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SPECTRAL PROPERTIES OF THE CAUCHY OPERATOR AND THE OPERATOR OF LOGARITHMIC POTENTIAL TYPE ON L^2 SPACE WITH RADIAL WEIGHT

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Abstract. We consider the Cauchy operator C and the operator of logarithmic potential type L on $L^2(D, d\mu)$, defined by

$$Cf(z) = -\frac{1}{\pi}\int_D \frac{f(\xi)}{\xi - z}\, d\mu(\xi), \quad Lf(z) = -\frac{1}{2\pi}\int_D \ln|z-\xi|\, f(\xi)\, d\mu(\xi),$$

where D is the unit disc in $\mathbf{C}$, $d\mu(\xi) = h(|\xi|)\, dA$, $h \in L^\infty(0,1)$ is a function, positive a.e. on $(0,1)$ and dA the Lebesgue measure on D. We describe all eigenvectors and eigenvalues of these operators in terms of some operators acting on $L^2(I, d\nu)$ with $I = [0,1]$, $d\nu(r) = rh(r)\, dr$.

Introduction

Let D denote the unit disc in $\mathbf{C}$ and let $h \in L^\infty(0,1)$ be a function such that $h > 0$ a.e. on $(0,1)$. Consider the operators C, $L\colon L^2(D, d\mu) \to L^2(D, d\mu)$, where $d\mu(\xi) = h(|\xi|)\, dA$ (dA – Lebesgue measure in D) defined by

$$Cf(z) = -\frac{1}{\pi}\int_D \frac{f(\xi)}{\xi - z}\, d\mu(\xi) \qquad \text{(Cauchy operator)},$$

$$Lf(z) = -\frac{1}{2\pi}\int_D \ln|z-\xi|\, f(\xi)\, d\mu(\xi) \qquad \text{(operator of logarithmic potential type)}.$$

It is well known that under these conditions $\operatorname{Ker} L = \{0\}$, $\operatorname{Ker} C = \{0\}$ and both operators are compact. Detailed structure of operators C and L with $h \equiv 1$ is given by J. M. Anderson, D. Khavinson and V. Lomonosov in [1] and by J. Arazy and D. Khavinson in [2]. Results obtained in [3] and [4] yield the following asymptotic formulae for singular values of the operators C and L:

$$s_n(C) \sim \left(\frac{2}{n}\int_0^1 rh^2(r)\, dr\right)^{1/2}, \quad s_n(L) \sim \frac{1}{2n}\int_0^1 rh(r)\, dr \qquad \text{as } n \to \infty.$$

In this paper we describe all eigenvectors and eigenvalues of operators C and L in terms of some operators acting on $L^2(I, d\nu)$ with $I = [0,1]$, $d\nu(r) = rh(r)\, dr$.

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Results

For $m \in \mathbf{N}$ let $K_m(x,r) = \frac{1}{m}(r/x)^m$, for $r \leqslant x$ and $K_m(x,r) = \frac{1}{m}(x/r)^m$, for $x \leqslant r$, and let $A_m \colon L^2(I, d\nu) \to L^2(I, d\nu)$ be an operator defined by

$$A_m f(x) = \int_0^1 K_m(x,r) f(r)\, d\nu(r).$$

Lemma 1. *Operator A_m is selfadjoint, Hilbert-Schmidt and it has trivial kernel* ($\operatorname{Ker} A_m = \{0\}$).

Proof. Since $d\nu(r)$ is a finite measure and $K_m(x,r) \leqslant 1/m$, we have

$$\int_0^1 \int_0^1 K_m^2(x,r)\, d\nu(x)\, d\nu(r) \leqslant \frac{1}{m^2}(\nu(I))^2 < \infty,$$

i.e. A_m is Hilbert-Schmidt.

Consider the equation $A_m f = 0$, $f \in L^2(I, d\nu)$, i.e.

$$x^{-m} \int_0^x r^m f(r) \cdot r h(r)\, dr + x^m \int_x^1 r^{-m} f(r) \cdot r h(r)\, dr = 0. \tag{1}$$

Since $f \in L^2(I, d\nu)$, we have $g(r) = r f(r) h(r) \in L^1$ and from (1) it follows

$$x^{-m} \int_0^x r^m g(r)\, dr + x^m \int_x^1 r^{-m} g(r)\, dr = 0.$$

Differentiating the above equation with respect to x and multiplying by x we get

$$-x^{-m} \int_0^x r^m g(r)\, dr + x^m \int_x^1 r^{-m} g(r)\, dr = 0. \tag{2}$$

From (1) and (2) we obtain $x^{-m} \int_0^x r^m g(r)\, dr = 0$ a.e., i.e. $\int_0^x r^m g(r)\, dr = 0$ a.e., which gives $g(x) = 0$ a.e. Since $h > 0$ a.e., it follows that $f = 0$ a.e. ∎

Let $A_0 \colon L^2(I, d\nu) \to L^2(I, d\nu)$ be a linear operator defined by

$$A_0 f(x) = \int_0^1 K_0(x,r) f(r)\, d\nu(r),$$

where $K_0(x,r) = -2 \ln x$, for $r \leqslant x$, and $K_0(x,r) = -2 \ln r$, for $x \leqslant r$. In a similar way as above one can prove

Lemma 2. *Operator A_0 is Hilbert-Schmidt and* $\operatorname{Ker} A_0 = \{0\}$.

Let $\{\Phi_{mn}(r)\}_{n=1}^{\infty}$ denote the system of eigenvectors of the operator A_m ($m = 0, 1, \dots$) which correspond to eigenvalues s_{mn}, normalized by $\int_0^1 |\Phi_{mn}(r)|^2\, d\nu(r) = (2\pi)^{-1}$. For $m = 1, 2, \dots$ set $\Phi_{-mn}(r) := \Phi_{mn}(r)$, $s_{-mn} := s_{mn}$ and let $g_{mn}(z) := \Phi_{mn}(|z|)(z/|z|)^m$ for $m \in \mathbf{Z}$, $n \in \mathbf{N}$.

Theorem 1. *Operator L satisfies* $L = \sum\limits_{m \in \mathbf{Z},\, n \in \mathbf{N}} s_{mn} (\cdot, g_{mn})_{L^2(D, d\mu)} g_{mn}$.

Proof. Let $m \in \mathbf{N}$. Then

$$Lg_{mn} = \int_0^1 \Phi_{mn}(r)\, d\nu(r)\Big(-\frac{1}{\pi}\int_0^{2\pi} e^{im\theta} \ln|z - re^{i\theta}|\, d\theta\Big). \tag{3}$$

Since the integral in parentheses is equal to $r^m/m\bar{z}^m$, for $r < |z|$, and z^m/mr^m, for $|z| < r$, it follows from (3) that $Lg_{mn} = g_{mn}\dfrac{A_m\Phi_{mn}}{\Phi_{mn}}$. This equation, together with $A_{mn}\Phi_{mn} = s_{mn}\Phi_{mn}$ gives

$$Lg_{mn} = s_{mn}g_{mn}.$$

In a similar way one shows that $Lg_{0n} = s_{0n}g_{0n}$. Also, one can check directly that $Lg_{-mn} = s_{mn}g_{-mn}$ $(m, n \in \mathbf{N})$, i.e. $Lg_{-mn} = s_{-mn}g_{-mn}$ (since $s_{-mn} = s_{mn}$). Therefore $Lg_{mn} = s_{mn}g_{mn}$, for all $m \in \mathbf{Z}$, $n \in \mathbf{N}$.

Since the operator L is positive ([5], Theorem 1.16), we get $s_{mn} > 0$. We will show that the system $\{g_{mn}\}_{m\in\mathbf{Z},\, n\in\mathbf{N}}$ is orthonormal in $L^2(D, d\mu)$. Indeed,

$$\int_D g_{mn}(z)\,\overline{g_{kn}(z)}\, d\mu(z) = 2\pi\delta_{mk}\int_0^1 \Phi_{mn}(r)\,\overline{\Phi_{kn}(r)}\, d\nu(r) = \delta_{mk}$$

(from the way the system $\{\Phi_{mn}\}$ is normalized). Furthermore,

$$\int_D g_{mn}(z)\,\overline{g_{mk}(z)}\, d\mu(z) = 2\pi\int_0^1 \Phi_{mn}(r)\,\overline{\Phi_{mk}(r)}\, d\nu(r) = \delta_{nk},$$

since Φ_{mn} are eigenvectors of a selfadjoint operator A_m. Since the systems $\{\Phi_{mn}\}_{n=1}^{\infty}$ are bases in $L^2(I, d\nu)$ for all $m \in \mathbf{Z}$ and since $\{e^{in\theta}\}_{n\in\mathbf{Z}}$ is a basis in $L^2(0, 2\pi)$, the system $\{g_{mn}\}_{m\in\mathbf{Z},\, n\in\mathbf{N}}$ is complete in $L^2(D, d\mu)$ (see [6], p. 66). Therefore, $\{g_{mn}\}$ is an orthonormal basis in $L^2(D, d\mu)$ and thus

$$f = \sum_{m,n} (f, g_{mn})_{L^2(D,d\mu)} g_{mn}.$$

Applying L to the above equation we finish the proof of Theorem 1. ■

Next we consider the Cauchy operator. Let B_m $(m \in \mathbf{N})$ be linear operators defined on $L^2(I, d\nu)$ by $B_m f(x) = \int_0^1 \mathcal{B}_m(x, r) f(r)\, d\nu(r)$, with

$$\mathcal{B}_m(x, r) = \frac{4}{(xr)^m} \cdot \begin{cases} \int_0^x t^{2m-2}\, d\nu(t), & x \leqslant r, \\ \int_0^r t^{2m-2}\, d\nu(t), & r \leqslant x. \end{cases}$$

LEMMA 3. *For all $m \in \mathbf{N}$, B_m is a selfadjoint, Hilbert-Schmidt operator and* $\operatorname{Ker} B_m = \{0\}$.

Proof. Indeed, from

$$\begin{aligned} \mathcal{B}_m(x, r) &= \frac{4}{(xr)^m} \cdot \begin{cases} \int_0^x t^{2m-1}h(t)\, dt, & x \leqslant r, \\ \int_0^r t^{2m-1}h(t)\, dt, & r \leqslant x \end{cases} \\ &\leqslant \frac{4\|h\|_\infty}{(xr)^m} \cdot \begin{cases} x^{2m}/2m, & x \leqslant r, \\ r^{2m}/2m, & r \leqslant x \end{cases} \leqslant \frac{2\|h\|_\infty}{m} \end{aligned}$$

it follows that B_m is a Hilbert-Schmidt operator.

Consider the equation $B_m f(x) = 0$, $f \in L^2(I, d\nu)$. It can be written in the form

$$\int_0^x r^{m-1}\, d\nu(r)\Big(r^{m-1}\int_r^1 \frac{f(s)}{s^m}\, d\nu(s)\Big) = 0,$$

i.e. $\displaystyle\int_r^1 \frac{f(s)}{s^m}\, sh(s)\, ds = 0$. Differentiating, we get $f = 0$ a.e., and thus $\operatorname{Ker} B_m = \{0\}$.

Denote by $\{\Psi_{mn}\}_{n=1}^{\infty}$ the eigenvectors of the operator B_m corresponding to the eigenvalues λ_{mn} (sequences $\{\lambda_{mn}\}_{n=1}^{\infty}$ are ordered as non-increasing for each m). Let $B_0\colon L^2(I, d\nu) \to L^2(I, d\nu)$ be a linear operator defined by $B_0 f(x) = \int_0^1 \mathcal{B}_0(x, r) f(r)\, d\nu(r)$ with

$$\mathcal{B}_0(x, r) = 4 \cdot \begin{cases} \int_x^1 d\nu(t)/t^2, & r \leqslant x, \\ \int_r^1 d\nu(t)/t^2, & x \leqslant r. \end{cases}$$

In a similar way as in Lemma 3 one can prove that B_0 is a Hilbert-Schmidt operator with trivial kernel. Denote its eigenvalues by $\{\lambda_{0n}\}_{n=1}^{\infty}$ and the corresponding eigenvectors by $\{\Psi_{0n}\}$. We normalize the system $\{\Psi_{mn}\}_{m=0,1,\dots;\, n=1,2,\dots}$ by $\int_0^1 |\Psi_{mn}(r)|^2\, d\nu(r) = (2\pi)^{-1}$. For $m \in \mathbf{Z}$, $m < 0$ we set $\Psi_{mn} := \Psi_{-mn}$, $\lambda_{mn} := \lambda_{-mn}$ and $h_{mn}(z) = \Psi_{mn}(|z|)(z/|z|)^m$, $m \in \mathbf{Z}$, $n \in \mathbf{N}$.

Theorem 2. *Operator C^*C satisfies $C^*C = \sum\limits_{m\in\mathbf{Z},\, n\in\mathbf{N}} \lambda_{mn}(\cdot, h_{mn})_{L^2(D,d\mu)} h_{mn}$.*

Proof. For $m \in \mathbf{N}$,

$$Ch_{mn} = \int_0^1 \Psi_{mn}(r)\, d\nu(r)\Big(-\frac{1}{\pi}\int_0^{2\pi} \frac{e^{im\theta}}{re^{i\theta} - z}\, d\theta\Big) = G(|z|)\Big(\frac{z}{|z|}\Big)^{m-1},$$

where $G(t) = -2t^{m-1}\displaystyle\int_t^1 \frac{\Psi_{mn}(r)}{r^m}\, d\nu(r)$. That gives

$$C^*Ch_{mn} = \int_0^1 G(r)\, d\nu(r)\Big(-\frac{1}{\pi}\int_0^{2\pi} \frac{e^{im\theta}}{\bar{z}e^{i\theta} - r}\, d\theta\Big) = h_{mn}\,\frac{B_m h_{mn}}{h_{mn}} = \lambda_{mn} h_{mn}.$$

In a similar way one gets $C^*Ch_{0n} = \lambda_{0n} h_{0n}$ and $C^*Ch_{-mn} = \lambda_{-mn} h_{-mn}$, for $m = 1, 2, \dots$. That gives rise to $\lambda_{mn} > 0$, since C^*C is a positive operator.

In the same way as in Theorem 1 one proves that $\{h_{mn}\}_{m\in\mathbf{Z},\, n\in\mathbf{N}}$ is an orthonormal complete system in $L^2(D, d\mu)$ and thus it is a basis in $L^2(D, d\mu)$. Applying C^*C to

$$f = \sum_{m,n} (f, h_{mn})_{L^2(D,d\mu)} h_{mn}$$

we get the conclusion. Since $\lambda_{mn} > 0$, setting $s'_{mn} = \sqrt{\lambda_{mn}}$, we obtain

$$C^*C = \sum_{m\in\mathbf{Z},\, n\in\mathbf{N}} s^2_{mn}(\cdot, h_{mn})_{L^2(D,d\mu)} h_{mn}.$$

That gives rise to the permuted Hilbert-Schimdt expansion

$$C = \sum_{m\in\mathbf{Z},\, n\in\mathbf{N}} s'_{mn}(\cdot, h_{mn})_{L^2(D,d\mu)} r_{mn}, \quad r_{mn} = \frac{Ch_{mn}}{s'_{mn}}. \quad \blacksquare$$

Remark. Regarding the operators C and L we can raise the following problem: for which weight h, $\|L\| = s_{01}$, $\|C\| = \sqrt{\lambda_{01}}$ $(= s'_{01})$?

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