P(\mathbf{\Theta}^p \in \cdot)\,.$$

5.2. Polynomials. We consider a sum $S_n = X_1 + \cdots + X_n$ of iid non-negative random variables X_i. Assume that X_1 is regularly varying with index $\alpha > 0$. By virtue of Lemma 3.2 this is equivalent to the fact that S_n is regularly varying and $P(S_n > x) \sim n\, P(X_1 > x)$. Then S_n^p for $p > 0$ is regularly varying with index α/p and

$$P(S_n^p > x) \sim n\, P(X_1^p > x) \sim P(X_1^p + \cdots + X_n^p > x)\,.$$

The latter relation has an interesting consequence for integers $k > 1$. Then one can write

$$S_n^k = \sum_{i=1}^n X_i^k + \sum X_{i_1} \cdots X_{i_k}\,,$$

where the second sum contains the off-diagonal products. It follows from the results in Section 4 that this sum consists of regularly varying summands whose index does not exceed $\alpha/(k-1)$. Hence, by Lemma 3.12, the influence of the off-diagonal sum on the tail of S_n^k is negligible. The regular variation of polynomial functions of the type

$$\sum_{1 \leqslant i_1, \ldots, i_k \leqslant n} c_{i_1 \ldots i_k}\, X_{i_1}^{p_{i_1}} \cdots X_{i_k}^{p_{i_k}}$$

for non-negative coefficients $c_{i_1 \ldots i_k}$ and integers $p_i \geqslant 0$ can be handled by similar ideas.

5.3. Maxima. Assume that $\mathbf{X} \in \mathrm{RV}(\alpha, \mu)$ and write $M_d = \max_{i=1,\ldots,d} X_i$ for the maximum of the components of $\mathbf{X}$. The set $A = \{\mathbf{x} : x_i > 1 \text{ for some } i\}$ is bounded away from zero and $\mu(\partial A) = 0$. Then

$$\frac{P(M_d > x)}{P(|\mathbf{X}| > x)} = \frac{P(x^{-1}\mathbf{X} \in A)}{P(|\mathbf{X}| > x)} \to \mu(A)\,.$$

If $\mu(A) > 0$, M_d is regularly varying with index α. In particular, if $\mathbf{X}$ has non-negative components and $|\cdot|$ is the max-norm, then $M_d = |\mathbf{X}|$ which is clearly regularly varying.

If $X_1, \ldots, X_n$ are independent, direct calculation with

$$\frac{P(X_i > x)}{P(|X_i| > x)} \to p_i \quad \text{and} \quad \frac{P(|X_i| > x)}{P(|\mathbf{X}| > x)} \to c_i\,,$$

yields the following limits

$$\frac{P(M_d > x)}{P(|\mathbf{X}| > x)} \sim \sum_{i=1}^d p_i\, \frac{P(|X_i| > x)}{P(|\mathbf{X}| > x)} \to \sum_{i=1}^d c_i\, p_i\,.$$

For iid X_i we obtain $\sum_{i=1}^d c_i\, p_i = d\, p$.

Next we consider maxima with a random index.

Lemma 5.1. *Assume that K is independent of the sequence (X_i) of iid random variables with distribution function F and right endpoint x_F.*

(1) *If $EK < \infty$ then*

$$P(M_K > x) \sim EK\, P(X_1 > x)\,, \quad x \uparrow x_F\,.$$

Hence X_1 is regularly varying with index α if and only if M_K is regularly varying with index α.

(2) *If $EK = \infty$ assume that $P(K > x) = L(x)x^{-\alpha}$ for some $\alpha \in [0,1)$ and a slowly varying function L. Then*

$$P(M_K > x) \sim (\overline{F}(x))^{\alpha} L(1/\overline{F}(x)), \quad x \uparrow x_F . \tag{5.1}$$

Hence X_1 is regularly varying with index $p > 0$ if and only if M_K is regularly varying with index $p\alpha$.

Proof. (1) Write $F(x) = P(X_i \leqslant x)$. Then by monotone convergence, as $x \uparrow x_F$,

$$P(M_K > x) = \overline{F}(x)\, E\left[1 + F(x) + \cdots + F^{K-1}(x)\right] \sim EK\, \overline{F}(x) .$$

(2) By Karamata's Tauberian theorem (see Feller [**21**, XIII, Section 5]) and a Taylor expansion argument as $x \uparrow x_F$

$$\begin{aligned} P(M_K > x) &= 1 - E(F^K(x)) = 1 - E\big(\mathrm{e}^{\log F(x)\, K}\big) \\ &\sim (-\log F(x))^{\alpha} L(1/(-\log F(x))) \\ &\sim (\overline{F}(x))^{\alpha} L(1/\overline{F}(x)) . \end{aligned}$$

Finally, if X_1 is regularly varying, $L(1/\overline{F}(x))$ is slowly varying and therefore $(\overline{F}(x))^{\alpha} L(1/\overline{F}(x))$ is regularly varying with index $-p\alpha$. □

5.4. Minima. For the minimum $m_d = \min(X_1, \ldots, X_d)$ of $\mathbf{X} \in \mathrm{RV}(\alpha, \mu)$ similar calculations apply by observing that $m_d = -\max(-X_1, \ldots, -X_d)$. This observation is not useful if some of the X_i's do not assume negative values. Nevertheless, in this situation

$$P(m_d > x) = P(X_1 > x\, , \ldots, X_d > x) = P(x^{-1}\mathbf{X} \in B)\, ,$$

where $B = \{\mathbf{x} : \min_{i=1,\ldots,d} x_i > 1\}$ which is bounded away from zero and $\mu(\partial B) = 0$, and therefore m_d is regularly varying with index α if $\mu(B) > 0$. However, for independent X_i, m_d is not regularly varying with index α since $\mu(B) = 0$ and

$$P(m_d > x) = \prod_{i=1}^{d} P(X_i > x) .$$

In particular, if all $X_i \in \mathrm{RV}(\alpha)$, then $m_d \in \mathrm{RV}(d\alpha)$.

For an integer-valued non-negative random variable K independent of the sequence (X_i) of iid non-negative regularly varying random variables we have

$$P(m_K > x) = \sum_{n=1}^{\infty} P(K = n)\, [P(X_1 > x)]^n .$$

Let n_0 be the smallest positive integer such that $P(K = n_0) > 0$. Then

$$P(m_K > x) \sim P(K = n_0)\, [P(X_1 > x)]^{n_0} ,$$

implying that m_K is regularly varying with index $n_0\alpha$.

5.5. Order statistics. Let $X_{(1)} \leqslant \cdots \leqslant X_{(d)}$ be the *order statistics* of the components of the vector $\mathbf{X} \in \mathrm{RV}(\alpha, \mu)$. The tail behavior of the order statistics has been studied in some special cases, including infinite variance α-stable random vectors which are regularly varying with index $\alpha < 2$, see Theorem 4.4.8 in Samorodnitsky and Taqqu [**56**]. It is shown in Samorodnitsky [**55**] (cf. Theorem 4.4.5 in Samorodnitsky and Taqqu [**56**]) that each $X_{(i)}$ as well as the order statistics of the $|X_i|$'s are regularly varying with index α.

For a general regularly varying vector $\mathbf{X}$ with index α similar results can be obtained. We assume that $\mathbf{X}$ has non-negative components. Write $x_{(1)} \leqslant \cdots \leqslant x_{(d)}$ for the ordered values of $x_1, \ldots, x_d$. Notice that the sets $A_i = \{\mathbf{x} : x_{(i)} > x\}$ are bounded away from zero. Hence the limits

$$\lim_{x\to\infty} \frac{P(X_{(i)} > x)}{P(|\mathbf{X}| > x)} = \mu(A_i)$$

exist and if $\mu(A_i) > 0$ then $X_{(i)}$ is regularly varying. This statement can be made more precise by the approach advocated in Samorodnitsky and Taqqu [**56**], Theorem 4.4.5, which also works for general regularly varying vectors:

$$\frac{P(X_{(d-i+1)} > x)}{P(|\mathbf{X}| > x)} \tag{5.2}$$

$$= \sum_{j=i}^{d} (-1)^{j-i} \binom{j-1}{i-1} \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant d} \frac{P(X_{i_1} > x\,, \ldots, X_{i_j} > x)}{P(|\mathbf{X}| > x)}$$

$$\to \sum_{j=i}^{d} (-1)^{j-i} \binom{j-1}{i-1} \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant d} \mu(\{\mathbf{x} : x_{i_1} > 1\,, \ldots, x_{i_j} > 1\}). \tag{5.3}$$

In the same way one can also show the joint regular variation of a vector of order statistics.

For iid positive X_i's the limits of the ratios $P(X_{(i)} > x)/P(|\mathbf{X}| > x)$ are zero with the exception of $i = 1$. However, one can can easily derive that $X_{(d-i+1)}$ is regularly varying with index $i\,\alpha$. Indeed, by virtue of (5.2),

$$\frac{P(X_{(d-i+1)} > x)}{[P(X_1 > x)]^i} \sim \frac{d \cdots (d-i+1)}{i!}\,.$$

5.6. General transformations. Since the notion of regular variation bears some resemblance with weak convergence it is natural to apply the continuous mapping theorem to a regularly varying vector $\mathbf{X}$ with index α. Assume that $f : \overline{\mathbb{R}}_0^d \to \overline{\mathbb{R}}_0^m$ for some $d, m \geqslant 1$ is an a.e. continuous function with respect to the limit measure μ such that the inverse image with respect to f of any set $A \in \mathcal{B}(\overline{\mathbb{R}}_0^m)$ which is bounded away from zero is also bounded away from zero in $\overline{\mathbb{R}}_0^d$. Then we may conclude that

$$\frac{P(f(x^{-1}\mathbf{X}) \in A)}{P(|\mathbf{X}| > x)} = \frac{P(x^{-1}\mathbf{X} \in f^{-1}(A))}{P(|\mathbf{X}| > x)} \to \mu(f^{-1}(A))\,,$$

provided $\mu(\partial f^{-1}(A)) = 0$.

This means that $f(x^{-1}\mathbf{X})$ can be regularly varying in $\overline{\mathbb{R}}_0^m$, usually with an index different from α. Think for example of the functions $f(\mathbf{x}) = x_1 \cdots x_d$, $\min_{i=1,\ldots,d} x_i$, $\max_{i=1,\ldots,d} x_i$, $(x_1^p, \ldots, x^p)$, $c_1 x_1 + \cdots + c_d x_d$. These are some of the examples of the previous sections. These functions have in common that they are homogeneous, i.e., $f(t\,\mathbf{x}) = t^q f(\mathbf{x})$ for some $q > 0$, all $t > 0$. Then $f(\mathbf{X})$ is regularly varying with index α/q.

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