МАТЕМАТИЧКИ ВЕСНИК
50 (1998), 111–123

UDK 515.122.2
оригинални научни рад
research paper

FUZZY STRONGLY PREOPEN SETS AND FUZZY STRONG PRECONTINUITY

Biljana Krsteska

Abstract. A new class of generalized fuzzy open sets, called fuzzy strongly preopen sets is introduced. Fuzzy strong precontinuous, fuzzy strongly preopen and fuzzy strongly preclosed mappings between fuzzy topological spaces are defined. Their properties and the relationships between these mappings and other mappings introduced previously are investigated.

1. Introduction

Since semiopen sets and their properties were introduced by Levine [3] in 1963, many studies have been done on this topic. The preopen sets were introduced by Mashour et al. [4] in 1982. The class of strongly semiopen sets was defined by Njastad [5] in 1965.

The concept of fuzzy semiopen sets was first introduced by Azad [1]. Bai Shi Zhong [7] and Singal [8] have introduced the fuzzy strongly semiopen sets and fuzzy preopen sets. But in fuzzy topology there exist many classes of such sets, which coincide if we consider them in ordinary topology.

In the Section 3 we introduce fuzzy strongly preopen sets and we study some of their properties. Also we discuss the relationships between this class and the classes defined previously. Using the fuzzy strongly preopen sets we introduce the concept of fuzzy SPO-extremely disconnected space. In the Sections 4 and 6 we produce characterizing theorems for fuzzy strong precontinuous, fuzzy strongly preopen and fuzzy strongly preclosed mappings. In the Section 5 we introduce the notions of a fuzzy P-set and a fuzzy P-continuous mapping, and we prove that a fuzzy mapping is continuous if and only if it is both fuzzy strong precontinuous and fuzzy P-continuous. Our decomposition of fuzzy continuity is different from others introduced previously.

AMS Subject Classification: 54 A 40

Keywords and phrases: Fuzzy topology, fuzzy semiopen set, fuzzy preopen set, fuzzy strongly semiopen set, fuzzy strongly preopen set, fuzyy strong precontinuous mapping, fuzzy strongly preopen mapping, fuzzy strongly preclosed mapping.

2. Preliminaries

Now we introduce some basic notions and results that are used in the sequel.

In this work by (X, τ) or simply by X we will denote a fuzzy topological space (*fts*) due to Chang [2]. The interior, closure and the complement of a fuzzy set A will be denoted by $\operatorname{int} A$, $\operatorname{cl} A$ and A^c, respectively.

DEFINITION 2.1. *A fuzzy set A of an fts X is called:*

(1) *fuzzy semiopen if and only if $A \leqslant \operatorname{cl}(\operatorname{int} A)$ [1];*

(2) *fuzzy preopen if and only if $A \leqslant \operatorname{int}(\operatorname{cl} A)$ [6];*

(3) *fuzzy strongly semiopen if and only if $A \leqslant \operatorname{int}(\operatorname{cl}(\operatorname{int} A))$ [7].*

DEFINITION 2.2. *A fuzzy set A of an fts X is called:*

(1) *fuzzy semiclosed if and only if A^c is a fuzzy semiopen set of X [1];*

(2) *fuzzy preclosed if and only if A^c is a fuzzy preopen set of X [6];*

(3) *fuzzy strongly semiclosed if and only if A^c is a fuzzy strongly semiopen set of X [7].*

DEFINITION 2.3. [7] *Let A be a fuzzy set of an fts X. Then,*

$\operatorname{pint} A = \bigvee\{ B \mid B \leqslant A,\ B$ *is a fuzzy preopen set of* $X \}$, *is called a fuzzy preinterior of A [6];*

$\operatorname{pcl} A = \bigwedge\{ B \mid B \geqslant A,\ B$ *is a fuzzy preclosed set of* $X \}$, *is called a fuzzy preclosure of A [6];*

$\operatorname{ssint} A = \bigvee\{ B \mid B \leqslant A,\ B$ *is a fuzzy strongly semiopen set of* $X \}$, *is called a fuzzy strong semi-interior of A [7];*

$\operatorname{sscl} A = \bigwedge\{ B \mid B \geqslant A,\ B$ *is a fuzzy strongly semiclosed set of* $X \}$, *is called a fuzzy strong semiclosure of A [7].*

THEOREM 2.1. *Let A be a fuzzy set of an fts X, Then,*

(1) $\operatorname{pcl} A \geqslant A \vee \operatorname{cl}(\operatorname{int} A)$*;*

(2) $\operatorname{pint} A \leqslant A \wedge \operatorname{int}(\operatorname{cl} A)$.

Proof. We will prove only the statement (1). Since $\operatorname{pcl} A$ is a fuzzy preclosed set, we have $\operatorname{cl}(\operatorname{int} A) \leqslant \operatorname{cl}(\operatorname{int}(\operatorname{pcl} A)) \leqslant \operatorname{pcl} A$. Thus $A \vee \operatorname{cl}(\operatorname{int} A) \leqslant \operatorname{pcl} A$. ■

In ordinary topology we have the relation $\operatorname{pcl} A = A \vee \operatorname{cl}(\operatorname{int} A)$, so $\operatorname{int}(\operatorname{pcl} A) = \operatorname{int}(\operatorname{cl}(\operatorname{int} A))$. The next example shows that in fuzzy topology the equality may not be satisfied.

EXAMPLE 2.1 Let $X = \{a, b, c\}$ and A, B, C are fuzzy sets of X defined as it follows:

$$A(a) = 0{,}3, \qquad A(b) = 0{,}2, \qquad A(c) = 0{,}7,$$
$$B(a) = 0{,}8, \qquad B(b) = 0{,}8, \qquad B(c) = 0{,}4,$$
$$C(a) = 0{,}8, \qquad C(b) = 0{,}7, \qquad C(c) = 0{,}6.$$

Let $\tau = \{0, A, B, A \wedge B, A \vee B, 1\}$. By easy verification it can be seen that $\operatorname{pcl} C > C \vee \operatorname{cl}(\operatorname{int} C)$ and $\operatorname{pint} C^c < C^c \wedge \operatorname{int}(\operatorname{cl} C^c)$.

The Theorem 2.1 gives the motivation to introduce the class which will be discussed.

LEMMA 2.2. [7] *Let A be a fuzzy set of an fts X. Then,*

(1) $\operatorname{pcl} A^c = (\operatorname{pint} A)^c$;

(2) $\operatorname{pint} A^c = (\operatorname{pcl} A)^c$. ■

LEMMA 2.3 [2] *Let $f\colon X \to Y$ be a mapping. For fuzzy sets A and B of X and Y respectively, the following statements hold:*

(1) $ff^{-1}(B) \leqslant B$;

(2) $f^{-1}f(A) \geqslant A$;

(3) $f(A^c) \geqslant f(A)^c$;

(4) $f^{-1}(B^c) = f^{-1}(B)^c$;

(5) *if f is injective, then* $f^{-1}f(A) = A$;

(6) *if f is surjective, then* $ff^{-1}(B) = B$;

(7) *if f is bijective, then* $f(A^c) = f(A)^c$. ■

DEFINITION 2.4 Let $f\colon (X, \tau_1) \to (Y, \tau_2)$ be a mapping from an *fts* (X, τ_1) into an *fts* (Y, τ_2). The mapping f is called:

(1) fuzzy continuous if $f^{-1}(B)$ is a fuzzy open set of X, for each $B \in \tau_2$ [2];

(2) fuzzy semicontinuous if $f^{-1}(B)$ is a fuzzy semiopen set of X, for each $B \in \tau_2$ [1];

(3) fuzzy precontinuous if $f^{-1}(B)$ is a fuzzy preopen set of X, for each $B \in \tau_2$ [6];

(4) fuzzy strong semicontinuous if $f^{-1}(B)$ is a fuzzy strongly semiopen set of X, for each $B \in \tau_2$ [7];

(5) fuzzy open (closed) if $f(A)$ is a fuzzy open (closed) set of Y, for each $A \in \tau_1$ $(A^c \in \tau_1)$ [2];

(6) fuzzy semiopen (semiclosed) if $f(A)$ is a fuzzy semiopen (semiclosed) set of Y, for each $A \in \tau_1$ $(A^c \in \tau_1)$ [1];

(7) fuzzy preopen (preclosed) if $f(A)$ is a fuzzy preopen (preclosed) set of Y, for each $A \in \tau_1$ $(A^c \in \tau_1)$ [6];

(8) fuzzy strongly semiopen (semiclosed) if $f(A)$ is a fuzzy strongly semiopen (semiclosed) set of Y, for each $A \in \tau_1$ $(A^c \in \tau_1)$ [1].

DEFINITION 2.5. [1] An *fts* (X, τ_1) is a product related to an *fts* (Y, τ_2) if for fuzzy sets A of X and B of Y whenever $C^c \geqslant A$ and $D^c \geqslant B$ implies $C^c \times 1 \vee 1 \times D^c \geqslant A \times B$, where $C \in \tau_1$ and $D \in \tau_2$, there exist $C_1 \in \tau_1$ and $D_1 \in \tau_2$ such that $C_1^c \geqslant A$ or $D_1^c \geqslant B$ and $C_1^c \times 1 \vee 1 \times D_1^c = C^c \times 1 \vee 1 \times D^c$.

Lemma 2.4. [1] *Let X and Y be fts's such that X is product related to Y. Then for fuzzy sets A of X and B of Y,*

(1) $\mathrm{cl}(A \times B) = \mathrm{cl}\, A \times \mathrm{cl}\, B$;

(2) $\mathrm{int}(A \times B) = \mathrm{int}\, A \times \mathrm{int}\, B$. ■

Remark. The relation $\mathrm{int}(A \times B) = \mathrm{int}\, A \times \mathrm{int}\, B$ holds for any fuzzy topological spaces X and Y, not only in the case when the spaces are product related.

Lemma 2.5. [1] *Let $g\colon X \to X \times Y$ be the graph of a mapping $f\colon X \to Y$. If A is a fuzzy set of X and B is a fuzzy set of Y, then $g^{-1}(A \times B) = A \wedge f^{-1}(B)$.* ■

3. Fuzzy strongly preopen sets and fuzzy strongly preclosed sets

Definition 3.1. A fuzzy set A of an *fts* X is called a fuzzy strongly preopen set if and only if $A \leqslant \mathrm{int}(\mathrm{pcl}\, A)$.

The family of all fuzzy strongly preopen sets of an *fts* (X, τ) will be denoted by $FSPO(\tau)$.

Lemma 3.1. *Let A be a fuzzy set of an fts X. Then the following properties hold:*

(1) $\mathrm{int}(\mathrm{pcl}\, A) \leqslant \mathrm{int}(\mathrm{cl}\, A)$;

(2) $\mathrm{int}(\mathrm{pcl}\, A) \geqslant \mathrm{int}(\mathrm{cl}(\mathrm{int}\, A))$.

Proof. (1) It follows from the relation $\mathrm{pcl}\, A \leqslant \mathrm{cl}\, A$, for any fuzzy set A of X.

(2) From Theorem 2.1 we have $\mathrm{int}(\mathrm{pcl}\, A) \geqslant \mathrm{int}(A \vee \mathrm{cl}(\mathrm{int}\, A)) \geqslant \mathrm{int}(\mathrm{cl}(\mathrm{int}\, A))$. ■

Theorem 3.2. *Let X be an fts. Then the following statements hold:*

(1) *every fuzzy open set is a fuzzy strongly preopen set;*

(2) *every fuzzy strongly semiopen set is a fuzzy strongly preopen set;*

(3) *every fuzzy strongly preopen set is a fuzzy preopen set.*

Proof. It follows easily from Lemma 3.1. ■

Example 3.1. Let (X, τ) and C are defined as in Example 2.1. Then C is a fuzzy strongly preopen set, but C is not a fuzzy strongly semiopen set. If we choose $\tau_1 = \{0, B, 1\}$, then A is a fuzzy preopen set which is not fuzzy strongly preopen.

Remark. From the example above it is not difficult to conclude that a fuzzy preopen set may be not a fuzzy strongly preopen set, and a fuzzy strongly preopen set may be not a fuzzy strongly semiopen set. Also the classes of fuzzy strongly preopen sets and fuzzy semiopen sets are independent.

Definition 3.2. Let A be a fuzzy set of an *fts* X. Then A is called a fuzzy strongly preclosed set if and only if A^c is a fuzzy strongly preopen set of X.

The family of all fuzzy strongly preclosed sets of an *fts* (X, τ) will be denoted by $FSPC(\tau)$.

Theorem 3.3. *Let A be a fuzzy set of an fts (X,τ). Then A is a fuzzy strongly preclosed set if and only if* $\mathrm{cl}(\mathrm{pint}\, A) \leqslant A$.

Proof. Let A be a fuzzy strongly preclosed set. Then A^c is a fuzzy strongly preopen set. From the relation $A^c \leqslant \mathrm{int}(\mathrm{pcl}\, A^c)$, we have $A \geqslant \mathrm{cl}(\mathrm{pint}\, A)$.

Conversely, let A be a fuzzy set such that $A \geqslant \mathrm{cl}(\mathrm{pint}\, A)$. From the relation $A^c \leqslant \mathrm{int}(\mathrm{pcl}\, A^c)$ it follows that A^c is a fuzzy strongly preopen set. Hence, A is a fuzzy strongly preclosed set. ∎

Lemma 3.4. *Let $\{A_\alpha\}_{\alpha\in I}$, be a family of fuzzy sets of an fts X. Then* $\bigvee_{\alpha\in I} \mathrm{pcl}(A_\alpha) \leqslant \mathrm{pcl}(\bigvee_{\alpha\in I} A_\alpha)$.

Proof. Since $\mathrm{pcl}(A_\alpha) \leqslant \mathrm{pcl}(\bigvee_{\alpha\in I} A_\alpha)$, for each $\alpha \in I$, the conclusion follows. ∎

Theorem 3.5. (1) *Any union of fuzzy strongly preopen sets is a fuzzy strongly preopen set.*

(2) *Any intersection of fuzzy strongly preclosed sets is a fuzzy strongly preclosed set.*

Proof. (1) Let $\{A_\alpha\}_{\alpha\in I}$ be any family of fuzzy strongly preopen sets. For each $\alpha \in I$, $A_\alpha \leqslant \mathrm{int}(\mathrm{pcl}\, A_\alpha)$. Hence form Lemma 3.4 we have $\bigvee_{\alpha\in I} A_\alpha \leqslant \bigvee_{\alpha\in I} \mathrm{int}(\mathrm{pcl}\, A_\alpha) \leqslant \mathrm{int}(\mathrm{pcl}(\bigvee_{\alpha\in I} A_\alpha))$.

(2) Let $\{A_\alpha\}_{\alpha\in I}$ be any family of fuzzy strongly preclosed sets. Thus $\{A^c_\alpha\}_{\alpha\in I}$ is a family of fuzzy strongly preopen sets. According to (1), $\bigvee_{\alpha\in I} A^c_\alpha$ is a fuzzy strongly preopen set. From $(\bigvee_{\alpha\in I} A^c_\alpha)^c = \bigwedge_{\alpha\in I} A_\alpha$ we obtain the conclusion. ∎

Remark. The intersection (union) of two fuzzy strongly preopen (preclosed) sets may not be fuzzy strongly preopen (preclosed). Even the intersection (union) of a fuzzy strongly preopen (preclosed) set with a fuzzy open (closed) set may fail to be a fuzzy strongly preopen (preclosed) set. We can verify this conclusion if we consider the *fts* (X,τ) in Example 2.1. The fuzzy set D defined as

$$D(a) = 0{,}4, \quad D(b) = 0{,}2, \quad D(c) = 0{,}8$$

is a strongly preopen set, but $B \wedge D$ is not a fuzzy strongly preopen set of (X,τ). Also $B^c \vee D^c$ is not a fuzzy strongly preclosed set of (X,τ).

Definition 3.3. Let A be a fuzzy set of an *fts* X.

(1) The union of all fuzzy strongly preopen sets contained in A is called the fuzzy strong preinterior of A, denoted by $\mathrm{spint}\, A$.

(2) The intersection of all fuzzy strongly preclosed sets containing A is called the fuzzy strong preclosure of A, denoted by $\mathrm{spcl}\, A$.

Theorem 3.6. *Let A and B are fuzzy sets of an fts X. Then:*

(1) $\mathrm{int}\, A \leqslant \mathrm{spint}\, A \leqslant A$; $\quad A \leqslant \mathrm{spcl}\, A \leqslant \mathrm{cl}\, A$;

(2) $\mathrm{spint}\, A \in FSPO(\tau)$; $\quad \mathrm{spcl}\, A \in FSPC(\tau)$;

(3) $A \in FSPO(\tau) \iff A = \mathrm{spint}\, A$; $\quad A \in FSPC(\tau) \iff A = \mathrm{spcl}\, A$;

(4)	$A \leqslant B \implies \operatorname{spint} A \leqslant \operatorname{spint} B$;	$A \leqslant B \implies \operatorname{spcl} A \leqslant \operatorname{spcl} B$;
(5)	$\operatorname{spint}(\operatorname{spint} A) = \operatorname{spint} A$;	$\operatorname{spcl}(\operatorname{spcl} A) = \operatorname{spcl} A$;
(6)	$\operatorname{spint} A \wedge \operatorname{spint} B \geqslant \operatorname{spint}(A \wedge B)$;	$\operatorname{spcl} A \vee \operatorname{spcl} B \leqslant \operatorname{spcl}(A \vee B)$;
(7)	$\operatorname{spint} A \vee \operatorname{spint} B \leqslant \operatorname{spint}(A \vee B)$;	$\operatorname{spcl}(A \wedge B) \leqslant \operatorname{spcl} A \wedge \operatorname{spcl} B$;
(8)	$\operatorname{spint} X = X$;	$\operatorname{spcl} \varnothing = \varnothing$.

Proof. It follows from Definition 3.3 and Theorem 3.5. ∎

THEOREM 3.7. (1) *A fuzzy set B of an fts X is fuzzy strongly preopen if and only if there exists a fuzzy set A of X such that* $A \leqslant B \leqslant \operatorname{int}(\operatorname{pcl} A)$.

(2) *A fuzzy set B of an fts X is a fuzzy strongly preclosed set if and only if there exists a fuzzy set A of X such that* $\operatorname{cl}(\operatorname{pint} A) \leqslant B \leqslant A$.

Proof. We prove only the statement (1). Let B be a fuzzy set of X. If a fuzzy set A of X such that $A \leqslant B \leqslant \operatorname{int}(\operatorname{pcl} A)$ exists, then $B \leqslant \operatorname{int}(\operatorname{pcl} A) \leqslant \operatorname{int}(\operatorname{pcl} B)$. Thus B is a fuzzy strongly preopen set.

Conversely, if B is any fuzzy strongly preopen set, then the result follows for $A = B$. ∎

The next statement gives the relationship between the operators of fuzzy strong preinterior and fuzzy strong preclosure.

THEOREM 3.8. *Let A be a fuzzy set of an fts X. Then:*

(1) $\operatorname{spcl} A^c = (\operatorname{spint} A)^c$*;*

(2) $\operatorname{spint} A^c = (\operatorname{spcl} A)^c$.

Proof. (1) $(\operatorname{spint} A)^c = \big(\bigvee\{\, d \mid d \leqslant A, d \in FSPO(\tau)\,\}\big)^c = \bigwedge\{\, d^c \mid d \leqslant A, d \in FSPO(\tau)\,\} = \bigwedge\{\, c \mid c \geqslant A^c, c \in FSPC(\tau)\,\} = \operatorname{spcl} A^c$.

(2) $(\operatorname{spcl} A)^c = (\operatorname{spcl}(A^c)^c)^c = ((\operatorname{spint} A^c)^c)^c = \operatorname{spint} A^c$. ∎

THEOREM 3.9. *Let A be a fuzzy set of an fts X. Then*

$$\operatorname{int} A \leqslant \operatorname{ssint} A \leqslant \operatorname{spint} A \leqslant \operatorname{pint} A \leqslant A \leqslant \operatorname{pcl} A \leqslant \operatorname{spcl} A \leqslant \operatorname{sscl} A \leqslant \operatorname{cl} A.$$

Proof. It follows from the definitions of corresponding operators. ∎

THEOREM 3.10. *Let A be a fuzzy set of an fts X.*

(1) *If A is a fuzzy strongly preopen set, then* $\operatorname{pcl} A = \operatorname{cl} A$.

(2) *If A is a fuzzy strongly preclosed set, then* $\operatorname{pint} A = \operatorname{int} A$.

Proof. We prove only the statement (1). Let A be any fuzzy strongly preopen set. From $A \leqslant \operatorname{int}(\operatorname{pcl} A)$ we obtain $\operatorname{cl} A \leqslant \operatorname{cl}(\operatorname{int}(\operatorname{pcl} A)) \leqslant \operatorname{pcl} A$. This relation together with the evident relation $\operatorname{cl} A \geqslant \operatorname{pcl} A$ gives the conclusion. ∎

The following theorem considers the operators of fuzzy strong preinterior and fuzzy strong preclosure and relates them with other operators.

THEOREM 3.11. *Let A be a fuzzy set of an fts X. Then*

(1) $\operatorname{spint} A \leqslant A \wedge \operatorname{int}(\operatorname{pcl} A)$*;*

(2) $\operatorname{spcl} A \geqslant A \vee \operatorname{cl}(\operatorname{pint} A)$.

Proof. We prove only the statement (1). Since $\operatorname{spint} A \in FSPO(\tau)$, we have $\operatorname{spint} A \leqslant \operatorname{int}(\operatorname{pcl}(\operatorname{spint} A)) \leqslant \operatorname{int}(\operatorname{pcl} A)$. This relation together with the evident relation $\operatorname{spint} A \leqslant A$ gives the conclusion. ∎

Based on the concept of fuzzy extremely disconnected space we produce the concept of fuzzy SPO-extremely disconnected space.

DEFINITION 3.4. *An fts X is fuzzy SPO-extremely disconnected if and only if* $\operatorname{spcl} A$ *is a fuzzy strongly preopen set, for each fuzzy strongly preopen set A of X.*

The following statement gives an interesting characterization of these spaces.

THEOREM 3.12. *Let X be an fts. Then the following statements are equivalent:*

(i) *X is SPO-extremely disconnected;*

(ii) $\operatorname{spint} A$ *is a fuzzy strongly preclosed set, for each fuzzy strongly preclosed set A of X;*

(iii) $\operatorname{spcl}(\operatorname{spcl} A)^c = (\operatorname{spcl} A)^c$*, for each fuzzy strongly preopen set A of X;*

(iv) $B = (\operatorname{spcl} A)^c$ *implies* $\operatorname{spcl} B = (\operatorname{spcl} A)^c$*, for each pair of fuzzy strongly preopen sets A, B of X.*

Proof. (i) $\Longrightarrow$ (ii). Let A be a fuzzy strongly preclosed set of X. Then A^c is a fuzzy strongly preopen set. According to the assumption, $\operatorname{spcl} A^c$ is a fuzzy strongly preopen set, so $\operatorname{spint} A$ is a fuzzy strongly preclosed set of X.

(ii) $\Longrightarrow$ (iii). Suppose that A is a fuzzy strongly preopen set. Then $\operatorname{spcl}(\operatorname{spcl} A)^c = \operatorname{spcl}(\operatorname{spint} A^c)$. According to the assumption, $\operatorname{spint} A^c$ is a fuzzy strongly preclosed set, so $\operatorname{spcl}(\operatorname{spint} A^c) = \operatorname{spint} A^c = (\operatorname{spcl} A)^c$.

(iii) $\Longrightarrow$ (iv). Let A and B be fuzzy strongly preopen sets of X such that $B = (\operatorname{spcl} A)^c$. From the assumption we have $\operatorname{spcl} B = \operatorname{spcl}(\operatorname{spcl} A)^c = (\operatorname{spcl} A)^c$.

(iv) $\Longrightarrow$ (i). Let A be a fuzzy strongly preopen set of X. We put $B = (\operatorname{spcl} A)^c$. From the assumption we obtain that $\operatorname{spcl} B = (\operatorname{spcl} A)^c$, so $(\operatorname{spcl} B)^c = \operatorname{spcl} A$. Hence $\operatorname{spint} B^c = \operatorname{spcl} A$. Thus $\operatorname{spcl} A$ is fuzzy strongly preopen set of X. ∎

THEOREM 3.13. *Let X and Y be fts's such that X is product related to Y. Then the product $A \times B$ of a fuzzy strongly preopen set A of X and a fuzzy strongly preopen set B of Y is a fuzzy preopen set of the fuzzy product space $X \times Y$.*

Proof. Since A and B are fuzzy strongly preopen sets, $A \leqslant \operatorname{int}(\operatorname{pcl} A)$ and $B \leqslant \operatorname{int}(\operatorname{pcl} B)$. From Lemma 2.4 and Theorem 3.10 we obtain $A \times B \leqslant \operatorname{int}(\operatorname{pcl} A) \times \operatorname{int}(\operatorname{pcl} B) = \operatorname{int}(\operatorname{pcl} A \times \operatorname{pcl} B) = \operatorname{int}(\operatorname{cl} A \times \operatorname{cl} B) = \operatorname{int}(\operatorname{cl}(A \times B))$, which shows that $A \times B$ is a fuzzy preopen set. ∎

4. Fuzzy strong precontinuity

DEFINITION 4.1. A mapping $f\colon (X,\tau_1) \to (Y,\tau_2)$ from an *fts* X into an *fts* Y is called fuzzy strong precontinuous if $f^{-1}(B) \in FSPO(\tau_1)$ for each $B \in \tau_2$.

The implications contained in the following diagram are true.

$$\begin{array}{ccc} \text{fuzzy continuity} & & \text{fuzzy strong semicontinuity} \\ \Downarrow & & \Downarrow \\ & \text{fuzzy strong precontinuity} & \\ & \Downarrow & \\ & \text{fuzzy precontinuity} & \end{array}$$

The following example shows that reverse may not be true.

EXAMPLE 4.1. We consider Example 3.1. If we put $\tau_2 = \{0, A, 1\}$ and $f = id\colon (X,\tau_1) \to (X,\tau_2)$ we conclude that f is fuzzy precontinuous but f is not a fuzzy strong precontinuous mapping. If we choose $\tau_3 = \{0, C, 1\}$, then $f = id\colon (X,\tau) \to (X,\tau_3)$ is fuzzy strong precontinuous, but f is neither fuzzy continuous nor fuzzy strong semicontinuous.

THEOREM 4.1. *Let $f\colon (X,\tau_1) \to (Y,\tau_2)$ be a mapping from an fts (X,τ_1) into an fts (Y,τ_2). Then the following statements are equivalent:*

(i) *f is a fuzzy strong precontinuous mapping;*

(ii) *$f^{-1}(B)$ is a fuzzy strongly preclosed set of X, for each fuzzy closed set B of Y;*

(iii) $f(\operatorname{spcl} A) \leqslant \operatorname{cl} f(A)$, *for each fuzzy set A of X;*

(iv) $\operatorname{spcl} f^{-1}(B) \leqslant f^{-1}(\operatorname{cl} B)$, *for each fuzzy set B of Y;*

(v) $f^{-1}(\operatorname{int} B) \leqslant \operatorname{spint}(f^{-1}(B))$, *for each fuzzy set B of Y;*

(vi) *there is a base β for τ_2 such that $f^{-1}(B)$ is a fuzzy strongly preopen set of X for each $B \in \beta$;*

(vii) *there is a base β for τ_2 such that $f^{-1}(B)$ is a fuzzy strongly preclosed set of X for each $B^c \in \beta$.*

Proof. The proof is standard and therefore omitted. ■

THEOREM 4.2. *Let $f\colon X \to Y$ be a mapping from an fts X into an fts Y. Then the following statements are equivalent:*

(i) *f is a fuzzy strong precontinuous mapping;*

(ii) $\operatorname{cl}(\operatorname{pint} f^{-1}(B)) \leqslant f^{-1}(\operatorname{cl} B)$, *for each fuzzy set B of Y;*

(iii) $f(\operatorname{cl}(\operatorname{pint} A)) \leqslant \operatorname{cl} f(A)$, *for each fuzzy set A of X.*

Proof. The proof is standard and therefore omitted. ■

THEOREM 4.3. *Let $f\colon X \to Y$ be a bijective mapping from an fts X into an fts Y. The mapping f is fuzzy strong precontinuous if and only if* $\operatorname{int} f(A) \leqslant f(\operatorname{spint} A)$, *for each fuzzy set A of X.*

Proof. We suppose that f is fuzzy strong precontinuous. For any fuzzy set A of X, $f^{-1}(\operatorname{int} f(A))$ is a fuzzy strongly preopen set. From Theorem 4.1 and the fact that f is injective we have $f^{-1}(\operatorname{int} f(A)) \leqslant \operatorname{spint} f^{-1}(\operatorname{int} f(A)) \leqslant \operatorname{spint} f^{-1} f(A) = \operatorname{spint} A$. Again, since f is surjective, we obtain $\operatorname{int} f(A) = ff^{-1}(\operatorname{int} f(A)) \leqslant f(\operatorname{spint} A)$.

Conversely, let B be a fuzzy open set of Y. Then $\operatorname{int} B = B$. According to assumption, $f(\operatorname{spint} f^{-1}(B)) \geqslant \operatorname{int} ff^{-1}(B) = \operatorname{int} B = B$. This implies that $f^{-1}f(\operatorname{spint}(f^{-1}(B))) \geqslant f^{-1}(B)$. Since f is injective we obtain $\operatorname{spint} f^{-1}(B) = f^{-1}f(\operatorname{spint} f^{-1}(B)) \geqslant f^{-1}(B)$. Hence $\operatorname{spint} f^{-1}(B) = f^{-1}(B)$, so $f^{-1}(B)$ is a fuzzy strongly preopen set. Thus f is fuzzy strong precontinuous. ∎

THEOREM 4.4. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be mappings, where X, Y and Z are fts's. If f is fuzzy strong precontinuous and g is fuzzy continuous, then gf is fuzzy strong precontinuous.*

Proof. It follows from the relation $(gf)^{-1}(B) = f^{-1}(g^{-1}(B))$, for each fuzzy set B of Z. ∎

COROLLARY 4.5. *Let X, X_1 and X_2 are fuzzy spaces and $p_i\colon X_1 \times X_2 \to X_i$ $(i = 1, 2)$ are the projections of $X_1 \times X_2$ onto X_i. If $f\colon X \to X_1 \times X_2$ is fuzzy strong precontinuous, then $p_i f$ are also fuzzy strong precontinuous mappings.*

Proof. It follows from the fact that p_i $(i = 1, 2)$ are fuzzy continuous mappings. ∎

THEOREM 4.6. *Let X_1, X_2, Y_1 and Y_2 be fts's such that X_1 is product related to X_2. Then the product $f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2$ of fuzzy strong precontinuous mappings $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$ is a fuzzy precontinuous mappings.*

Proof. Let $B = \bigvee(U_\alpha \times V_\beta)$, where U_α and V_β are fuzzy open sets of Y_1 and Y_2 respectively. From $(f_1 \times f_2)^{-1}(B) = \bigvee(f_1^{-1}(U_\alpha) \times f_2^{-1}(V_\beta))$ it follows that $(f_1 \times f_2)^{-1}(B)$ is a fuzzy preopen set as a union of product of fuzzy strongly preopen sets. ∎

THEOREM 4.7. *Let $f\colon X \to Y$ be a mapping from an fts X into an fts Y. If the graph $g\colon X \to X \times Y$ of f is fuzzy strong precontinuous, then f is fuzzy strong precontinuous.*

Proof. From Lemma 2.5, for each fuzzy open set B of Y, $f^{-1}(B) = 1 \wedge f^{-1}(B) = g^{-1}(1 \times B)$. Since g is fuzzy strong precontinuous and $1 \times B$ is a fuzzy open set of $X \times Y$, $f^{-1}(B)$ is a fuzzy strongly preopen set of X, so f is fuzzy strong precontinuous. ∎

5. Decomposition of fuzzy continuiuty

DEFINITION 5.1. A fuzzy set A of an *fts* X is called a P-set if and only if $A = U \wedge C$, where $U \in \tau$ and C is a fuzzy preclosed set of X.

THEOREM 5.1. *A fuzzy set A of an fts X is open if and only if it is both a fuzzy strongly preopen set and a P-set.*

Proof. Let A be a fuzzy open set of X. Then from $A = A \wedge X$ follows that A is a fuzzy P-set. Also A is a fuzzy strongly preopen set by Theorem 3.2(1).

Conversely, let A be both a fuzzy P-set and a fuzzy strongly preopen set. Then $A \leqslant \operatorname{int}(\operatorname{pcl} A)$ and $A = U \wedge C$, where $U \in \tau$ and C is fuzzy preclosed. Therefore

$$\begin{aligned} U \wedge C &\leqslant \operatorname{int}(\operatorname{pcl}(U \wedge C)) \leqslant \operatorname{int}(\operatorname{pcl} U \wedge \operatorname{pcl} C) = \operatorname{int}(\operatorname{pcl} U) \wedge \operatorname{int}(\operatorname{pcl} C) \\ &= \operatorname{int}(\operatorname{pcl} U) \wedge \operatorname{int} C. \end{aligned}$$

Hence $U \wedge C = (U \wedge C) \wedge U \leqslant \operatorname{int}(\operatorname{pcl} U) \wedge \operatorname{int} C \wedge U = U \wedge \operatorname{int} C$. Noticing that $U \wedge C \geqslant U \wedge \operatorname{int} C$, we obtain $U \wedge C = U \wedge \operatorname{int} C$, thus $A = U \wedge C$ is a fuzzy open set. ∎

DEFINITION 5.2. A mapping $f\colon (X, \tau_1) \to (Y, \tau_2)$ is called fuzzy P-continuous if $f^{-1}(B)$ is a fuzzy P-set for each $B \in \tau_2$.

According to Theorem 5.1, we have the following decomposition of continuity.

THEOREM 5.2. *A mapping $f\colon X \to Y$ is fuzzy continuous if and only if it is both fuzzy strong precontinuous and fuzzy P-continuous.* ∎

6. Fuzzy strongly preopen and fuzzy strongly preclosed mappings

DEFINITION 6.1 A mapping $f\colon (X, \tau_1) \to (Y, \tau_2)$ is called:

(1) fuzzy strongly preopen if $f(A) \in FSPO(\tau_2)$ for each $A \in \tau_1$;

(2) fuzzy strongly preclosed if $f(A) \in FSPC(\tau_2)$ for each $A^c \in \tau_1$.

We have the following diagram of implications:

$$\begin{array}{ccc} \text{fuzzy open (closed) mapping} & & \text{fuzzy strongly semiopen (semiclosed) mapping} \\ \Downarrow & & \Downarrow \\ & \text{fuzzy strongly preopen (preclosed) mapping} & \\ & \Downarrow & \\ & \text{fuzzy preopen (preclosed) mapping} & \end{array}$$

The following example shows that reverse may not be true.

EXAMPLE 6.1. We consider Example 4.1. The mapping $f = id\colon (X, \tau_3) \to (X, \tau)$ is fuzzy strongly preopen (preclosed) but it is not fuzzy strongly semiopen (semiclosed). Also, f is not fuzzy open (closed). Similarly, the mapping $f = id\colon (X, \tau_2) \to (X, \tau_1)$ is fuzzy preopen (preclosed), but it is not fuzzy strongly preopen (preclosed).

THEOREM 6.1. *Let $f\colon (X, \tau_1) \to (Y, \tau_2)$ be a mapping from an fts X into an fts Y. Then the following statements are equivalent:*

(i) *f is a fuzzy strongly preopen mapping;*

(ii) $f(\operatorname{int} A) \leqslant \operatorname{spint} f(A)$, *for each fuzzy set* A *of* X*;*

(iii) $\operatorname{int} f^{-1}(B) \leqslant f^{-1}(\operatorname{spint} B)$, *for each fuzzy set* B *of* Y*;*

(iv) $f^{-1}(\operatorname{spcl} B) \leqslant \operatorname{cl} f^{-1}(B)$, *for each fuzzy set* B *of* Y*;*

(v) *there is a base* α *for* τ_1 *such that* $f(A)$ *is fuzzy strongly preopen set of* Y *for each* $A \in \alpha$.

Proof is standard and therefore omitted. ∎

Theorem 6.2. *A mapping* $f\colon X \to Y$ *from an fts* X *into an fts* Y *is fuzzy strongly preclosed if and only if* $\operatorname{spcl} f(A) \leqslant f(\operatorname{cl} A)$ *for each fuzzy set* A *of* X.

Proof is standard and therefore omitted. ∎

Theorem 6.3. *Let* $f\colon X \to Y$ *be a bijective mapping from an fts* X *into an fts* Y*. Then* f *is fuzzy strongly preopen if and only if it is fuzzy strongly preclosed.*

Proof. It follows from Lemma 2.3. ∎

Corollary 6.4. *Let* $f\colon X \to Y$ *be a bijective mapping from an fts* X *into an fts* Y*. Then* f *is fuzzy strongly preopen if and only if* $\operatorname{spcl} f(A) \leqslant f(\operatorname{cl} A)$ *for each fuzzy set* A *of* X.

Corollary 6.5. *Let* $f\colon X \to Y$ *be a bijective mapping from an fts* X *into an fts* Y*. Then the following statements are equivalent:*

(i) f *is a fuzzy strongly preclosed mapping;*

(ii) $f(\operatorname{int} A) \leqslant \operatorname{spint} f(A)$, *for each fuzzy set* A *of* X*;*

(iii) $\operatorname{int} f^{-1}(B) \leqslant f^{-1}(\operatorname{spint} B)$, *for each fuzzy set* B *of* Y*;*

(iv) $f^{-1}(\operatorname{spcl} B) \leqslant \operatorname{cl} f^{-1}(B)$, *for each fuzzy set* B *of* Y*;*

(v) *there is a base* α *for* τ_1 *such that* $f(A)$ *is fuzzy strongly preclosed set of* Y *for each* $A^c \in \alpha$. ∎

Theorem 6.6. *A mapping* $f\colon X \to Y$ *from an fts* X *into an fts* Y *is fuzzy strongly preopen if and only if* $f(\operatorname{int} A) \leqslant \operatorname{int}(\operatorname{pcl} f(A))$, *for each fuzzy set* A *of* X.

Proof. We suppose that f is a fuzzy strongly preopen mapping. For any fuzzy set A of X, $f(\operatorname{int} A)$ is a fuzzy strongly preopen set of Y. Thus $f(\operatorname{int} A) \leqslant \operatorname{int}(\operatorname{pcl} f(\operatorname{int} A)) \leqslant \operatorname{int}(\operatorname{pcl} f(A))$.

Conversely, let A be a fuzzy open set of X. From $f(A) = f(\operatorname{int} A) \leqslant \operatorname{int}(\operatorname{pcl} f(A))$, we conclude that f is a fuzzy strongly preopen mapping. ∎

Theorem 6.7. *A mapping* $f\colon X \to Y$ *from an fts* X *into an fts* Y *is fuzzy strongly preclosed if and only if* $\operatorname{cl}(\operatorname{pint} f(A)) \leqslant f(\operatorname{cl} A)$, *for each fuzzy set* A *of* X.

Proof. It can be proved in a similar manner as Theorem 6.6. ∎

Theorem 6.8. *Let* $f\colon X \to Y$ *be a mapping from an fts* X *into an fts* Y.

(1) *If* $f(\operatorname{int}(\operatorname{pcl} A)) \leqslant \operatorname{int}(\operatorname{pcl} f(A))$, *for each fuzzy open set* A *of* X*, then* f *is a fuzzy strongly preopen mapping.*

(2) *If* $f(\mathrm{cl}(\mathrm{pint}\, A)) \geqslant \mathrm{cl}(\mathrm{pint}\, f(A))$, *for each fuzzy closed set* A *of* X, *then* f *is a fuzzy strongly preclosed mapping.*

Proof. We prove only the statement (1). Let A be any fuzzy open set of X. Then $A \leqslant \mathrm{int}(\mathrm{pcl}\, A)$. According to the assumption $f(A) \leqslant f(\mathrm{int}(\mathrm{pcl}\, A)) \leqslant \mathrm{int}(\mathrm{pcl}\, f(A))$, so $f(A)$ is a fuzzy strongly preopen set of Y. ∎

THEOREM 6.9. *Let* $f\colon X \to Y$ *be a mapping from an fts* X *into an fts* Y. *Then* f *is fuzzy strongly preopen if and only if for each fuzzy set* B *of* Y *and each fuzzy closed set* A *of* X, $f^{-1}(B) \leqslant A$, *there exists a fuzzy strongly preclosed set* C *of* Y *such that* $B \leqslant C$ *and* $f^{-1}(C) \leqslant A$.

Proof. Let B be any fuzzy set of Y and A be a fuzzy closed set of X such that $f^{-1}(B) \leqslant A$. Then $A^c \leqslant f^{-1}(B^c)$, or $f(A^c) \leqslant ff^{-1}(B^c) \leqslant B^c$. Since A^c is a fuzzy open set, $f(A^c)$ is a fuzzy strongly preopen set, so $f(A^c) \leqslant \mathrm{spint}\, B^c$. Hence $A^c \leqslant f^{-1}f(A^c) \leqslant f^{-1}(\mathrm{spint}\, B^c)$. Thus $A \geqslant f^{-1}(\mathrm{spint}\, B^c)^c = f^{-1}(\mathrm{spcl}\, B)$. The result follows for $C = \mathrm{spcl}\, B$.

Conversely, let U be a fuzzy open set of X. We show that $f(U)$ is a fuzzy strongly preopen set. From $U \leqslant f^{-1}f(U)$ follows that $U^c \geqslant (f^{-1}f(U))^c \geqslant f^{-1}f(U)^c$ where U^c is a fuzzy closed set of X. Hence there is a fuzzy strongly preclosed set B of Y such that $B \geqslant f(U)^c$ and $f^{-1}(B) \leqslant U^c$. From $B \geqslant f(U)^c$ follows $B \geqslant \mathrm{spcl}(f(U)^c)$, so $B^c \leqslant (\mathrm{spcl}\, f(U)^c)^c \leqslant \mathrm{spint}\, f(U)$. From $f^{-1}(B) \leqslant U^c$ we obtain $f^{-1}(B^c) \geqslant U$, so $B^c \geqslant ff^{-1}(B^c) \geqslant f(U)$. Hence $f(U) = \mathrm{spint}(f(U))$. Thus $f(U)$ is a fuzzy strongly preopen set, so f is a fuzzy strong preopen mapping. ∎

COROLLARY 6.10. *If* $f\colon X \to Y$ *is a fuzzy strongly preopen mapping, then:*

(i) $f^{-1}(\mathrm{cl}(\mathrm{pint}\, B)) \leqslant \mathrm{cl}\, f^{-1}(B)$ *for each fuzzy set* B *of* Y;

(ii) $f^{-1}(\mathrm{cl}\, B) \leqslant \mathrm{cl}\, f^{-1}(B)$ *for each fuzzy preopen set* B *of* Y.

Proof. (i) Let B be a fuzzy set of Y. Then $\mathrm{cl}\, f^{-1}(B)$ is a fuzzy closed set of X containing $f^{-1}(B)$. According to Theorem 6.9 there exists a fuzzy strongly preclosed set C of Y, $B \leqslant C$, such that $f^{-1}(C) \leqslant \mathrm{cl}\, f^{-1}(B)$. Thus $f^{-1}(\mathrm{cl}(\mathrm{pint}\, B)) \leqslant f^{-1}(\mathrm{cl}(\mathrm{pint}\, C)) \leqslant f^{-1}(C) \leqslant \mathrm{cl}\, f^{-1}(B)$.

(ii) It follows immediately from (i). ∎

THEOREM 6.11. *Let* $f\colon X \to Y$ *be a mapping from an fts* X *into an fts* Y. *Then* f *is fuzzy strongly preclosed if and only if for each fuzzy set* B *of* Y *and each fuzzy open set* A *of* X, *when* $f^{-1}(B) \leqslant A$, *there exists a fuzzy strongly preopen set* C *of* Y *such that* $B \leqslant C$ *and* $f^{-1}(C) \leqslant A$.

Proof. It can be proved in a similar manner as Theorem 6.9. ∎

THEOREM 6.12. *Let* $f\colon X \to Y$ *and* $g\colon Y \to Z$ *be mappings, where* X, Y *and* Z *are fts's. If* g *is fuzzy strongly preopen (preclosed) and* f *is fuzzy open (closed), then* gf *is fuzzy strongly preopen (preclosed).*

Proof. For any fuzzy open (closed) set A of X, it holds $(gf)(A) = g(f(A))$. Since f is a fuzzy open (closed) mapping and g is fuzzy strongly preopen (preclosed), we obtain that $(gf)(A)$ is a fuzzy storngly preopen (preclosed) set of Z. ∎

REFERENCES

[**1**] K. K. Azad, *On fuzzy precontinuity, fuzzy almost continuity and fuzzy weakly continuity*, J. Math. Anal. Appl. **82** (1981), 14–32.

[**2**] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24** (1968), 182–190.

[**3**] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 34–41.

[**4**] A. S. Mashour, M. E. Abd El-Monsef and S. N. El-Deep, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc, Egypt **53** (1982), 47–53.

[**5**] O. Njastad, *On some classes of nearly open sets*, Pacific J. Math. **15** (3) (1965), 961–970.

[**6**] M. K. Singal and N. Prakash, *Fuzzy preopen sets and fuzzy preseparation axioms*, Fuzzy Sets and Systems **44** (1991), 273–281.

[**7**] Bai Shi Zhong, *Fuzzy strongly preopen sets and fuzzy strong precontinuity*, Fuzzy Sets and Systems **52** (1992), 345–351.

[**8**] L. A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), 338–353.

(received 02.04.1998, in revised form 14.10.1998.)

ul. Polog 14, 91000 Skopje, Macedonia

E-mail: biljana@iunona.pmf.ukim.edu.mk