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ON NEARLY PARACOMPACT SPACES AND NEARLY FULL NORMALITY

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Abstract. This paper is a continuation of the study of nearly paracompact spaces, initiated by Singal and Arya in [5]. After suitably defining the generalized versions of normality and full normality in our setting, we achieve, as our final objective, an analogue of the celebrated theorem of A. H. Stone on paracompactness viz. "a Hausdorff topological spaces is paracompact iff it is fully normal". Incidentally, in course of the deliberation, we obtain a few extended forms of certain well known results on paracompactness.

The concept of nearly paracompact spaces was initiated by Singal and Arya [5]. Although quite a good number of varied forms of paracompactness has been introduced and studied so far, the most widely studied paracompactness-like notion is near paracompactness. In addition to [5], one may refer to [1] and [4] for further descriptions concerning such a concept. In [5], Singal has furnished a comparative study of near paracompactness vis-a-vis certain other variant forms of paracompactness.

In the present article, our sole purpose is to bring forth a generalized version of the celebrated theorem of A. H. Stone that in a Hausdorff topological space paracompactness is equivalent to the full normality of the space. To this end, we have suitably defined a weaker version of fully normal space, that suits our purpose. This notion along with a corresponding introduced form of normality is developed to the extent that we need for achieving our aim. In course of the proceedings, some general forms of certain results of paracompactness are incidentally obtained.

To make the exposition self-contained, as far as practicable, we clarify certain prerequisites as follows. By a space X we shall always mean a topological space (X,τ) in which no separation axiom is presumed, and for a subset A of X, $\operatorname{int} X$ and $\operatorname{cl} A$ will respectively denote the interior and closure of A in (X,τ). By Λ we shall always mean an index set. We recall that a set A in a space X for which $A = \operatorname{int}\operatorname{cl} A$, is called a regular open set, and complements of such sets are

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known as regular closed sets. It is known that the class of all regular open sets in a space (X,τ) is an open base for a topology on X coarser than τ, called the semiregularization topology on X, to be denoted by τ_s [2]. The members of τ_s are known as δ-open sets of X and the complements of such sets are called δ-closed sets [7]. A cover $\mathcal{U}$ of a space X by open (regular open) sets is called an open cover (regular open cover) of X. For a cover $\mathcal{U}$ of a space X and any $A \subseteq X$, the set $S(A,\mathcal{U})$ is defined by $S(A,\mathcal{U}) = \{\, U \in \mathcal{U} : A \cap U \neq \varnothing \,\}$; in particular, we shall use the notation $S(x,\mathcal{U})$ when $A = \{x\}$. If $\mathcal{U}$ and $\mathcal{V}$ are two covers of a space X, by $\mathcal{U} < \mathcal{V}$, we shall mean, as usual, that $\mathcal{U}$ is a refinement of $\mathcal{V}$; we shall adopt the usual notation $\mathcal{U}^* < \mathcal{V}$ ($\mathcal{U}^\Delta < \mathcal{V}$) to mean that $\mathcal{U}$ is a star refinement (resp. delta refinement) of $\mathcal{V}$, i.e., if the covering $\{\, S(U,\mathcal{U}) : U \in \mathcal{U} \,\}$ (resp. $\{\, S(x,\mathcal{U}) : x \in X \,\}$) refines $\mathcal{V}$. For the well known definitions of locally finite and point finite covers, and relevant results on paracompactness, one may refer to any standard text book of general topology (e.g. see Dugundji [3]).

Singal and Arya [6] called a topological space X to be almost regular if for any regular closed set F and any point $x \in X \setminus F$, there exist disjoint open sets containing F and x respectively. The notion of near paracompactness was also initiated by the same authors in the following way:

Definition 1. [5] A topological space X is nearly paracompact if every regular open cover of X has a locally finite open refinement.

We now introduce the following definition:

Definition 2. A space X is said to be nearly fully normal if for every regular open cover $\mathcal{U}$ of X, there exists an open cover $\mathcal{V}$ of X such taht $\mathcal{V}^* < \mathcal{U}$.

Remark 3. It is easy to see that a space X is nearly fully normal iff for every regular open cover $\mathcal{U}$ of X, there exists a regular open cover $\mathcal{V}$ of X such that $\mathcal{V}^* < \mathcal{U}$.

Theorem 4. *A space X is nearly fully normal iff every regular open cover of X has a regular open δ-refinement.*

Proof. If X is nearly fully normal, then by Remark 3 and the fact that for any cover $\mathcal{V}$ of X, $\mathcal{V}^\Delta < \mathcal{V}^*$, the requirement is met.

Conversely, for a given regular open cover $\mathcal{U}$ of X, there exists a regular open cover $\mathcal{V}$ such that $\mathcal{V}^\Delta < \mathcal{U}$. Now, for the regular open cover $\mathcal{V}$ of X, there is a regular open cover $\mathcal{W}$ of X such that $\mathcal{W}^\Delta < \mathcal{V}$. So $\mathcal{W} < \mathcal{W}^\Delta < \mathcal{V}$ and hence $\mathcal{W}^\Delta < \mathcal{W}^{\Delta\Delta} < \mathcal{V}^\Delta$. Since $\mathcal{W}^\Delta < \mathcal{W}^* < \mathcal{W}^{\Delta\Delta}$, we get the regular open cover $\mathcal{W}$ of X such that $\mathcal{W}^* < \mathcal{U}$, proving X to be nearly fully normal. ∎

Definition 5. A space X is said to be nearly normal if for any two disjoint sets F and G, one of which is δ-closed and the other regular closed, there exist open sets U, V such that $F \subseteq U$, $G \subseteq V$ and $U \cap V = \varnothing$.

Theorem 6. *Any nearly fully normal space X is nearly normal.*

Proof. Let F and G be two disjoint δ-closed sets in a nearly fully normal space X, with F a regular closed set. Then $X \setminus F$ is regular open while $X \setminus G$ is δ-open.

Then there exists a family $\{ G_\alpha : \alpha \in \Lambda \}$ of regular open sets in X such that $X \setminus G = \bigcup_{\alpha\in\Lambda} G_\alpha$. Thus $\mathcal{U} = \{ X \setminus F, G_\alpha : \alpha \in \Lambda \}$ is a regular open cover of X. By nearly full normality of X and Theorem 4, there exists a regular open cover $\mathcal{B}$ of X such that $\mathcal{B}^\Delta < \mathcal{U}$. We consider $U = S(F, \mathcal{B})$ and $V = S(G, \mathcal{B})$. Then U, V are open sets in X such that $F \subseteq U$ and $G \subseteq V$. In order to separate F and G strongly, we must have $U \cap V = \varnothing$. If possible, let $U \cap V \neq \varnothing$. Then there exist V_1, $V_2 \in \mathcal{B}$ such that $V_1 \cap F \neq \varnothing$, $V_2 \cap F \neq \varnothing$ and $V_1 \cap V_2 \neq \varnothing$. Hence there is $p \in V_1 \cap V_2$. Clearly, $V_1 \cup V_2 \subseteq S(p, \mathcal{B})$ showing that $S(p, \mathcal{B}) \cap F \neq \varnothing$ and $S(p, \mathcal{B}) \cap G \neq \varnothing$. Consequently, $S(p, \mathcal{B}) \nsubseteq X \setminus F$ and $S(p, \mathcal{B}) \nsubseteq X \setminus G = \bigcup_{\alpha\in\Lambda} G_\alpha$, i.e., $S(p, \mathcal{B}) \nsubseteq G_\alpha$, for all $\alpha \in \Lambda$. This is a contradiction, since $\mathcal{B}^\Delta < \mathcal{U}$. Hence $U \cap V = \varnothing$, proving the space to be nearly normal. ■

In a Hausdorff space every singleton is clearly δ-closed. Thus in a Hausdorff nearly normal space, every regular closed set and a point outside it can be strongly separated by open sets. This leads us to infer:

COROLLARY 7. *A Hausdorff nearly normal space is almost regular.*

COROLLARY 8. *A Hausdorff nearly fully normal space is almost regular.*

THEOREM 9. *A nearly paracompact Hausdorff space is nearly normal.*

Proof. We first show that a nearly paracompact Hausdorff space is almost regular. Let F be a regular closed set and $p \in X \setminus F$. For each $q \in F$, by Hausdorffness of X there exist open sets $V(q)$ and $U_q(p)$ containing q and p respectively such that $V(q) \cap U_q(p) = \varnothing$, so that $p \notin \operatorname{cl} V(q)$. Consequently, $\mathcal{V} = \{ X \setminus F, \operatorname{int}\operatorname{cl} V(q) : q \in F \}$ becomes a regular open cover of X. By near paracompactness of X, there exists a locally finite open cover $\mathcal{U}$ of X such that $\mathcal{U} < \mathcal{V}$. Now, the set $S(F, \mathcal{U})$ ($= U$, say) is an open set with $F \subseteq U$. Thus it is enough to show that $p \notin U$. In fact, if $U' \in \mathcal{U}$ is such that $U' \cap F \neq \varnothing$, then $U' \nsubseteq X \setminus F$. So as $\mathcal{U} < \mathcal{V}$, there exists $q \in F$ such that $\operatorname{int}\operatorname{cl} V(q) \supseteq U'$. Thus $\operatorname{cl} U' \subseteq \operatorname{cl} V(q)$. As $p \notin \operatorname{cl} V(q)$, $p \notin \operatorname{cl} U'$, i.e., $p \notin \bigcup \{ \operatorname{cl} U' : U' \in \mathcal{U}, U' \cap F \neq \varnothing \} = \operatorname{cl}[\bigcup \{ U' \in \mathcal{U} : U' \cap F \neq \varnothing \}]$ (since $\mathcal{U}$ is a locally finite family) $= \operatorname{cl} U$.

Next, we show that X is nearly normal. Let F and G be two disjoint δ-closed sets in X with G a regular closed set. Then by almost regularity of X, to each $p \in F$ there corresponds an open set $V(p)$ such that

$$\operatorname{cl} V(p) \cap G = \varnothing. \tag{1}$$

As $X \setminus F$ is δ-open, $X \setminus F = \bigcup_{\alpha\in\Lambda} G_\alpha$, where $\{ G_\alpha : \alpha \in \Lambda \}$ is a family of regular open sets in X. Now, $\mathcal{V} = \{ G_\alpha, \operatorname{int}\operatorname{cl} V(p) : \alpha \in \Lambda, p \in F \}$ is a regular open cover of X. By near paracompactness of X, there exists a locally finite open cover $\mathcal{U}$ of X such that $\mathcal{U} < \mathcal{V}$. We put $U = S(F, \mathcal{U})$. Then

$$\operatorname{cl} U = \operatorname{cl}\Big[\bigcup \{ U' \in \mathcal{U} : U' \cap F \neq \varnothing \}\Big] = \bigcup \{ \operatorname{cl} U' : U' \in \mathcal{U}, U' \cap F \neq \varnothing \}. \tag{2}$$

Now, $U' \cap F \neq \varnothing \implies U' \nsubseteq X \setminus F = \bigcup_{\alpha\in\Lambda} G_\alpha \implies U' \nsubseteq G_\alpha$, for all $\alpha \in \Lambda$. So as $\mathcal{U} < \mathcal{V}$, there exists $p \in F$ such that $U' \subseteq \operatorname{int}\operatorname{cl} V(p)$. Then by (1), $\operatorname{cl} U' \subseteq \operatorname{cl} V(p)$. Thus by (2), $\operatorname{cl} U \subseteq X \setminus G$. Consequently, $F \subseteq U \subseteq \operatorname{cl} U \subseteq X \setminus G$, proving the space to be nearly normal. ■

LEMMA 10. *Let X be a nearly normal space. Then for every point finite regular open cover $\Omega = \{ G_\alpha : \alpha \in \Lambda \}$ of X, there is a regular open cover $\{ V_\alpha : \alpha \in \Lambda \}$ of X such that $\operatorname{cl} V_\alpha \subseteq G_\alpha$, for each $\alpha \in \Lambda$; moreover, $V_\alpha \neq \varnothing$ whenever $G_\alpha \neq \varnothing$.*

Proof. Let X be nearly normal. By Zermelo's theorem, we can choose a well-order $<$ on Λ so that $\Omega = \{ G_\alpha : \alpha < T \}$ where T is some definite ordinal number. We shall now define a regular open set V_α, for every $\alpha < T$ such that

$$X \setminus \Big[\bigcup \{ V_\beta : \beta < \alpha \} \cup \bigcup \{ G_\gamma : \gamma > \alpha \} \Big] \subseteq V_\alpha \subseteq \operatorname{cl} V_\alpha \subseteq G_\alpha. \tag{1}$$

For this, we apply transfinite induction on the ordinal α. We define V_0 first. We have, $X \setminus [\bigcup \{ G_\gamma : \gamma > 0 \}]$ is a δ-closed set in X, contained in G_0 (as Ω is a cover of X). By near normality, there is a regular open set V_0 such that $X \setminus [\bigcup \{ G_\gamma : \gamma > 0 \}] \subseteq V_0 \subseteq \operatorname{cl} V_0 \subseteq G_0$. Suppose that we have already defined V_β, for all $\beta < \alpha$. Then $\{ V_\beta : \beta < \alpha \} \cup \{ G_\gamma : \gamma \geqslant \alpha \}$ forms a covering of X. In fact, if $p \in X$ is such that $p \notin G_\gamma$, for all $\gamma \geqslant \alpha$, by point finiteness of Ω there exists the last ordinal $\beta < \alpha$ such that $p \in G_\beta$. If $p \notin V_{\beta'}$ for all $\beta' < \beta$ we have by (1), $p \in X \setminus [\bigcup \{ V_{\beta'} : \beta' < \beta \} \cup \bigcup \{ G_{\beta''} : \beta'' > \beta \}] \subseteq V_\beta$, so that $p \in V_\beta$. Now,

$$X \setminus \Big[\bigcup \{ V_\beta : \beta < \alpha \} \cup \bigcup \{ G_\gamma : \gamma > \alpha \} \Big] \subseteq G_\alpha.$$

By near normality of X, there exists a regular open set V_α such that

$$X \setminus \Big[\bigcup \{ V_\beta : \beta < \alpha \} \cup \bigcup \{ G_\gamma : \gamma > \alpha \} \Big] \subseteq V_\alpha \subseteq \operatorname{cl} V_\alpha \subseteq G_\alpha.$$

Thus we can construct V_α satisfying (1), for each $\alpha < T$, i.e., V_α is a regular open set satisfying $\operatorname{cl} V_\alpha \subseteq G_\alpha$, for each $\alpha < T$. Finally, we are to show that $\{ V_\alpha : \alpha < T \}$ covers X. Choose $p \in X$. The covering Ω of X being point finite, there is the last ordinal number $\beta < T$ such that $p \in G_\beta$. Then $p \notin \bigcup \{ G_\gamma : \gamma > \beta \}$. Now, if $p \notin V_{\beta'}$, for each $\beta' < \beta$, we have $p \in X \setminus [[\bigcup \{ G_\gamma : \gamma > \beta \}] \cup [\bigcup \{ V_{\beta'} : \beta' < \beta \}]] \subseteq V_\beta$. Thus $p \in V_\beta$. Consequently, we get a regular open cover $\{ V_\alpha : \alpha \in \Lambda \}$ with $\operatorname{cl} V_\alpha \subseteq G_\alpha$, for each $\alpha \in \Lambda$. ∎

LEMMA 11. *Every regular open locally finite covering of a nearly normal space has a regular open delta refinement.*

Proof. Suppose $\mathcal{V} = \{ V_\alpha : \alpha \in \Lambda \}$ is a locally finite regular open cover of a nearly normal space X. We construct, by Lemma 10, a regular open cover $\mathcal{W} = \{ W_\alpha : \alpha \in \Lambda \}$ such that $\operatorname{cl} W_\alpha \subseteq V_\alpha$, for all $\alpha \in \Lambda$. Let Λ' be a subset of the index set Λ. We define

$$P(\Lambda') = \Big[\bigcap \{ V_\alpha : \alpha \in \Lambda' \} \Big] \cap \Big[\bigcap \{ X \setminus \operatorname{cl} W_\alpha : \alpha \in \Lambda \setminus \Lambda' \} \Big].$$

Let us prove that $P(\Lambda')$ is a regular open set. Since $\mathcal{V}$ is locally finite, $\overline{\mathcal{W}}$ ($= \{ \operatorname{cl} W : W \in \mathcal{W} \}$) is also so. Then

$$\begin{aligned} \bigcap \{ X \setminus \operatorname{cl} W_\alpha : \alpha \in \Lambda \setminus \Lambda' \} &= X \setminus \bigcup \{ \operatorname{cl} W_\alpha : \alpha \in \Lambda \setminus \Lambda' \} \\ &= X \setminus \operatorname{cl} \Big[\bigcup \{ W_\alpha : \alpha \in \Lambda \setminus \Lambda' \} \Big]. \end{aligned}$$

On the other hand, as $\mathcal{V}$ is locally finite, $\bigcap\{\, V_\alpha : \alpha \in \Lambda' \,\} = \varnothing$, if Λ' is infinite, and is regular open if Λ' is finite. Thus $P(\Lambda')$ is a regular open set, for each subset Λ' of Λ. For any $p \in X$, $p \in P(\Lambda')$ whenever $\Lambda' = \{\, \alpha \in \Lambda : p \in \operatorname{cl} W_\alpha \,\}$. Thus $\mathcal{P} = \{\, P(\Lambda') : \Lambda' \subseteq \Lambda \,\}$ is a regular open cover of X. To prove $\mathcal{P}^\Delta < \mathcal{V}$, we consider a given point p of X. Since $\mathcal{W}$ is a covering of X, $p \in W_\beta$ for some $\beta \in \Lambda$. Let $p \in P(\Lambda')$, then in view of the definition of $P(\Lambda')$, we know that $\beta \in \Lambda'$; because if $\beta \in \Lambda \setminus \Lambda'$, then $p \notin X \setminus \operatorname{cl} W_\beta \supseteq P(\Lambda')$, a contradiction as $p \in P(\Lambda')$. Thus $P(\Lambda') \subseteq V_\beta$. As the inclusion holds for all $P(\Lambda')$ which contains p, $S(p, \mathcal{P}) \subseteq V_\beta$, proving that $\mathcal{P}^\Delta < \mathcal{V}$. ■

As a last step towards our prerequisites for attaining our desired goal we recall the following result from [5].

LEMMA 12. *An almost regular space X is nearly paracompact iff each regular open cover of X has an open σ-locally finite refinement.*

With the deliberations so far, we are now set to prove the desired analogue of the celebrated Stone's theorem as follows.

THEOREM 13. *A Hausdorff space X is nearly paracompact iff it is nearly fully normal.*

Proof. Let us first assume that X is Hausdorff and nearly paracompact. Then by Theorem 9, X is nearly normal. If $\mathcal{U}$ is any regular open cover of X, then $\mathcal{U}$ has a regular open locally finite refinement $\mathcal{V}$. By Lemma 11, there exists a regular open delta refinement $\mathcal{W}$ of $\mathcal{V}$, i.e., $\mathcal{W}^\Delta < \mathcal{V}$. Then $\mathcal{W}^\Delta < \mathcal{U}$. Then by Theorem 4, X is nearly fully normal.

Conversely, let X be nearly fully normal Hausdorff space. Let $\Omega = \{\, G_\alpha : \alpha \in I \,\}$ (I being an index set) be a regular open cover of X. We define a sequence of regular open covers $\{\Omega_i\}$ of X taking $\Omega_0 = \Omega$ and Ω_{n+1} to be a regular open star refinement of Ω_n for $n = 0, 1, 2, \ldots$. Let $G_{\alpha,n} = \{\, x \in X :$ there is a regular open set V containing x such that $S(V, \Omega_n) \subseteq G_\alpha \,\}$, for every $\alpha \in I$ and $n = 1, 2, \ldots$. We show that for each n, the family $\{G_{\alpha,n} : \alpha \in I\,\}$ is an open cover of X, which is a refinement of Ω. We apply induction on n to prove that $\{\, G_{\alpha,n} : \alpha \in I \,\}$ is a covering of X for each $n \in \mathbf{N}$ ($\mathbf{N}$ denoting the set of all natural numbers). Let $x \in X$. Then for some $V^1 \in \Omega_1$, $x \in V^1$. Since $\Omega_1^* < \Omega$, there is $G_\alpha \in \Omega$ such that $S(V^1, \Omega_1) \subseteq G_\alpha$. Hence by definition of $G_{\alpha,1}$ we have $x \in G_{\alpha,1}$. Thus $\{\, G_{\alpha,1} : \alpha \in I \,\}$ is a cover of X.

Suppose that $\{\, G_{\alpha,n-1} : \alpha \in I \,\}$ is a cover of X. Then for any $x \in X$, there is a regular open set V containing x such that $S(V, \Omega_{n-1}) \subseteq G_\alpha$, for some $\alpha \in I$, i.e., $x \in G_{\alpha,n-1}$. But Ω is a cover of X and hence $x \in V^n$, for some $V^n \in \Omega$. As $\Omega_n^* < \Omega_{n-1}$, there is $T^{n-1} \in \Omega_{n-1}$ such that $S(V^n, \Omega_n) \subseteq T^{n-1}$. Since $x \in S(V^n, \Omega_n)$, we have $x \in T^{n-1} \in \Omega_{n-1}$. Hence $x \in V \cap T^{n-1}$, i.e., $V \cap T^{n-1} \neq \varnothing$. Thus $T^{n-1} \subseteq S(V, \Omega_{n-1}) \subseteq G_\alpha$, so that $S(V^n, \Omega_n) \subseteq T^{n-1} \subseteq G_\alpha$. Thus $x \in G_{\alpha,n}$ and hence $\{\, G_{\alpha,n} : \alpha \in I \,\}$ is a cover of X, for each $n \in \mathbf{N}$. We prove that if $x \in G_{\alpha,n}$ and $y \notin G_{\alpha,n+1}$, then there does not exist any $G \in \Omega_{n+1}$ such that $x, y \in G$. (1)

For every $G \in \Omega_{n+1}$, there exists $H \in \Omega_n$, such that $S(G, \Omega_{n+1}) \subseteq H$ (since $\Omega^*_{n+1} < \Omega_n$). Thus $x \in G \cap G_{\alpha,n} \implies H \subseteq S(x, \Omega_n) \subseteq G_\alpha$. In fact, as $x \in G$, $x \in S(G, \Omega_{n+1})$ and hence $x \in H$, where $H \in \Omega_n$ and hence $H \subseteq S(x, \Omega_n)$. Again $x \in G_{\alpha,n}$ implies that there exists some open neighbourhood V of x such that $S(V, \Omega_n) \subseteq G_\alpha$. As $x \in V$, all those members of Ω_n which contain x have nonempty intersection with V and hence each is contained in $S(V, \Omega_n)$; then $S(x, \Omega_n) \subseteq G_\alpha$. Now, as $S(G, \Omega_{n+1}) \subseteq H$, for some $H \in \Omega_n$, and $H \subseteq G_\alpha$, we obtain $S(G, \Omega_{n+1}) \subseteq H$. Now for $z \in G$, as G is a regular open set containing x with $S(G, \Omega_{n+1}) \subseteq G_\alpha$, we have $z \in G_{\alpha,n+1}$ so that $G \subseteq G_{\alpha,n+1}$. Thus $y \notin G$.

Let us now suppose that the set I is well-ordered by the relation '$<$' and let

$$H_{\beta,n} = G_{\beta,n} \setminus \mathrm{cl}\Big[\bigcup_{\alpha<\beta} G_{\alpha,n+1}\Big], \quad \text{for every } \beta \in I \text{ and each } n \in \mathbf{N}.$$

For any pair of distinct elements $\gamma, \eta \in I$, we have either $\gamma < \eta$ or $\gamma > \eta$ and correspondingly we have $H_{\eta,n} \subseteq X \setminus G_{\gamma,n+1}$ or $H_{\gamma,n} \subseteq X \setminus G_{\eta,n+1}$. Now, if $x \in H_{\gamma,n}$ and $y \in H_{\eta,n}$ where $\gamma \neq \eta$ (suppose $\gamma < \eta$), there is no set $G \in \Omega_{n+1}$ which would contain both x and y. Indeed, $x \in H_{\gamma,n}$ means $x \in G_{\gamma,n}$. Again $y \in H_{\eta,n} \subseteq X \setminus G_{\gamma,n+1}$ means that $y \notin G_{\gamma,n+1}$. Then by (1), there is no $G \in \Omega_{n+1}$ which contains both x and y. Thus we see that for every $x \in X$, there is some $G \in \Omega_{n+1}$ such that $x \in G$ and G intersects at most one $H_{\alpha,n}$, for a fixed $n \in \mathbf{N}$ ($H_{\alpha,n}$'s being mutually disjoint). Hence $\{\, H_{\alpha,n} : \alpha \in I \,\}$ is a discrete family of open sets, for $n = 1, 2, \dots$.

Finally we shall show that $\{\, H_{\alpha,n} : \alpha \in I, n \in \mathbf{N} \,\}$ is a cover of X. Let $y \in X$, and let α_n is the first index such that $y \in G_{\alpha_n,n}$ (that $\{\, G_{\alpha,n} : \alpha \in I \,\}$ is a cover of X, for each $n \in \mathbf{N}$, has already been proved). Taking $\inf\{\, \alpha_n : n \in \mathbf{N} \,\} = \alpha(y)$ (say), we obtain $y \in G_{\alpha(y),n}$, for some $n \in \mathbf{N}$. So for $\alpha < \alpha(y) \leqslant \alpha_{n+2}$, $y \notin G_{\alpha,n+2}$. For $\alpha < \alpha(y)$, let $x \in G_{\alpha,n+1}$. Then there does not exist any $G \in \Omega_{n+2}$ such that x and y both belong to G. So the collection of sets in Ω_{n+2} which contain y cannot intersect $\bigcup\{\, G_{\alpha,n+1} : \alpha < \alpha(y) \,\}$. Now, $H_{\alpha(y),n} = G_{\alpha(y),n} \setminus \mathrm{cl}[\bigcup_{\alpha<\alpha(y)} G_{\alpha,n+1}]$ and $y \in G_{\alpha(y),n}$. But since $S(y, \Omega_{n+2}) \cap [\bigcup\{\, G_{\alpha,n+1} : \alpha < \alpha(y) \,\}] = \varnothing$, $y \notin \mathrm{cl}[\bigcup_{\alpha<\alpha(y)} G_{\alpha,n+1}]$. Thus $y \in H_{\alpha(y),n}$. Hence every regular open cover of X has an open σ-discrete refinement. Every σ-discrete refinement being σ-locally finite, it follows by virtue of Corollary 8 and Lemma 12 that X is nearly paracompact. ■

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