МАТЕМАТИЧКИ ВЕСНИК
50 (1998), 83–87

UDK 517.535
оригинални научни рад
research paper

BOUNDARY BEHAVIOR OF SUBHARMONIC FUNCTIONS ON THE UNIT DISK

Žarko Pavićević and Jela Šušić

Abstract. In this paper, we prove some boundary properties of subharmonic functions in the open unit disk of the complex plane.

1. Introduction, preliminary notations and definitions

In this paper we investigate boundary properties of harmonic and subharmonic functions. These functions were firstly investigated by I. I. Privalov [1], and later by E. D. Solomencev ([2], [3]) and M. Tsuji [4] and by many other mathematicians. The results obtained show that the investigation of boundary properties of subharmonic functions is more complicated than that of harmonic and meromorphic functions. This shows a function $u(z)$ subharmonic on the disk $\Delta = \{\, z \mid |z| < 1 \,\}$ satisfies the condition ([4])

$$\int_0^{2\pi} \bigl| u\bigl(r \exp(i\theta)\bigr) \bigr| \, d\theta = \mathcal{O}(1), \quad r \to 1. \tag{1}$$

According to the classical result of J. E. Littlewood, for a function $u(z)$ there holds

$$u(z) = \nu(z) + \omega(z),$$

where $\nu(z)$ is a harmonic function on Δ which satisfies (1), and

$$\omega(z) = \iint_{|a|<1} \ln \left| \frac{1 - \bar{a} z}{z - a} \right| \, d\mu(a)$$

is the Green potential of measure $d\mu(a)$ satisfying the condition

$$\iint_{|a|<1} \bigl(1 - |a|\bigr) \, d\mu(a) < +\infty.$$

The harmonic function $\nu(z)$ has the angular boundary values almost everywhere on $\partial\Delta = \{\, z \mid |z| = 1 \,\}$, while the Green potential $\omega(z)$ has the radial limits

AMS Subject Classification: 30 E 25, 30 D 40, 31 A 05

Communicated at the 4th Symposium on Mathematical Analysis and Its Applications, Aranđelovac 1997.

almost everywhere on $\partial\Delta$, but does not necessarily have angular boundary values on $\partial\Delta$ (see [4]).

The property of the normality in the sense of Montel, with respect to the group of all conformal automorphisms of the disk Δ, gives interesting boundary properties of meromorphic functions, while this is not the case for subharmonic functions. For more information about these questions, see [1], [2], [3], [4], [5], [6], [7].

In this paper, we investigate locally boundary properties of harmonic and subharmonic functions on the open unit disk D of the complex plane. Statements which we will formulate in this paper are related to some results of Berberian ([5]).

Notations

$\mathbf{C}$ — complex plane;

$\overline{\mathbf{C}} = \Omega$ — extended complex plane, $\mathbf{C} \cup \{\infty\}$, identified with the Riemann sphere;

$D = \{\, z \in \mathbf{C} \mid |z| < 1 \,\}$ — the open unit disk; $\partial D = \{\, z \mid |z| = 1 \,\}$ — unit circle;

$h(\xi, \varphi)$ — chord at $\xi = \exp(i\theta) \in \partial D$ that makes the angle φ, $-\pi/2 < \varphi < \pi/2$ with the radius $h(\xi, 0)$;

$\Delta(\xi, \varphi_1, \varphi_2)$ — Stolz angle at $\xi = \exp(i\theta)$, between the chords $h(\xi, \varphi_1)$ and $h(\xi, \varphi_2)$;

G — group of all conformal automorphisms of the disk D;

$g \in G$, $g^n(z) = \underbrace{(g \circ g \circ \cdots \circ g)}_{n}(z)$

$T_H^\theta = \left\{ S_H^\theta \,\middle|\, S_H^\theta = \dfrac{z + a\exp(i\theta)}{1 + a\exp(-i\theta)z},\ a \in (-1, 1) \right\}$, θ fixed, $0 \leqslant \theta < \pi$, — hyperbolic subgroup of the group G with two fixed points $\exp(i\theta)$ and $-\exp(i\theta)$;

$\Delta(\omega, r) = \left\{ z \in D \,\middle|\, \left| \dfrac{z - \omega}{1 - \bar{\omega} z} \right| < r \right\}$, $\omega \in D$;

$\Delta_H^\theta = \bigcup\limits_{a \in (-1,1)} \Delta\big(a\exp(i\theta), r\big)$;

$\sigma(z_1, z_2)$ — hyperbolic distance in D;

$\psi(z_1, z_2)$ — pseudohyperbolic distance in D

$$\psi(z_1, z_2) = \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right| = \tanh\big(\sigma(z_1, z_2)\big);$$

$N_\delta(\xi) = \{\, z \in \mathbf{C} \mid |z - \xi| < \delta \,\}$, $\delta > 0$.

Definition 1. A function f defined in D is said to be normal in D if the family $\{\, f \circ S \mid S \in G \,\}$ is a normal family in the sense of Montel.

Definition 2. A function f defined in D belongs to the class N if f is normal in D.

Definition 3. A function f defined in D belongs to the class N^θ, $0 \leqslant \theta < \pi$, if the family $\{ f \circ S \mid S \in T_H^\theta \}$ is normal in the sense of Montel.

Definition 4. Let $K \subset D$ such that $K \cap \partial D = \exp(i\theta) \in \partial D$. Then the set $C\big(f, K, \exp(i\theta)\big) = \big\{ a \in \Omega \mid \exists (z_n) \subset K$ with $\lim_{n\to\infty} z_n = \exp(i\theta)$ and $\lim_{n\to\infty} f(z_n) = a \big\}$ is said to be the cluster set for a function f with respect to the set K in $\exp(i\theta)$.

Definition 5. A point $\exp(i\theta) \in \partial D$ is said to be a Fatou's point of f if the set $\bigcup C\big(f, \Delta(\exp(i\theta), \varphi_1, \varphi_2), \exp(i\theta)\big)$ consists of a single value $a = f\big(\exp(i\theta)\big) \in \Omega$, where the union is taken over all angles $\Delta\big(\exp(i\theta), \varphi_1, \varphi_2\big)$ at $\exp(i\theta)$. Then $a = f\big(\exp(i\theta)\big)$ is an angular limit of f at point $\exp(i\theta)$.

We denote by $F(f)$ the set of all Fatou's points of f.

2. Preliminary results

Theorem A. (Berberian [5]) *Let $\xi = \exp(i\theta)$ with $0 \leqslant \theta < \pi$, and let u be a subharmonic function in D satisfying the following conditions:*

(i) $u \in N^\theta$;

(ii) *there is a $\delta > 0$ such that for all $z \in N_\delta(\xi) \cap D$ there holds $u(z) \leqslant \alpha < +\infty$;*

(iii) *there is a sequence $(z_n) \subset \Delta_H^\theta(r)$ such that $\lim_{n\to\infty} z_n = \xi = \exp(i\theta)$, $\lim_{n\to\infty} \sigma(z_n, z_{n+1}) = M < +\infty$, and $\lim_{n\to\infty} u(z_n) = \alpha$.*

Then $\xi = \exp(i\theta) \in F(u)$ and $u\big(\exp(i\theta)\big) = \alpha$.

Remark 1. Replacing in Theorem A the sequence (z_n) by a curve in $\Delta_H^\theta(r)$, we obtain the same statement proved by J. Meek [6] for the class N. J. Meek showed also that the condition (ii) of locally boundedness of u cannot be omitted.

Remark 2. An example of the modular function given by Berberian shows that the condition (iii) from Theorem A also cannot be omitted.

Theorem B. (Berberian [5]) *Suppose that u is a harmonic function in D satisfying the following conditions:*

(i) $u \in N^\theta$ *with fixed* $0 \leqslant \theta < \pi$;

(ii) *α is an exceptional point for u in the sense of Picard;*

(iii) *there exists a sequence $(z_n) \subset \Delta_H^\theta(r)$ such that $\overline{\lim}_{n\to\infty} \sigma(z_n, z_{n+1}) = M < +\infty$, $\lim_{n\to\infty} z_n = \xi = \exp(i\theta)$ and $\lim_{n\to\infty} u(z_n) = \alpha$.*

Then $\xi = \exp(i\theta) \in F(u)$ and $u\big(\exp(i\theta)\big) = \alpha$.

Theorem C. (Berberian [5]) *Let u be a subharmonic function on D which belongs to N^θ with fixed $0 \leqslant \theta < \pi$, and suppose that for some sequence $(z_n) \subset \Delta_H^\theta(r)$ with $\overline{\lim}_{n\to\infty} \sigma(z_n, z_{n+1}) = M < +\infty$ and $\lim_{n\to\infty} z_n = \xi = \exp(i\theta)$, u has a bounded above cluster set $C\big(u, (z_n)\big)$. Then a function u is bounded above on each angle $\Delta\big(\exp(i\theta), \varphi_1, \varphi_2\big) \subset D$.*

Theorem D. (Berberian [5]) *Let u be a continuous function on D belonging to N^θ with fixed $0 \leqslant \theta\pi$. Suppose that for some sequence $(z_n) \subset \Delta_H^\theta(r)$ with $\overline{\lim}_{n\to\infty} \sigma(z_n, z_{n+1}) = M < +\infty$ and $\lim_{n\to\infty} z_n = \xi = \exp(i\theta)$, u has a bounded cluster set $C\big(u,(z_n)\big)$. Then the function u is bounded on each angle $\Delta\big(\exp(i\theta), \varphi_1, \varphi_2\big) \subset D$.*

Lemma. There exists an element $g \in T_H^\theta$ with fixed $0 \leqslant \theta < \pi$, such that for each $z \in \overline{\mathbf{C}}$ there holds:

(i) $\lim_{n\to\infty} g^n(z) = \exp(i\theta)$;

(ii) $\lim_{n\to\infty} \left(g^{-1}\right)^n(z) = \exp(i\theta)$.

Proof. The assertion of our Lemma follows immediately from Theorem 4.3.10 in [8]. ■

The points $\exp(i\theta)$ and $-\exp(i\theta)$ from the above lemma are called attractive points for the elements g and g^{-1} of the group T_H^θ, respectively.

Every element from T_H^θ, except the identity, has either $\exp(i\theta)$ or $-\exp(i\theta)$ as an attractive point. If $\exp(i\theta)$ is an attractive point for $g \in T_H^\theta$, then $-\exp(i\theta)$ is an attractive point for g^{-1}.

3. Main results and their proofs

Theorem 1. *Suppose that a function u subharmonic in D belongs to N^θ, $0 \leqslant \theta < \pi$, and $u(z) \leqslant \alpha < +\infty$ for all $z \in N_\delta(\xi) \cap D$, $\xi = \exp(i\theta)$. Then the following statements are equivalent.*

(i) *$\xi = \exp(i\theta) \in F(u)$ and $u\big(\exp(i\theta)\big) = \alpha$;*

(ii) *there exist $g \in T_H^\theta$ and $z \in D$ so that $\exp(i\theta)$ is an attractive point for g, and $\lim_{n\to\infty} u\big(g^n(z)\big) = \alpha$;*

(iii) *for any $g \in T_H^\theta$, $\exp(i\theta)$ is an attractive point for g, and for each $z \in D$ there holds $\lim_{n\to\infty} u\big(g^n(z)\big) = \alpha$.*

Theorem 2. *Suppose that a function u harmonic in D satisfies the following conditions:*

(i) *$u \in N^\theta$ with fixed $0 \leqslant \theta < \pi$;*

(ii) *α is an exceptional value for u in the sense of Picard.*

Then the following statements are equivalent:

(i) *$\xi = \exp(i\theta) \in F(u)$ and $u\big(\exp(i\theta)\big) = \alpha$;*

(ii) *there exist $g \in T_H^\theta$ and $z \in D$, such that $\exp(i\theta)$ is an attractive point for g and $\lim_{n\to\infty} u\big(g^n(u)\big) = \alpha$;*

(iii) *for each $g \in T_H^\theta$, such that $\exp(i\theta)$ is an attractive point for g, and $\lim_{n\to\infty} u\big(g^n(z)\big) = \alpha$ for each $z \in D$.*

Theorem 3 *Suppose that a function u subharmonic in D belongs to N^θ for a fixed $0 \leqslant \theta < \pi$. Then the following conditions are equivalent:*

(i) *u is bounded above on each angle* $\Delta(\xi,\varphi_1,\varphi_2)\subset D$, $\xi=\exp(i\theta)$*;*

(ii) *there exist* $g\in T_H^\theta$ *and* $z\in D$, *such that* $\xi=\exp(i\theta)$ *is an attractive point for* g, *and* $C\big(u,g^n(z)\big)$ *is a bounded above set;*

(iii) *for any* $g\in T_H^\theta$, *such that* $\xi=\exp(i\theta)$ *is an attractive point for* g, $C\big(u,g^n(z)\big)$ *is a bounded above set for each* $z\in D$.

THEOREM 4. *Suppose that a function* $u\in N^\theta$, $0\leqslant\theta<\pi$, *is continuous on* D. *Then the following conditions are equivalent:*

(i) *u is bounded on each angle* $\Delta(\xi,\varphi_1,\varphi_2)\subset D$, $\xi=\exp(i\theta)$*;*

(ii) *there exist* $g\in T_H^\theta$ *and* $z\in D$ *such that* $\xi=\exp(i\theta)$ *is an attractive point for* g, *and* $C\big(u,g^n(z)\big)$ *is a bounded set;*

(iii) *for any* $g\in T_H^\theta$, *such that* $\xi=\exp(i\theta)$ *is an attractive point for* g, $C\big(u,g^n(z)\big)$ *is a bounded set for each* $z\in D$.

Proofs of Theorems 1, 2, 3 and 4. Since the hyperbolic and pseudohyperbolic distances in D are invariant with respect to the elements of the group G, it follows that for sequences $\big(g^n(z)\big)$, $z\in D$ is fixed, there holds

$$\sigma\big(g^n(z),g^{n+1}(z)\big)=\sigma\big(z,g(z)\big)=M<+\infty \text{ for all } n\in\mathbf{N}.$$

For any fixed $z\in D$ we can find an r, $|z|<r<1$ such that $g^n(z)\in\Delta_H^\theta(r)$ for all $n\in\mathbf{N}$. From this fact and our Lemma, it follows that every sequence $\big(g^n(z)\big)$ satisfies all conditions of Theorems A, B, C and D, which implies the implications (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) of Theorems 1, 2, 3 and 4, respectively. The implications (iii) $\Rightarrow$ (ii), (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are trivial. This completes the proof. ■

REFERENCES

[**1**] I. I. Privalov, *Subharmonic Functions*, Moscow, 1937 (in Russian).

[**2**] E. D. Solomencev, *On boundary values of subharmonic functions*, Ch. Math. J. **8** (1958), 520–536 (in Russian).

[**3**] E. D. Solomencev, *An example of a bounded and continuous subharmonic function which does not have angular boundary values*, Sib. Math. J. **1** (1960), 488–491 (in Russian).

[**4**] M. Tsuji, *Littlewood theorem on subharmonic functions in a unit circle*, Comment. Math. Univ. St. Pauli **5** (1956), 3–16.

[**5**] S. L. Berberian, *Boundary behavior of subharmonic and meromorphic functions*, Kandidat. Dissertation, Moscow State University, Moscow, 1979 (in Russian).

[**6**] J. Meek, *On Fatou's points of normal subharmonic functions*, Math. Japonica **22**, 3 (1977), 309–314.

[**7**] J. Meek, *Subharmonic versions of Fatou's theorem*, Proc. Amer. math. Soc. **30** (l971), 313–317.

[**8**] A. F. Berdon, *The Geometry of Discrete Groups*, Springer-Verlag, New York–Heidelberg–Berlin, 1983.

(received 01.08.1997, in revised form 09.03.1998.)

University of Montenegro, Faculty of Mathematical and Natural Sciences, P. O. Box 211, 81000 Podgorica, Yugoslavia

E-mail: zarko@rc.pmf.cg.ac.yu